Random Walks and Electrical Networks 3

**Reminder:** \( G = (V, E) \) graph, \( e \in V \)

\[ \sum_{e \in E} c_e \] (c(e) \in \mathbb{E}) Positive numbers (conductance)

**Random Walk:**
\[ P(X_{t+1} = V | X_t = u) = \frac{c_{uv}}{\sum_{w : w \in V} c_{wu}} \]

**Directed Edges**

**Flow:** \( \Theta : E \rightarrow \mathbb{R} \) satisfying node law

\( \Theta \) at any vertex \( v \neq a, z \) and is antisymmetric.

\[ \|\Theta\|_1 = \sum_{u \in V} \Theta_{au} \] Strength

**Theorem (Thomson’s Principle):**

\[ R_{eff}(a \rightarrow z) = \inf \{ \Theta(\Theta) : \Theta \text{ flow, } \|\Theta\|_1 = 1 \} \]

\[ e_{(\Theta)} = \frac{1}{2} \sum_{e \in E} \Theta(e)^2 = \sum_{e \in E} c_{ee} \Theta(e)^2 \]

We saw an extension to infinite graphs where we had \( R_{eff}(a \rightarrow \infty) \).

**Connection to Random Walk:**

\[ R_{eff}(a \rightarrow z) = \frac{1}{P_{a \rightarrow z}|z \rightarrow a(z)} \sum_{u \in V} \Theta_{au} \]

Get to \( z \) before returning to \( a \)

Similarly,

\[ R_{eff}(a \rightarrow \infty) = \frac{1}{P_{a \rightarrow \infty}\text{ (walk never) } \sum_{u \in V} \Theta_{au}} \]

**Corollary:** Random walk is recurrent

\[ R_{eff}(a \rightarrow a) = \infty. \]
**Corollary:** Random walk is recurrent if \( P(\text{Remp}(a \leftrightarrow 0) = \infty) \).

**Nash-Williams inequality**

A cutset between \( a \) and \( z \) is a set of edges \( T \) separating \( a \) from \( z \) (every path from \( a \) to \( z \) must use an edge of \( T \)).

**Prop.:** For every flow from \( a \) to \( z \) and every cutset \( T \),

\[
\sum_{e \in T} \delta(e) \geq 1.
\]

(Sketch: reduce \( T \) to a minimal cutset, then a unit flow must leave the side of \( a \) of the cutset towards the side of \( z \).)

**Thm. (Nash-Williams):** Let \( (T_n) \) be disjoint cutsets separating \( a \) from \( z \).

Then \( \text{Remp}(a \leftrightarrow 0) \geq \sum_{T_n} \delta(e) \).

Similarly if \( z = \infty \),

\[ \text{Remp}(a \leftrightarrow \infty) \geq \sum_{T_n} \delta(e) \]

**Example:** \( \mathbb{Z}^2 \) is recurrent.

\( T_n \) = boundary of the ball or radius \( n \) around 0.

\[ |T_n| = \Theta(n^2) \]

\[ \implies \text{Remp}(0 \leftrightarrow \infty) = \sum_{T_n} \frac{1}{|T_n|} = \infty. \]

\(-2\) is recurrent.
\( \Rightarrow \exists^2 \) is recurrent.

**Proof of theorem:** Let \( \theta \) be a flow with \( \|\theta\|_1 = 1 \).

\[
\varepsilon(\theta) = \sum_{e \in E} \text{Re} \theta(e)^2 \geq \sum_{n \in G_T} \text{Re} \theta(e)^2.
\]

\[\Rightarrow \sum_{e \in G_T} \text{Re} \theta(e)^2 \geq \frac{1}{\sum_{e \in G_T} C_e}.\]

Proving the theorem.

**Remark:** There are recurrent graphs for which the Nash-Williams does not prove recurrence.

**Random paths**

This is a method to generate unit flows. For \( \lambda \), it may even generate the unit current flow.

**Proof:** Let \( P \) be a probability distribution over paths from \( a \) to \( z \).

For a path \( \gamma \) define

\[
\Theta_\gamma(z) = \text{number of times } z \text{ is traversed by } \gamma - \text{number of times } z \text{ is traversed by } \gamma.
\]

**Claim:** \( \Theta \) is a flow from \( a \) to \( z \), \( \|\Theta\|_1 = 1 \).

**Proof:** Check that for each path \( \gamma \) from \( a \) to \( z \), \( \Theta_\gamma \) is a flow of unit strength.

This passes to the expected value.

\( \sum_{g \in G_T} \Theta_g \) is transient.
This passes to the expected value.

**Example:** $\mathbb{R}^3$ is transient.

**Proof sketch:** given a unit vector $\vec{n} \in S^2 \subset \mathbb{R}^3$.

Let $\gamma_\vec{n}$ be a lattice approximation of the straight line from 0 to $\infty$ in the direction $\vec{n}$.

Let $\Theta(\vec{e}) = \sum_{\vec{e} \in \gamma_\vec{n}} \delta_{\vec{e}}$

When $\vec{n}$ is chosen uniformly on $S^2$. Idea: $|\Theta(\vec{e})| \sim \frac{1}{d(0, \vec{e})^2}$

$\implies \Theta(\vec{n}) \equiv \sum_{\vec{e} \in \Theta(\vec{n})} \delta_{\vec{e}} \sim \sum_{\vec{e} \in \Theta(\vec{n})} \frac{1}{d(0, \vec{e})^2} < \infty$

The number of edges in boundary of the ball of radius $n$ around origin.

By Thomson's principle, $\text{Re} \rho(0 \leftrightarrow \infty) < \infty$

So $\mathbb{R}^3$ is transient.

**Galton-Watson trees** (E.g., Lyons-Peres book)

**Background:** Francis Galton was interested in the disappearance of family names.

Model: Say that a person has a random number of children, sampled according to a distribution $\nu$ ($\nu$ is supported on $\{1, 2, 3\}$).

Say that this is independent from person to person.

Reverend Watson analyzed this in 1874.

Basic theorem - family tree will die out a.s. if the average number of children is smaller, or equal, to 1 unless $\nu$ is supported on $\{1, 2, 3\}$. 

Almost surely.
almost surely

Formal Statement:

Let $\mu$ be supp. on $\xi_0, \xi_1, \xi_2, \ldots$. We just discuss the number of children at every generation instead of the full tree.

Let $Z_0 = 1$. For each $n \geq 1$,

let $Z_{n+1} = \frac{Z_n}{\xi_n} X_n^{(k)}$ number of children of $k$th person in generation $n$

where $(X_n^{(k)})_{n \geq 0, k \geq 1}$ are i.i.d. samples of $\mu$.

Basic question: Survival probability $P(\forall n, Z_n > 0) > 0$?

Theorem: $P(\forall n, Z_n > 0) > 0$ if and only if

average $m = \mathbb{E}X > 1$ or $P(X = 1) = 1$

number of children where $X \sim \mu$.

Example: Percolation on a binary tree.

Each edge is kept with prob. $p \in [0, 1]$, indep. among edges.

$\{0, 1\} \rightarrow \mathbb{P}(\exists \text{ an infinite connected component}) = \sum_{p > \frac{1}{2}} \mathbb{P}(\text{conn. comp. of root is a GW tree with } \mu = \text{Bin}(2, p))$ by Kolmogorov's $0-1$ law

Proof of thm.: The case $m < 1$:

Note that $\mathbb{E}Z_n = \mathbb{E}(\mathbb{E}(Z_n | Z_{n-1})) = (n+1)$

$= \mathbb{E}(\mathbb{E}(\prod_{k=1}^{Z_{n-1}} X_k^{(k)} | Z_{n-1})) = \mathbb{E}(m^{Z_{n-1}})$

$= m^n$. $P(Z_n > 1)$
\[ n = 7 \implies m^n. \quad \Pr(Z_n \geq 1) \]

By Markov's ineq., \( \Pr(Z_n > 0) \leq m^n \)

Note \( \forall n, Z_n > 0 \Rightarrow \bigcup \{ Z_n > 0 \} = \bigcap \{ Z_n > 0 \} = \lim_{n \to \infty} \Pr(Z_n > 0) \to 0 \)

The case \( m = 1 \):

Note that, for any finite \( m \),

\[ M_n := \frac{Z_n}{m^n} \text{ is a martingale}. \]

Indeed, \( \mathbb{E}(M_n | Z_{n-1}, Z_{n-2}) = (n > 1) \)

\[ = \frac{1}{m^n} \mathbb{E}(Z_n | Z_{n-1}) = \frac{1}{m^n} m Z_{n-1} = M_{n-1} \]

In particular, when \( m = 1 \), \( Z_n \) is a mart.

By the mart. conv. theorem \( (Z_n > 0) \)

\[ \exists \Omega \text{ s.t. } Z_n \to \infty \text{ almost surely.} \]

Since \( Z_n \) is integer-valued we conclude that, a.s., \( (Z_n) \) is eventually constant.

If \( \Pr(X = 1) \neq 1 \) then this is only possible if \( (Z_n) \) is eventually zero.

The general case:

We may assume that \( \mathbb{E}X < \infty \) since otherwise we may truncate \( X \)
(replace it by \( \min \{ X, M \} \) for \( M \) large enough) and analyze \( \Pr(X > 0) \) for the truncated process.

**Probability generating function**:

\[ f(s) := \mathbb{E}(s^X) \quad \text{for } 0 \leq s \leq 1 \]

with \( f(0) := \Pr(X = 0), f(1) := 1. \)
with \( \phi(0) = p(X=0) \), \( \phi(1) = 1 \).

\[ \phi(s) = \sum_{k=0}^{\infty} p(X=k) s^k. \]

**Properties:**

1. \( \phi \) is non-decreasing.
2. \( \phi \) is convex.
3. \( \phi'(1) = E(X) = m. \)

**Proof:**

**Define** \( \psi_n(s) = E(s^{Z_n}) \) so that \( \psi_1 = \phi \).

**Claim:** \( \psi_{n+1}(s) = \phi(\psi_n(s)) \) for \( s \geq 0, 1 \).

**Proof:**

\[ \psi_{n+1}(s) = E(s^{Z_{n+1}}) = E(E(s^{Z_n} | Z_n)) = E(\psi_n(s)) \]

**Thus** \( \psi_n(s) = \phi(\phi(...(\phi(\psi(1))))...) \)

**By Def.** \( \phi(Z_n = 0) = \psi_n(0) \)

By claim = \( \phi(\psi_{n-1}(0)) = \phi(p(Z_{n-1} = 0)) \)

In particular, \( \phi(Z_n = 0) \) \( \rightarrow \) \( p \) (process dies out) = \( q \)

Thus we would like to investigate \( \phi(\phi(...(\phi(\psi(1))))...) \)

as we apply \( \phi \) more and more.

\( \phi \) is convex, hence continuous,

\[ \phi(\frac{1}{2} \phi(0)) = \phi(p(Z_{1/2} = 0) = \phi(p(Z_1 = 0)) \)
If $f$ is convex, then $\nu_n \to \nu$.

So since $\nu_2 = 0 = f(\nu_1)$

$$9 = f(9)$$

Extension problem is a fixed point of prob. gen. fcn. In fact, it is the smallest fixed point.

$m > 1$:

Thus $\nu_n > 0$.

$1 - \nu$ where $\nu$ is the smallest fixed point of $f$.

$m = 1$:

$\nu(1) \neq 1$

$m < 1$:

Non-negative.
Sub-critical case: \( m < 1 \).

We saw that \( P(\varepsilon_n > 0) \leq m^n \).

How sharp is this?

**Theorem (Heathcote, Seneta-Jones 1967):**

\[ \forall m, \frac{P(\varepsilon_n > 0)}{m^n} \] is non-increasing and thus converges.

For \( m < 1 \), the limit is positive iff \( \mathbb{E}(\log X) < \infty \)  
where \( o(\log(1/m)) = 0 \).

Super-critical case: \( m > 1 \).

Notice that \( \mathbb{E}(\varepsilon_n) = m^n \). Is \( \varepsilon_n \) really of order \( m^n \) for all \( n \) on the event of survival?

Recall that \( M_n = \frac{\varepsilon_n}{m^n} \) is a martingale.

So \( M_n \to M_\infty \) a.s. Is \( M_\infty > 0 \) a.s. on survival?

**Thm. (Kesten-Stigum 1966):** When \( m > 1 \) the following are equivalent:

i) \( P(M_\infty > 0) = P(\forall n, \varepsilon_n > 0) \).

ii) \( \mathbb{E}(M_\infty) = 1 \).

iii) \( \mathbb{E}(X \log X) < \infty \).

If these cond. are violated, \( M_\infty = 0 \) a.s.

It is easy to see that the cond. in the Kesten-Stigum Thm. hold if \( \mathbb{E}(X^2) < \infty \).

- \( 1 \) is the case since
This is the case since
\[ \text{Var}(Z_n) = \text{Var}(X) \leq \frac{m^n(m^n - 1)}{m^2 - m} \quad n \neq 1 \]

E.g., by cond. on \( Z_{n-1} \)
and using the total variance formula.

Thus, when \( m > 1 \), \( \sup_n \text{Var}(M_n^2) < \infty \).

Hence \( (M_n) \) is a martingale bounded in \( L^2 \).

So \( M_n \to M_\infty \) in \( L^2 \).
It follows that \( \mathbb{E}(M_\infty) = 1 \).

In addition,
\[
P(M_\infty = 0) = \mathbb{E}(P(M_\infty = 0) 1_{Z_1}) =
\]
\[
= \mathbb{E}(P(M_\infty = 0) Z_1) = P(P(M_\infty = 0)).
\]

So \( P(M_\infty = 0) \) is a fixed point of \( \mathbb{R} \).

Combined with \( \mathbb{E}M_\infty = 1 \Rightarrow P(M_\infty > 0) =
\]
\[
= P(\forall n, Z_n > 0).
\]