Random Walks and Brownian Motion	Instructor: Ron Peled
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Lecture 5

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In today's lecture we return to the Chung-Fuchs theorem regarding recurrence of general random walks on $\mathbb{R}^d$, and provide a proof for the $\mathbb{Z}^d$ case. We then move on to present a brief review of Martingales, mentioning several of their properties. In particular, we relate to the optional sampling theorem, Doob's maximal inequality and the martingale convergence theorem. Towards the end we begin to discuss Harmonic functions on graphs, define a Liouville graph, and finish with concluding that $\mathbb{Z}^2$ is Liouville, using the martingale convergence theorem.

Tags for today's lecture: Recurrence criteria for general RWs, Chung-Fuchs theorem, Martingales, Harmonic functions, Liouville graph.

1 Chung-Fuchs Theorem

Theorem (Chung-Fuchs). *Let S_n be a RW on $\mathbb{R}^d$.*

1. *Suppose $d = 1$: If the weak law of large numbers holds in the form $S_n/n \to 0$ in probability, then S_n is neighborhood-recurrent.*
2. *Suppose $d = 2$: If the central limit theorem holds in the form $S_n/\sqrt{n} \Rightarrow$ normal distribution, then S_n is neighborhood-recurrent.*
3. *Suppose $d = 3$: If S_n is not contained in a plane (meaning that the group of possible values for S_n does not rest on a plane) then S_n is neighborhood-transient.*

proof.

(i) The $d = 1$ case has been proved in the previous lecture (under first moment assumptions and for RWs on $\mathbb{Z}$).

(ii) $d = 2$: We will prove the theorem for RWs on $\mathbb{Z}^2$.

We need to show $\sum_{n=0}^{\infty} \mathrm{P}(S_n = 0) = \infty$ (according to proposition (3) in last week's lecture).

We have, from the assumption,

$$\mathrm{P}(\frac{\|S_n\|}{\sqrt{n}} \le b) \underset{n\to\infty}{\longrightarrow} \int\limits_{\|x\|\le b} n(y)dy$$

($\|\cdot\|$ refers to the Euclidean norm on $\mathbb{R}^d$, and $n(y)$ is the limiting normal distribution. Notice that we can assume $n(y)$ is non-degenerate, otherwise we're back to the 1-dimension problem).

Remark. A local limit theorem would give $\mathrm{P}(S_n = 0) \sim \frac{c}{n}$, which would yield the wanted divergence of the sum $\sum\limits_{n=0}^{\infty} \mathrm{P}(S_n = 0)$.

Lemma. Let

$$G(x,y) = \mathbb{E}^x[\#\, of\, visits\, to\, y] = \sum_{n=0}^{\infty} \mathrm{P}^x(S_n = y),$$

then

$$G(x,y) = \mathrm{P}^x(visit\, y) \cdot G(y,y)\,.$$

Notice that $G(y,y) = G(x,x)$.

Proof. Define $T = min(n \mid S_n = y)$ or $T = \infty$ if y not visited.

Now,

$$G(x,y) = \sum_{n=0}^{\infty} \mathrm{P}^x(S_n = y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{P}^x(S_n = y, T = k)$$

(notice that $\mathrm{P}^x(S_n = y, T = k) = 0\ \forall k > n$). Every element in the sum is positive, so we can change the order of summing:

$$G(x,y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{P}^x(S_n = y, T = k)\,.$$

Using the strong Markov property we get

$$G(x,y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{P}^y(S_n = y) \cdot P^x(T = k) = \sum_{k=0}^{\infty} \mathrm{P}^x(T = k) \cdot G(y,y) = \mathrm{P}^x(visit\, y) \cdot G(y,y)\,.$$

Corollary. Since $\mathrm{P}^x(visit\, y) \le 1$ it follows that

$$G(x,y) \le G(x,x)\,,$$

hence for any $A \subseteq \mathbb{Z}^d$

$$\sum_{n=0}^{\infty} \mathrm{P}^x(S_n \in A) \leq |A| \cdot G(x,x)$$

(by summing upon all $y \in A$).

Back to the **proof of (2)**: Note that for every m

$$(\star) \qquad \sum_{n=0}^{\infty} \mathrm{P}(S_n = 0) = G(0,0) \geq \frac{c}{m^2} \sum_{n=0}^{\infty} \mathrm{P}(\|S_n\| \leq m)$$

for some $c > 0$, from the corollary (here $A = \{x \mid \|x\| \leq m\}$ and $|A| = c' \cdot m^2$).
We can change the sum into an integral

$$(\star\star) \qquad \frac{1}{m^2} \sum_{n=0}^{\infty} \mathrm{P}(\|S_n\| \leq m) = \int_0^{\infty} \mathrm{P}(\left\|S_{\lfloor \theta m^2 \rfloor}\right\| \leq m) d\theta,$$

because when $\frac{n}{m^2} \leq \theta \leq \frac{n+1}{m^2}$ then $\lfloor \theta m^2 \rfloor = n$ and each segment of the integral is of length $\frac{1}{m^2}$. From the assumption

$$\mathrm{P}(\left\|S_{\lfloor \theta m^2 \rfloor}\right\| \leq m) = \mathrm{P}(\frac{\left\|S_{\lfloor \theta m^2 \rfloor}\right\|}{\sqrt{\theta} m} \leq \frac{1}{\sqrt{\theta}}) \underset{m \to \infty}{\longrightarrow} \int_{\|x\| \leq \theta^{-1/2}} n(y) dy\,,$$

and by Fatou's lemma,

$$\liminf_{m \to \infty} \int_0^{\infty} \mathrm{P}(\left\|S_{\lfloor \theta m^2 \rfloor}\right\| \leq m) d\theta \geq \int_0^{\infty} \liminf_{m \to \infty} \mathrm{P}(\left\|S_{\lfloor \theta m^2 \rfloor}\right\| \leq m) d\theta = \int_0^{\infty} \int_{\|x\| \leq \theta^{-1/2}} n(y) dy\,.$$

Now, denote $\{\|x\| \leq \theta^{-1/2}\} := B_\theta$. Notice that $\int_{B_\theta} n(y)dy \sim n(0) \cdot |B_\theta|$ as $\theta \to \infty$, and $n(0) \cdot |B_\theta| \geq \frac{c}{\theta}$ for some $c > 0$. Thus, using $(\star\star)$ from above,

$$\liminf_{\mathrm{m} \to \infty} \frac{1}{m^2} \sum_{n=0}^{\infty} \mathrm{P}(\|S_n\| \leq m) \geq \int_{C'}^{\infty} \frac{c}{\theta} = \infty$$

for some $c, C' > 0$. Returning to $(\star)$ we get

$$\sum_{n=0}^{\infty} \mathrm{P}(S_n = 0) = \infty\,. \qquad \square$$

(iii) Proof outline for the 3-dimensional case of RWs on $\mathbb{Z}^3$.

Reminder. *A RW on $\mathbb{Z}^d$ is recurrent iff*

$$\sup_{r<1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left[\frac{1}{1-r\cdot\varphi(t)}\right] dt = \infty$$

where $\varphi(t) = \mathbb{E}[e^{it\cdot X_1}]$ is the characteristic function of X_1.

Remark. If $z \in \mathbb{C}$, $\mathrm{Re}\,[z] \le 1$ then $\mathrm{Re}\left[\frac{1}{1-z}\right] \le \frac{1}{\mathrm{Re}[1-z]}$. (exercise)

Thus,

$$\sup_{r<1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left[\frac{1}{1-r\cdot\varphi(t)}\right] dt \le \sup_{r<1} \int_{[-\pi,\pi]^d} \frac{1}{\mathrm{Re}[1-r\cdot\varphi(t)]} dt\,.$$

But $\forall r<1:\ \frac{1}{Re[1-r\varphi(t)]} \le \frac{1}{Re[1-\varphi(t)]}$, so the integral on the right hand side is ∞ only if

$$\int_{[-\pi,\pi]^d} \frac{1}{\mathrm{Re}[1-\varphi(t)]} dt = \infty.$$

Accordingly, it is sufficient to show that $\int_{[-\pi,\pi]^3} \frac{1}{\mathrm{Re}[1-\varphi(t)]} dt < \infty$ in order to prove the transience of the walk.
Notice that if $\varphi(t) = 1 + \Theta(\|t\|^2)$ as $t \to 0$, then $\int_{[-\pi,\pi]^3} \frac{1}{\mathrm{Re}[1-\varphi(t)]} = \int_{[-\pi,\pi]^3} \frac{1}{\Theta(\|t\|^2)} < \infty$ as $t \to 0$.
So, for the integral to be ∞ (and hence the RW be recurrent) we need $\varphi(t) = 1 + o(\|t\|^2)$ as $t \to 0$.
But, for a 1-dimensional RV Y we have that (using the Taylor series)

$$\mathbb{E}[\mathrm{e}^{i\lambda Y}] = 1 + i\mu\lambda - \frac{a}{2}\lambda^2 + o(\lambda^2)\,,$$

where $\mathbb{E}[Y] = \mu$, $\mathbb{E}[Y^2] = a$. Hence, if $\varphi(t) = \mathbb{E}[\mathrm{e}^{it\cdot X_1}] = 1 + o(\|t\|^2)$, then by writing $t = \|t\|\,\mathrm{e}^{i\theta}$ we get that $\mathrm{e}^{i\theta}\cdot X_1$ is a 1-dimensional RV with both $\mathbb{E}[\mathrm{e}^{i\theta}\cdot X_1] = 0$ and $Var[\mathrm{e}^{i\theta}\cdot X_1] = 0$, and therefore $\mathrm{e}^{i\theta}\cdot X_1 \equiv 0$ (a.s.).
Thus, in order to have $\int_{[-\pi,\pi]^3} \frac{1}{\mathrm{Re}[1-\varphi(t)]} dt = \infty$, we must have that $\mathrm{e}^{i\theta}\cdot X_1 \equiv 0$ for almost all θ. Consequently, a RW on $\mathbb{Z}^3$ is recurrent only if it is contained in a plane, and otherwise transient. □

2 Martingales

We now supply a quick review of martingales in discrete time. A more complete discussion on the subject appears in [1].

2.1 Definitions

A *filtration* is a sequence of σ-fields $\{\mathcal{F}_n\}_{n\geq 0}$ s.t. $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ...$

A *martingale* (with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$) is a sequence $\{M_n\}_{n\geq 0}$ of integrable RVs souch that M_n is measurable with respect to $\mathcal{F}_n$, and

$$\mathbb{E}[M_m \mid \mathcal{F}_n] = M_n \quad \forall m \geq n\,.$$

Notice that the last requirement is trivially equivalent to

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n \quad \forall n\,.$$

Some intuition to this definition: We can think of M_n as the fortune earned by a gambler at time n. The last requirement suggests that the game is fair - the expected fortune at any time is the same as the gambler's initial fortune (or the fortune given at a certain time in the past).

Submartingale is defined the same way, changing the last requirement into $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \geq M_n$ $\forall n$ (a submartingale is a good game to play...).

Accordingly, *supermartingale* is defined with the last requirement $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \leq M_n$ $\forall n$ ("there's nothing super about a supermartingale").

Remark. If no filtration is given, then $\mathcal{F}_n = \sigma(M_0, ..., M_n)$.

Example. A RW with mean 0 is a martingale (assuming X_1 is integrable).

2.2 Optional Stopping

Definition. T is a *stopping time* with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ if T takes values in $\{0, 1, 2, ...\} \cup \{\infty\}$ and $\{T \leq n\}$ is measurable with respect to $\mathcal{F}_n$.

Intuition: The gambler's decision to stop gambling is a function only of the results that happened in the past.

Proposition. *If M_n is a (super/sub)martingale and T a stopping time, then $\{M_{T\wedge n}\}_{n\geq 0}$ is also a (super/sub)martingale.*

In particular, $\mathbb{E}M_{T\wedge n} = \mathbb{E}M_0$ $\forall n$ (and unequal accordingly for a super/sub-martingale).

Proof. (exercise)

Example. $M_n := S_n$ for a SRW $\{S_n\}_{n\geq 0}$ on $\mathbb{Z}$ starting at 0, and $T = \min(n \mid M_n = 10)$ (first hitting time of 10). Notice that $\mathbb{E}M_n = \mathbb{E}M_0 = 0$, $\mathrm{P}(T < \infty) = 1$ (by recurrence), and $\mathbb{E}M_T = 10$, so $\mathbb{E}M_T \neq \mathbb{E}M_0$. However, $\mathbb{E}M_{T\wedge n} = 0$ by the above proposition.

When is $\mathbb{E}M_T = \mathbb{E}M_0$?

Theorem (optional stopping theorem). *let $\{M_n\}_{n\geq 0}$ be a martingale, T a stopping time, and suppose $\mathrm{P}(T<\infty)=1$. Then $\mathbb{E}M_T = \mathbb{E}M_0$ if any of the following holds:*

1. *T is bounded ($\exists k \ \mathrm{P}(T \leq k) = 1$).*
2. *(dominated convergence) $\exists RV\ Y$ with $\mathbb{E}\,|Y| < \infty$ and $|M_{T\wedge n}| \leq Y \ \forall n$.*
3. *$\mathbb{E}T < \infty$ and the differences $|M_n - M_{n-1}|$ are uniformly bounded ($|M_n - M_{n-1}| \leq k \ \forall n$ a.s. for some k).*
4. *$\mathbb{E}\,|M_T| < \infty$ and $\lim\limits_{n\to\infty} \mathbb{E}[|M_n| \cdot 1_{(T>n)}] = 0$.*
5. *$\{M_n\}$ is* uniformly integrable, *i.e. $\forall \varepsilon > 0 \ \exists k_\varepsilon$ s.t. $\mathbb{E}[|M_n| \cdot 1_{(M_n \geq k_\varepsilon)}] \leq \varepsilon \ \forall n$.*
6. *$\exists p > 1$ s.t. $\{M_n\}$ are bounded in $\mathrm{L_p}$, that is $\exists k \ \mathbb{E}\,|M_n|^p < k \ \forall n$.*

Proof.

1. By the assumption $M_T = M_{T\wedge k}$ (a.s.), so $\mathbb{E}M_T = \mathbb{E}M_{T\wedge k} = \mathbb{E}M_0$.
2. $M_{T\wedge n} \underset{n\to\infty}{\longrightarrow} M_T$, thus (by dom. conv.) $\mathbb{E}M_T = \lim\limits_{n\to\infty} \mathbb{E}M_{T\wedge k} = \mathbb{E}M_0$.
3. We have$|M_{T\wedge n} - M_0| = \left|\sum\limits_{k=1}^{T\wedge n}(M_k - M_{k-1})\right| \leq T \cdot k$ where the right hand side is integrable by assumption. Thus, by dominated convergence we obtain
$$\mathbb{E}[M_T - M_0] = \lim_{n\to\infty} \mathbb{E}[M_{T\wedge k} - M_0] = 0$$
.
4. Notice that $M_T = M_{T\wedge n} + (M_T - M_n)\cdot 1_{(T>n)}$, therefore
$$\mathbb{E}M_T = \mathbb{E}[M_{T\wedge n}] + \mathbb{E}[M_T \cdot 1_{(T>n)}] - \mathbb{E}[M_n \cdot 1_{(T>n)}]\,.$$
Taking $n\to\infty$ and using the second part of the assumption we are left with $\mathbb{E}M_T = \mathbb{E}_0 + \lim\limits_{n\to\infty} \mathbb{E}[M_T \cdot 1_{(T>n)}]$.
The assumption $\mathbb{E}\,|M_T| < \infty$ allows us to use the dominated convergence theorem once again to conclude that $\lim\limits_{n\to\infty} \mathbb{E}[M_T \cdot 1_{(T>n)}] = 0$ (here assuming $\mathrm{P}(T=\infty)=0$ is necessary) , and so $\mathbb{E}M_T = \mathbb{E}_0$.
5. We do not provide a proof for (5) and (6), but mention that once we prove (6), we can obtain (5) easily. (Exercise).

Remark. for a supermartingale, all the above holds with the slight change that $\mathbb{E}M_T \leq \mathbb{E}M_0$.

2.3 Doob's Maximal Inequality

Theorem (Doob's maximal inequality). *If $\{M_n\}_{n\geq 0}$ is a non-negative submartingale, then*

$$P(\max_{0\leq j\leq n} M_j \geq \lambda) \leq \frac{\mathbb{E}M_n}{\lambda}.$$

This theorem is often combined with the next proposition to get a stronger result:

Proposition (Jensen). *If $\{M_n\}_{n\geq 0}$ is a martingale, $f : \mathbb{R} \to \mathbb{R}$ convex, then $\{f(M_n)\}_{n\geq 0}$ is a submartingale (assuming additionally that $\mathbb{E}\,|f(M_n)| < \infty\ \forall n$).*

In particular, we can take $f(x) = |x|^p$ for some $p \geq 1$, or $f(x) = e^{bx}$ for some $b \in \mathbb{R}$. Given a martingale $\{M_n\}$, we can now use Doob's inequality for the non-negative submartingale $\{f(M_n)\}_{n\geq 0}$, and obtain:

$$P(\max_{0\leq j\leq n} M_j \geq \lambda) \leq \frac{\mathbb{E}\,|M_n|^p}{\lambda^p} \quad \forall p \geq 1$$

or

$$P(\max_{0\leq j\leq n} M_j \geq \lambda) \leq \frac{\mathbb{E}e^{bM_n}}{e^{b\lambda}} \quad \forall b \in \mathbb{R}.$$

This way we can replace the original linear bound with a higher polynomial or exponential bound.

Proofs for the above theorems can be found in[1].

2.4 Martingale Convergence Theorem

To start the (brief) discussion on the martingale convergence theorem, we present the next quote from the book "Probability of Martingales" by D. Williams, on the significance of this theorem:

> △ signifies something important.
> △△ signifies something very important.
> △△△ is the martingale convergence theorem.

Theorem. *Let $\{M_n\}_{n\geq 0}$ be a supermartingale, and suppose $\sup_n(M_n^-) < \infty$ (where $M_n^- = \max(-M_n, 0)$), then there exists a RV M such that $M_n \to M$ a.s.*

E.g. a non-negative martingale always converges (a.s.)

Remark. The limit M is integrable, but the convergence is not necessarily in L_1.

For example, consider $T = min(n \mid S_n = -10)$ where S_n is a SRW (T is the first crossing time of -10). Then, the martingale $S_{T\wedge n}$ converges to the constant $M = -10$ which is integrable, but the convergence is not in L_1, since $\mathbb{E}S_{T\wedge n} = \mathbb{E}S_0 = 0$ (using the fact that $S_{T\wedge n}$ is a martingale), hence $\mathbb{E}[S_{T\wedge n} - M] = 10 \underset{n\to\infty}{\not\to} 0$.

An interesting example of the use of the martingale convergence theorem is Polya's Urn: An urn contains b blue balls and r red balls. At each time we draw a ball out, then replace it with c more balls of the color drawn. The proportion of red balls (for instance) in the urn is a non-negative martingale, and hence this proportion converges to a RV (which must take values in $[0, 1]$). In the special case where initially $r = b = 1$ and $c = 2$ it turns out that the limiting RV is uniformly distributed on $[0, 1]$.

Theorem (uniformly integrable submartingale). *For a submartingale $\{M_n\}_{n\geq 0}$, the following are equivalent:*

- *It is uniformly integrable.*
- *It converges a.s. and in L_1.*
- *It converges in L_1.*

If $\{M_n\}$ is a martingale then also:

- *$\exists$ an integrable RV M s.t. $M_n = \mathbb{E}[M \mid \mathcal{F}_n]$.*

Corollary (Levy 0-1 law). *Denote $\mathcal{F}_n \nearrow \mathcal{F}_\infty$ (that is $\mathcal{F}_\infty = \sigma(\cup\mathcal{F}_n)$, the smallest σ-field containing the union), then for every integrable RV X it holds*

$$\mathbb{E}[X \mid \mathcal{F}_n] \longrightarrow \mathbb{E}[X \mid \mathcal{F}_\infty] \quad \text{a.s. } \textit{and in } L_1$$

The special case where $X = 1_A$ for $A \in \mathcal{F}_\infty$ and then $P(A \mid \mathcal{F}_n) \longrightarrow 1_A$ generalizes the Kolmogorov 0-1 law (this last remark is left as food for thought for the reader).

3 Harmonic Functions

From this point on we assume G to be a locally finite graph (that is $\deg(v) < \infty \quad \forall v \in G$).

Definition. $h : G \to \mathbb{R}$ is said to be *harmonic* if

$$h(x) = \frac{1}{|N(x)|} \sum_{y\in N(x)} h(y)$$

where $N(x) = \{y \mid y \,\text{is a neighbor of}\, x \,\text{in}\, G\}$.

In other words, for every x, if we start a SRW on G $\{Z_n\}_{n\geq 0}$ beginning at x, then $h(x) = \mathbb{E}[h(Z_1)]$. Notice that this means that $\{h(Z_n)\}_{n\geq 0}$ is a martingale for any starting vertex x (acknowledging that $h(Z_n)$ takes only finitely many values and thus is integrable).

Examples.

1. Constant functions are harmonic.
2. On $\mathbb{Z}^d$, linear functions are harmonic. (exercise)

Definition. The space of bounded harmonic functions is called the *Poisson Boundary of G*. G is called *Liouville* if it has no non-constant bounded harmonic functions.

Remark. All the above can be defined with respect to other RWs than the SRW.

Example. Consider $\mathbb{T}^2$ the infinite binary tree (for our purposes we define the tree to be infinite both upwards and downwards from the origin, in order to be coherent with the general case of *Cayley graphs*, which would not be further discussed here). We define an *end* of the tree to be an infinite simple path from the origin. Then, for a collection of ends A we can define $h(x) = \mathrm{P}^x(infinite\, trajectory\, beginning\, at\, x\, agrees\, with\, some\, end\, in\, A\, i.o.)$. It is easy to check that h is a bounded harmonic function, indeed

$$h(x) = \mathrm{P}^x(\text{"}ending\, in\, A\text{"}) = \mathbb{E}[\mathrm{P}^x(\text{"}ending\, in\, A\text{"} \mid Z_1)] = \mathbb{E}[\mathrm{P}^{Z_1}(\text{"}ending\, in\, A\text{"})] = \mathbb{E}[h(Z_1)]\,.$$

But, h is non-constant, for certain choices of the group A. For instance, if we choose A to be the bottom half of the graph, then the probability of "ending in A" is different for vertexes from the top half and vertexes in A. A RW beginning at the top half has a $2/3$ probability of going up at every step (until it, possibly, crosses the origin downwards), as where on the bottom half the $2/3$ probability is "directed" down. Thus, the further up we begin the RW (with respect to the origin) - the smaller the probability to end at the bottom half. Hence, $\mathbb{T}^2$ is not Liouville.

Remark. $\mathbb{Z}$ is Liouville.

Proof. For h harmonic on $\mathbb{Z}$, the definition implies that $h(x+1) - h(x) = h(x) - h(x-1)$. So, if h is non-constant, there exists a y s.t. $h(y+1) - h(y) = a \neq 0$, which means h has a fixed slope, and thus cannot be bounded.

Question is, what about $\mathbb{Z}^2$? Is it Liouville?

Proposition. *On any connected recurrent graph (meaning a SRW on it is recurrent), any non-negative harmonic function is constant.*

In particular, the graph is Liouville (because any bounded harmonic function must be constant, otherwise by adding a large enough constant we would obtain a non-negative non-constant harmonic function).

E.g. $\mathbb{Z}^2$ is Liouville.

Proof. Fix $x, y \in G$ and let h be a non-negative harmonic function. We need to show that $h(x) = h(y)$. Define $\{Z_n\}_{n\geq 0}$ to be a SRW on G starting at x. $\{h(Z_n)\}_{n\geq 0}$ is now a non-negative martingale, and so the martingale convergence theorem holds. Consequently, $h(Z_n) \longrightarrow H$ a.s. (where H is some RV). We have assumed G is recurrent, thus $\mathrm{P}(Z_n = x \ i.o.) = 1$, so $h(Z_n)$ must converge to $h(x)$ a.s. But by recurrence it also holds that $\mathrm{P}(Z_n = y \ i.o.) = 1$, and similarily $h(Z_n) \to h(y)$ a.s. Therefore $h(x) = h(y)$ and we're done. $\square$

In the next lecture we will discuss the question - is $\mathbb{Z}^d$ Liouville for all d?

References

[1] Rick Durret, Probability: Theory and examples, January 2010
4.2 (Chung-Fuchs theorem), 5 (martingales) and 6.7 (harmonic functions).