Random walks and electrical networks

$G = (V, E)$ finite, connected graph

$C = (c(e))_{e \in E}$ positive reals (conductances)

$R_e = \frac{1}{c(e)}$ resistance

$\Pi(V) = \sum_{u: u \in V} c_{uv}$

Random walk: $(X_n)$ taking values in $V$

$P(X_{n+1} = v \mid X_n = u) = \frac{c_{uv}}{\Pi(u)}$.

Reminder

$h : V \to \mathbb{R}$ is harmonic at $u \in V$ if

$h(u) = \frac{1}{\Pi(u)} \sum_{v : v \in V} c_{uv} h(v)

That is $\sum_{v : v \in V} h(X_v) = h(u)$

Start with $X_0 = u$

Voltage: Fix $\alpha, \beta \in V$. A function

$h : V \to \mathbb{R}$ that is harmonic at all $v \in \{\alpha, \beta\}$

is called a voltage.

Lemma: For every $\alpha, \beta \in V$ there exists a unique voltage $h$ s.t.

$h(\alpha) = \alpha,$

$h(\beta) = \beta$.

Also if $\alpha = 0, \beta = 1$ then the unique voltage is given by

$h(v) = P_{\alpha} (\tau_1 < \tau_2)$

where $\tau_x = \min \{ n \geq 0 : X_n = x \}$. 

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Flow: A flow \((\text{from} \, a \, \text{to} \, z)\) is a function \(\Theta : E \rightarrow \mathbb{R}\) s.t. directed edges:

1) Antisymmetry: \(\Theta(u,v) = -\Theta(v,u), u \neq v\).
2) Node law: \(\forall u \in V, \sum_{v \in \partial^+ u} \Theta(u,v) = 0\).

Current flow: If \(h\) is a voltage then \(\Theta(u,v) = \text{ca}_{\overrightarrow{uv}}(h(v) - h(u))\) is the current flow of \(h\).

Kirchhoff cycle law: A flow \(\Theta\) satisfies the cycle law if for any directed cycle \(\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_m}\) it holds that \(\sum_{i=1}^{m} \Theta(e_i, \overrightarrow{e_i}) = 0\).

**Lemma:** A flow is the current flow of a voltage iff it satisfies the cycle law. Then the voltage is unique up to an additive constant.

**Strength:** The strength of a flow \(\Theta\) is \(\|\Theta\| := \sum_{v \in V} \Theta(v)\).

**Lemma:** There is a unique flow satisfying the cycle law with a given strength.

**Def.:** Unit current flow is the unique such flow with strength 1.

Effective resistance: The ratio \(\frac{h(z) - h(a)}{\text{flow}}\).
Lemma: The ratio \( \frac{h(z) - h(a)}{11\|h\|} \) is the same for all non-constant \( h \) and \( \Theta_\ell \) the current flow of \( h \), and is positive.

Proof: This constant is the effective resistance of the network and denotes \( \text{Reff}(a \leftrightarrow z) = \text{Reff}(a \leftrightarrow z; G, (\Theta_\ell)) \).

Let \( h_1, h_2 \) be non-constant voltages.

Let \( \Theta_1, \Theta_2 \) be their current flows.

Notice that \( \|\Theta_1\| \rightarrow 0 \) (e.g., by uniqueness of the current flow for a given strength).

Normalize \( \overline{\Theta}_1 = \frac{\Theta_1}{\|\Theta_1\|} \), so that the current flow \( \overline{\Theta}_1 \) has strength \( 1 \).

Uniqueness now gives that \( \overline{\Theta}_1 = \Theta_2 \)

and \( \exists c \) s.t. \( \overline{h}_1 = \overline{h}_2 + c \).

In particular,

\[
\frac{h_1(z) - h_1(a)}{\|\Theta_1\|} = \frac{\overline{h}_1(z) - \overline{h}_2(a)}{\|\Theta_2\|} = \frac{\overline{h}_2(z) - \overline{h}_2(a)}{\|\Theta_2\|} = \frac{h_2(z) - h_2(a)}{\|\Theta_2\|}.
\]

The ratio is positive, e.g., by considering the voltage \( h(x) = 0 \) \((T_\geq \leq T_\leq)\).

Connection to the random walk:

Return time \( T_x^r = \min \delta h \geq T : x_n = x \).

Lemma: \( \text{Reff}(a \leftrightarrow z) = \frac{1}{\pi(x) P_a(z \geq T_\geq)} \) \( \pi(x)P_a(z \leq T_\leq) \)
Proof: Let \( h(x) = P_X(T_2 < T_a) \).
So that \( P_{R_{\theta}}(a \leftrightarrow x) = \frac{1}{\|\Theta\|} \cdot \) 
\[
\frac{1}{\Theta_h(a,v)} = \frac{1}{\sum_{v \in N_a} c_{av} (h(v)-h(a))} 
= \frac{1}{\sum_{v \in N_a} c_{av} \delta_v(T_2 < T_a)} 
\overset{\text{Total Prob. Formula}}{=} \frac{1}{\prod_{v \in N_a} \frac{P_a(T_2 < T_a)}{P_a(T_2 < T_a + 1)}}. 
\]

Network simplifications:

**Parallel law:**

\[ u \rightarrow \frac{1}{c_1} \rightarrow v \quad \Rightarrow \quad \frac{1}{c_1 + c_2} \rightarrow u \rightarrow v \]

If \( e_1, e_2 \) are parallel edges, the effective resistance stays if they are replaced by a single edge with the sum of conductances.

**Exercise to check.**

**Series law:**

\[ \frac{1}{r_1 + \frac{1}{r_2}} \rightarrow u \rightarrow v \quad \Rightarrow \quad \frac{1}{r_1 + r_2} \rightarrow u \rightarrow v \]

**Exercise**

Giving: If a non-constant voltage \( h \) is constant on a subset \( S \subseteq V \), then giving all vertices in \( S \) does not change the effective resistance.

(Identify the vertices in \( S \) into one vertex and all edges remain (maybe creating parallel edges or self loops).)

...
Parallel edges or 2d dimensions.

**Proof:** The same voltage function as on the original graph is still a voltage on the glued graph, and has the same strength.

**Example:** Spherically-

Symmetric tree → \( d_0 = 3 \) children

Let \( T_n \) be all vertices at level \( n \). Start a simple random walk on the tree from \( \rho \).

What is \( P(T_n < T_0^+) \)

Minimal time to reach level \( n \)

Truncate the tree after level \( n \), and identify all vertices at level \( n \) to a single vertex.

Put unit conductances on edges.

The voltage on this graph is constant on each level, by symmetry, so we glue the vertex on each level.

Between \( \ell_k \) and \( \ell_{k+1} \) we have \( \ell_{k+1} = d_0 \cdots d_k \) parallel edges.

By parallel law, these can be replace by single edge with resistance \( \frac{1}{\ell_{k+1}} \).

Sum these by series law to get

\[
R_{\text{res}} (\rho \leftrightarrow \ell_n) = \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^n}
\]
\[ \text{Ref}_\nu (\rho \leftrightarrow \mathbb{Z}_n) = \frac{2^{-1}}{2 \Gamma(\nu+1)} \frac{2^{\nu-1} \Gamma(\nu)}{\Gamma(\nu+1)} \]

and
\[ \rho_\nu (T_{\mathbb{Z}_n} < T_{\rho}^+) = \frac{1}{\text{Ref}_\nu (\rho \leftrightarrow \mathbb{Z}_n)} \]

Note also
\[ \lim_{n \to \infty} \rho_\nu (T_{\mathbb{Z}_n} < T_{\rho}^+) = \rho \text{ (random walk never returns to } \rho) \]

\[ = \frac{2^{-1} \Gamma(\nu+1)}{2 \Gamma(\nu+1)} = \frac{1}{2} \]

and this says the tree is recurrent if
\[ \sum_{\nu = 1}^{\infty} \frac{1}{\nu} = \infty. \]

The commute time identity
\[ \mathbb{E}_{\mathbb{A}} (T_{\mathbb{A}}) + \mathbb{E}_{\mathbb{Z}} (T_{\mathbb{Z}}) = 2 \text{Ref}_\nu (\mathbb{A} \leftrightarrow \mathbb{Z}) \sum_{\nu \in \mathbb{E}} \]

We will not prove this now (exercise).

Energy

We now turn the space of \( \Theta: \mathbb{E} \to \mathbb{R} \) to a Hilbert space.

**Def.** The energy of a flow \( \Theta \) is

\[ E(\Theta) = \frac{1}{2} \sum_{e \in \mathbb{E}} \Theta(e)^2 = \sum_{e \in \mathbb{E}} \Theta(e)^2 \]

Each edge is taken with both orientations.

It is generally hard to calculate \( \text{Ref}_\nu \) and difficult to find current flow. Due to this, the following theorem is extremely useful.

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William Thomson, Lord Kelvin
Theorem (Thomson's principle):

$$\text{Reff}(a \leftrightarrow \emptyset) := \inf \{ \Theta(a) : \Theta \text{ is a flow from } a \text{ to } \emptyset \text{ with } \|\Theta\| = 1 \}$$

and the unique minimizer is the unit current flow.

Proof: First, we will show that for the unit current flow $I$, it holds that

$$\text{Reff}(a \leftrightarrow \emptyset) = \Theta(I) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V \setminus x} I_{xy} I_{yx} = \frac{1}{2} \sum_{x \in V \setminus x} \sum_{y \in V \setminus y} h(x) I_{xy} = \sum_{x \in V \setminus x} h(x) I_{xy} = (\ast)$$

Node law: $\sum_{y \in V} h(y) I_{xy} = h(a) I_{ax} + h(z) I_{zx} = (\ast)$

Similarly, using the antisymmetry,

$$\sum_{y \in V} h(y) I_{xy} = - (h(a) - h(z))$$

$$\Rightarrow (\ast) = -(h(a) - h(z)) = \frac{h(z) - h(a)}{\|I\|} = \text{Reff}(a \leftrightarrow \emptyset)$$

We now show that $\Theta(I) \geq \Theta(J)$ for every flow $J$ with $\|J\| = 1$.

Set $\Theta := J - I$, so that $\|\Theta\| = 0$.

$$\Theta(I) = \frac{1}{2} \sum_{x \in V \setminus x} \sum_{y \in V \setminus y} R_{xy} (I_{xy} + \Theta_{xy})^2 = \frac{1}{2} \sum_{x \in V \setminus x} \sum_{y \in V \setminus y} R_{xy} (I_{xy}^2 + 2I_{xy} \Theta_{xy} + \Theta_{xy}^2)$$

$$= \frac{1}{2} \sum_{x \in V \setminus x} \sum_{y \in V \setminus y} R_{xy} I_{xy} \Theta_{xy}$$
\[
\frac{1}{3} \sum_{x \in V} \sum_{y \in V, y \sim x} xy (xy - x) = \frac{1}{3} \sum_{x \in V} \sum_{y \in V, y \sim x} xy \theta xy (xy - x)
\]

It suffices to show that the last sum is 0.

\[
\sum_{x \in V} \sum_{y \in V, y \sim x} xy \theta xy (h(y) - h(x)) = \frac{1}{3} \sum_{x \in V} \sum_{y \in V, y \sim x} \theta xy (h(y) - h(x)) = \frac{1}{3} (h(z) - h(a)) \mathbf{1} \mathbf{1} = 0.
\]

Since \( \mathbf{1} \mathbf{1} = 0 \)

**Corollary (Rayleigh monotonicity law):**

Consider a finite, connected graph \( G = (V, E) \) with a set \( V \) of distinct vertices and two resistances \( (V_e) \) and \( (V'_e) \). Suppose \( V'_e \geq V_e \).

Then \( \text{Reff} (a \leftrightarrow z; G, (V_e)) \leq \text{Reff} (a \leftrightarrow z; G, (V'_e)) \).

Notice that the limit \( r'_e \to \infty \) corresponds to removing the edge \( e \) and the limit \( r_e \to 0 \) corresponds to gluing the endpoints of \( e \).

Consequently, \( \Theta_a (T_a < T_a^+) \) cannot increase when removing an edge and cannot decrease when gluing the endpoints of an edge.

**Proof:** For each flow \( (\theta) \),

\[
E(\theta) \geq E_{(r_e)}(\theta)
\]

Since \( E(\theta) = \frac{1}{2} r_e \Theta_e (r_e) \)

Now use Thomson's principle.

**Corollary:** Gluing on arbitrary subset of vertices (without gluing \( a \) to \( z \))
Corollary: Given any two vertices of a graph without giving a to z, the minimum of vertices cannot increase the effective resistance from a to z.

Proof: Every flow on the original network is still a flow on the glued network with the same strength and energy. Thus the infimum in Thomson's principle cannot increase.

Thomson's principle allows to upper bound the effective resistance. We now give a way to lower bound it.

Per: The energy of a function $h : V \to \mathbb{R}$ is $E(h) = \sum_{x \sim y} c_{xy} (h(x)-h(y))^2$ (the $L^2$-norm of the gradient of $h$ with conductances as coefficients).

Theorem (Dirichlet's principle):

\[
\frac{1}{\text{Resp}(a \leftrightarrow z)} = \inf \left\{ E(h) : h : V \to \mathbb{R}, h(a)=0, h(z)=1 \right\}
\]

The unique infimum is the voltage with $h(a)=0, h(z)=1$ (with $h(x)=\mathbb{P}(z < x)$).

Proof: (Sketch): It is straightforward that the minimal function is harmonic at all vertices except, maybe, $a$ and $z$. (Given $h(y)$ for all $y \neq x$, the value \[ h(x) \] minimizes \[ \sum_{y : y \sim x} c_{xy} (h(y)-h(x))^2 \].

A calculation shows that for
A simple calculation shows that for the voltage with \( h(a) = 0, \ h(\omega) = 1 \),

\[
e(V) = \frac{1}{\text{Reff}(a \rightarrow \omega)}.
\]

**Infinite networks**

Let \( G = (V, E) \) infinite, connected graph, 
with \( C = \{(c_e)_{e \in E} \} \) positive (conductances) 
with \( \exists c_{xy} < \infty \ \forall x, y \).

The equality \( \text{Reff}(a \rightarrow \omega) = \frac{1}{\pi(a) \rho(a)} \) motivates us to define \( \text{Reff}(a \rightarrow \infty) \) so that we can find \( \rho_a \) (random walk never returns to \( a \)).

Let \( G_n \subseteq G \) to be finite induced subgraphs which increase to \( G \) \((G_n \subseteq G_{n+1}) U G_n = G\).

Define \( \overline{G_n} \) by gluing in \( G \) all the vertices outside \( G_n \) to a single vertex \( \varepsilon_n \), and removing self loops at \( \varepsilon_n \).

\[
G = \mathbb{Z}^2,
\]

For every \( a \in \mathbb{Z}^2 \), let \( \text{Reff}(a \rightarrow \varepsilon_n ; \overline{G_n}) \) be non-decreasing.

**Proof:** \( \overline{G_n} \) is formed from \( \overline{G}_{n+1} \) by gluing...
Proof. \( G \) is recurrent if and only if \( \lim_{n \to \infty} \text{Reff}(a \to 2_n, \infty) = \infty \).

Therefore we can define

\[
\lim_{n \to \infty} \text{Reff}(a \to 2_n, \infty) = \text{Reff}(a \to \infty),
\]

This definition does not depend on \( \langle 2_n \rangle \) (since we can form an "interlaced" sequence from a pair \( \langle 2_n \rangle \) and \( \langle 2_n' \rangle \)).

Since \( P_{2_n}(\tau^+ = \infty) = \lim_{n \to \infty} P_{2_n}(\tau^+ < \tau^+) \),

random walk never returns to a

\[
= \lim_{n \to \infty} \frac{1}{\prod \text{Reff}(a \to 2_n, \infty)}.
\]

Conclusion: \( G \) is recurrent if and only if \( \text{Reff}(a \to \infty) = \infty \).

For some (and then every) \( a \in V_0 \), Thomson's principle continues to hold on infinite networks:

\[
\text{Reff}(a \to \infty) = \inf \{ E(\theta) : \theta \text{ is a flow from } a \text{ to } \infty \text{ with } \|\theta\| = 1 \}
\]

\( \theta \) is a flow from \( a \) to \( \infty \) if it is asymmetric and the node low holds at all vertices exceed, maybe, a.

Conclusion: \( G \) is transient if and only if there exists a flow from \( a \) to \( \infty \) of finite energy.
exists a new form of finite energy.