Random Walks and Brownian Motion	Instructor: Ron Peled
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Lecture 13	
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Tags for today's lecture: Donsker's invariance priciple, Stochastic integration, Itô's formula

In this lecture we show an application of Donsker's invariance principle and then proceed to the construction of Itô's stochastic integral.

1 Donsker's invariance principle

We recall the definitions and give a simple example of an application of the invariance principle. Consider a random walk $S_n = \Sigma_{i=1}^n x_i$ with $\mathbf{E}(x) = 0$, $\mathbf{E}(x^2) = 1$. Let $S(t)$ be its linear interpolation and define

$$S_n^*(t) = \frac{S(nt)}{\sqrt{n}} \qquad t \in [0,1]$$

Theorem 1 *(Convergence to Brownian motion): $S_n^* \xrightarrow{d} B\big|_{[0,1]}$ on $\mathcal{C}[0,1]$*

Remark 2 *The meaning of the above statement is that for any bounded continuous function $G : \mathcal{C}[0,1] \longrightarrow \mathbb{R}$ the following holds $\mathbf{E}G(S_n^*) \overset{n\to\infty}{\longrightarrow} \mathbf{E}G(B\big|_{[0,1]})$*

Example 3 *Application to to maxima of Brownian motion.*

Let $S_n = \Sigma_{i=1}^n x_i$ with $\mathbf{E}(x) = 0$, $\mathbf{E}(x^2) = \sigma^2$. Denote $M_n = max\{S_k | 0 \le k \le n\}$ then

$$\lim_{n\to\infty} \mathbf{P}(M_n \ge x\sqrt{n}) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_x^\infty \exp\big[-\frac{y^2}{2\sigma^2}\big] dy \overset{N\sim\mathcal{N}(0,1)}{=} 2\mathbf{P}(N \ge x)$$

Proof It is enough to show this for $\sigma^2 = 1$ and to show that for any continuous bounded function $g : \mathbb{R} \to \mathbb{R}$

$$\mathbf{E}(g(\frac{M_n}{\sqrt{n}} \overset{n\to\infty}{\longrightarrow} \mathbf{E}g(\max B(t)) \sim 2\mathbf{P}(N \ge x)$$

the last step is an application of the reflection principle. (compare to the discrete walk study and the reflection principle application.)

consider $g_\lambda(x) = \begin{cases} \cos(\lambda x) \\ \sin(\lambda x) \end{cases}$ For this choice we define $F : \mathbb{R} \to \mathbb{R}$,

$$G(F) = g(\max F(t))$$

$$G(S_n^*) = g(\max S_n^*) \qquad \underbrace{= g(\max_k \frac{S_k}{\sqrt{n}})}_{\text{max of a piece linear f.is at the end points}} \qquad \square$$

2 Stochastic integration

The motivation behind stochastic integration is modelling noise. Let us illustrate this idea via an example: Let us consider S_n - a simple random walk in one dimension $x_i = \pm 1$

$R_n = \sum^n b_i x_i$ is a possible gambling strategy where R_i denotes the gain by time n, $b_i \in \sigma(x_1 ... x_{i-1})$ are random but predictable weights. R_n is obviously a martingale. we denote

$$R = \text{"} \int_1^n b \, \mathrm{d}s\text{"} = \sum^n b_i \cdot \underbrace{(S_i - S_{i-1})}_{X_i}$$

So, for Brownian motion this would be white noise.

We wish to define $\int H(s) \mathrm{d}B(s)$ for a random H. Since we know $B(s)$ does not have a bounded variation we can not define this as a Stieltjes integral, instead we will construct a stochastic process $M(t)$ which will have the following properties

- it will be a.s. continuous
- it will be a martingale

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, suppose $\{\mathcal{F}_t\}_t$ is a complete filtration (meaning that if $A \in \mathcal{A}$ with $\mathbf{P}(A) = 0$ then $A \in F_t$ for all t), such that the Brownian motion is adapted to it and the strong Markov property holds e.g. $\mathcal{F}(t)$ is the completion $(\mathcal{F}^+)$.

Definition 4 $\{\mathcal{H}(t, \omega) | t \geq 0 \ \omega \in \Omega\}$ *is progressively measurable if* $\forall t \geq 0$ *the map* $(t, \omega) \longrightarrow \mathcal{H}(t, \omega)$ *is measurable with respect to* $\mathcal{B}([0, t]) \times \mathcal{F}(t)$. *($\mathcal{B}$ is the Borel σ-field.)*

The measureability requirement corresponds to the predictability property we discussed in the discrete case example and it implies that $\mathcal{H}(t,\omega) \in \mathcal{F}(t) \quad \forall t$, which is adaptedness of the process.

Theorem 5 *Any $\mathcal{H}$ which is adapted and a.s. left-continuous or right-continuous is progressively measurable.*

Proof Without loss of generality suppose $\mathcal{H}$ is right-continuous fix $t \geq 0$ Define

$$H_n = \begin{cases} \mathcal{H}(0,\omega) & t = 0 \\ \mathcal{H}(\frac{k+1}{2^n}t, \omega) & kt2^{-n} < s \leq (k+1)t2^{-n} \end{cases}$$

$\mathcal{H}_n(t,\omega)$ is measurable with respect to $\mathcal{B}([0,t]) \times \mathcal{F}(t)$ hence is also its limit $\mathcal{H}$. $\square$

We will define $\int_0^\infty \mathcal{H}(s)\mathrm{d}B(s)$ for progressively a measurable $\mathcal{H}$ satisfying $\|\mathcal{H}\|_2^2 = \mathbf{E}\int_0^\infty \mathcal{H}^2(t,\omega)\mathrm{d}t < \infty$ for such integrals we get an isometry (the Itô Isometry):

$$\mathbf{E}\left[\int_0^\infty \mathcal{H}(s)\mathrm{d}B(s)\right]^2 = \|\mathcal{H}\|_2^2$$

and $\{\int_0^t \mathcal{H}(s)\mathrm{d}B(s)\}_{t\geq 0}$ will be a continuous martingale with respect to $\mathcal{F}$

Definition 6 *for a progressive measurable step process*

$$\mathcal{H}(t,\omega) = \sum^k A_i 1_{(t_i,t_{i+1}]}(t) \qquad 0 \leq t_1 \leq ... \leq t_{k+1}$$

define

$$\int_0^\infty \mathcal{H}dB(t) = \sum^k A_i \cdot \big(B(t_{i+1}) - B(t_i)\big)$$

We now proceed to the general case, defining an integral of a progressively measurable process we find an approximating sequence $\mathcal{H}_n$ of progressively measurable step processes with $\|\mathcal{H} - \mathcal{H}_n\|_2 \longrightarrow 0$ and using the isometry property show $\int_0^\infty \mathcal{H}(s)\mathrm{d}B(s)$ is well defined as the L_2 limit of $\int_0^\infty \mathcal{H}_n(s)\mathrm{d}B(s)$.

Lemma 7 *Given a progressively measurable process $\mathcal{H}$ with $\|\mathcal{H}\|_2 < \infty$ there exists a sequence $\mathcal{H}_n$ of progressively measurable step process with $\|\mathcal{H} - \mathcal{H}_n\|_2 \to 0$*

Sketch of Proof We can assume $\mathcal{H}$ is uniformly bounded zero outside a compact interval by truncation. We may assume $\mathcal{H}$ is continuous by replacing

$$\mathcal{H}_n(s) = n\int_{s-1/n}^{s} \mathcal{H}(s)\mathrm{d}s \qquad \mathcal{H}_n \xrightarrow{L_2} \mathcal{H}$$

since $\mathcal{H}_n \to \mathcal{H}$ for a.e. s by Lebesgue theorem, $\mathcal{H}_n$ is still progressively measurable being an integral over the past.

A continuous function can be approximated by a series of step functions $\square$

Lemma 8 *(isometry property): for a progressively measurable step process* $\mathcal{H}(s,\omega) = \sum A_i(\omega)1_{(t_i,t_{i+1}]}(s)$ *with* $\|\mathcal{H}\|_2 < \infty$ *we have* $\mathbf{E}\big[\int^\infty \mathcal{H}(s)\,dB(s)\big]^2 = \|\mathcal{H}\|_2^2$

Proof

$$\begin{aligned}
\mathbf{E}\Big[\int_0^\infty \mathcal{H}(s)\mathrm{d}B(s)\Big]^2 &= \mathbf{E}\sum_{i,j} A_iA_j\cdot\big(B(t_{i+1})-B(t_i)\big)\big(B(t_{j+1})-B(t_j)\big) \\
&= 2\,\underbrace{\mathbf{E}\sum_{i<j} A_iA_j\cdot\big(B(t_{i+1})-B(t_i)\big)\big(B(t_{j+1})-B(t_j)\big)}_{0} + \\
&\quad +\mathbf{E}\quad \underbrace{\sum_{i=j} A_i^2\cdot\big(B(t_{i+1})-B(t_i)\big)^2}_{\sum \mathbf{E}A_i^2\cdot(t_{i+1}-t_i)=\mathbf{E}\int\mathcal{H}^2(s)\mathrm{d}s=\|H\|_2^2}
\end{aligned}$$

$\square$

Corollary 9 *if* $\{\mathcal{H}_n\}$ *is a sequence of progressively measurable step processes with* $\|\mathcal{H}_n - \mathcal{H}_m\|_2 \to 0$ *(a Cauchy sequence in* L_2*) then*

$$\mathbf{E}\Big[\int_0^\infty \mathcal{H}_n(s)\mathcal{H}_m(s)\,dB(s)\Big]^2 \longrightarrow 0$$

Theorem 10 *the stochastic integral is well defined. If* $\mathcal{H}$ *is progressively measurable,* $\|\mathcal{H}\|_2 < \infty$ *and* $\{\mathcal{H}_n\}_n$ *are progressively measurable step processes* $\|\mathcal{H} - \mathcal{H}_n\|_2 \to 0$ *then the limit*

$$\lim_n \int_0^\infty \mathcal{H}_n(s)\,dB(s) \equiv \int_0^\infty \mathcal{H}(s)\,dB(s)$$

exists, as a limit in L_2*, is independent of* $\{\mathcal{H}_n\}$ *and satisfies the isometry property:*

$$\mathbf{E}\int_0^\infty \mathcal{H}(s)\,dB(s) = \|\mathcal{H}\|_2^2$$

Proof the limit exists since $\{\mathcal{H}_\backslash\}$ is a Cauchy sequence therefore has a limit by the completeness of L_2. This also gives the isometry. Independence of the choice of sequence follows from the isometry property.$\square$

Definition 11 *For a progressively measurable $\mathcal{H}$ with $\mathbf{E}\big[\int_0^\infty \mathcal{H}^2(s)ds\big] < \infty$ for arbitrary t we define*

$$\int_0^t \mathcal{H}(s)dB(s) \equiv \int_0^\infty \mathcal{H}(s)1_{(0,t]}(s)$$

Definition 12 *A stochastic process X is a* <u>*modification*</u> *of another Process Y if*

$$\forall t \qquad \mathbf{P}(X(t) = Y(t)) = 1$$

(essentially this means X and Y have the same finite dimensional distributions.)

Theorem 13 *if $\mathcal{H}$ is progressively measurable satisfying $\mathbf{E}\int_0^t \mathcal{H}^2 ds < \infty \quad \forall t > 0$, then there is an a.s. continuous martingale and in particular $\mathbf{E}\int_0^t \mathcal{H}(s)dB(s) = 0 \qquad \forall t$*

Sketch of Proof Fix t_0, find an approximating sequence of step processes $\mathcal{H}_n$ such that $\|\mathcal{H}_n - \mathcal{H}\|_2 \to 0$.

$\{\int_0^t \mathcal{H}_n(s)\mathrm{d}B(s) | 0 \leq t \leq t_0\}$ has a modification which is a continuous martingale. To transfer to the limit we need Doobs max inequality in L_2[1] $\square$

define $X(t) = \mathbf{E}\Big[\int_0^{t_0} \mathcal{H}(s)\mathrm{d}B(s)|\mathcal{F}(t)\Big]$ Note that $X(t_0) = \int_0^{t_0} \mathcal{H}(s)\mathrm{d}s$ (we expect to have $X(t) = \int_0^t \mathcal{H}\mathrm{d}B(s)$) X is a martingale up to time t_0 by definition. by L_2 max inequality

$$\mathbf{E}\Big\{\sup_{0\leq s\leq t_0}\Big[\int_0^s \underbrace{\mathcal{H}_n\mathrm{d}B(s)}_{\text{martingale}} - \underbrace{X(s)}_{\text{martingale}}\Big]\Big\}^2 \leq 4\mathbf{E}\Big(\int_0^{t_0} \mathcal{H}_n\mathrm{d}B(s) - \int_0^{t_0}\mathcal{H}\mathrm{d}B(s)\Big)^2$$
$$= 4\|\mathcal{H}_n - \mathcal{H}\|_2^2 \overset{n\to\infty}{\longrightarrow} 0$$

Taking $\mathcal{H}_n \to \mathcal{H}$ fast enough implies $\{\int_0^s \mathcal{H}_n\mathrm{d}B(s)\}_{0\leq s\leq t_0} \longrightarrow X$ both in sup norm and a.s. thus $X(t) = \int_0^t \mathcal{H}\mathrm{d}B(s)$ a.s.

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[1] Doob's inequality in L_p

$$\mathbf{E}\sup_{0\leq s\leq t}|X(s)|^p \leq \left[\frac{p}{p-1}\right]^p \mathbf{E}(X(t))^p$$

Proof for the discrete case this follows from the discrete statement, and then approximating gives the general result.$\square$

3 Itô's formula

Recall the fundamental theorem of calculus

$$f(x(t)) - f(x(0)) = \int_0^t f'(x(s))\mathrm{d}x(s) \tag{1}$$

here f is a continuous differentiable function, x is of bounded variation. In this case we have a Taylor expansion for f with the second term giving quadratic contribution.What happens in the case x is a Brownian Motion?

For a Brownian Motion, X, we have

$$B(t_i) - B(t_{i-1}) \approx o\sqrt{t_i - t_{i-1}}$$
$$\Big(B(t_i) - B(t_{i-1}\Big)^2 \approx o(t_i - t_{i-1})$$
$$\Big(B(t_i) - B(t_{i-1}\Big)^3 \approx o(t_i - t_{i-1})^{3/2}$$

Thus comparing this to formula (1) suggests:

Theorem 14 *Itôs formula (I): Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $\mathcal{C}_2$ with $\mathbf{E}\int_0^t f'(B(s))ds < \infty$ for some $t > 0$, then a.s. for $0 < s < t$ holds:*

$$f(B(s)) - f(B(0)) = \int_0^s f'(B(u))dB(u) + \frac{1}{2}\int_0^s f''(B(u))dB(u)$$

Theorem 15 *Itô's formula (II): suppose $\{\zeta(s)\}_{s>0}$ is a.s. an increasing continuous adapted process, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathcal{C}_2$ function and $\mathbf{E}\int_0^t \partial_x f(B(u), \zeta(u))^2 du < \infty$ for some $t > 0$ Then a.s. for all $0 < s < t$*

$$\begin{aligned} f(B(s), \zeta(s)) - f(B(0), \zeta(0)) = \int_0^s \partial_x f(B(u), \zeta(u))dB(u) + \frac{1}{2}\int_0^s \partial_{xx} f(B(u), \zeta(u))dB(u) \\ + \int_0^s \partial_\zeta f(B(u), \zeta(u))d\zeta(u) \end{aligned}$$

4 Multidimensional Itô's formula

let $f : \mathbb{R}^{d+m} \longrightarrow$ be $\mathcal{C}\underbrace{2,2...2}_{\text{d}},\underbrace{1,1...1}_{\text{m}}$ for $f(\underbrace{B(u)}_{\text{d-dim. BM}}, \underbrace{\zeta(u)}_{\text{m-dim time}})$ we have similarly to the 1-d case:

$$f(B(s),\zeta(s)) - f(B(0),\zeta(0)) = \sum_{i=1}^{d}\int_0^s \partial_x f(B(u),\zeta(u))\mathrm{d}B_i(u) + \frac{1}{2}\sum_{i=1}^{d}\int_0^s \partial_{xx} f(B(u),\zeta(u))\mathrm{d}B_i(u) + \int_0^s \partial_\zeta f(B(u),\zeta(u))\mathrm{d}\zeta(u)$$

this can be re-written suggestively as:

$$f(B(s),\zeta(s)) - f(B(0),\zeta(0)) = \int_0^s \nabla f(B,\zeta)\cdot(\mathrm{d}B,\mathrm{d}\zeta) + \int_0^s \Delta_x f(B(u),\zeta(u))\mathrm{d}B(u)$$

Example 16 *Important application*

Definition 17 *An adopted process $\{X(t)\}_{t\leq T}$ for a stopping time T is called a local martingale if there exist stopping times $T_n \nearrow T$ a.s. and such that $\{X(T_n \wedge t)\}_{t\geq 0}$ is a martingale.*

Theorem 18 *let D be a domain in $\mathbb{R}^d$, $f : D \longrightarrow \mathbb{R}$ harmonic on D and B a Brownian motion started in D and stopped at T, the first exit time from D. Then $\{f(B(t))\}_{t\leq T}$ is a local martingale.*

Explanation: Stochastic integral part is martingale - (we know it by the construction of Itô's integral.) f is harmonic $\Rightarrow$ second term is 0, the last term vanishes since it does not depend on time.

Example 19

$$f(x) = \begin{cases} \log|x| & d=2 \\ |x|^{2-d} & d\geq 3 \end{cases}$$

is harmonic on $\mathbb{R}^d - \{0\}$. $f(B(t))$ is a local martingale, yet it is not a martingale. Indeed,

$$\mathbf{E}f(B(t)) = \begin{cases} \infty & d=2 \\ 0 & d\geq 3 \end{cases} \quad \forall t\geq 0$$

5 Conformal Invariance of planar Brownian Motion

Theorem 20 *Let U be a domain in $\mathcal{C}(\cong \mathbb{R}^2)$ $x \in U$ $f : U \longrightarrow \mathcal{C}$ analytic. Let B be a planar Brownian motion started at x and let*

$$\tau_U = \inf\{t \geq 0 | B(t) \notin U\}$$

then the process $\{f(B(t)) \mid 0 < t < \tau_U\}$ is a time changed Brownian motion, That is there exists a planar Brownian motion $\tilde{B}$ such that

$$f(B(t)) = \tilde{B}(\zeta(t))$$

where $\zeta(t) = \int_0^t |f'(B(s))|^2 ds$. If in addition f is conformal then $\zeta(\tau_U)$ is a first exit time from $f(U)$ by $\tilde{B}$