Random Walks and Brownian Motion | Instructor: Ron Peled
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Lecture 10

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In this lesson we talk about Hölder continuity of BM and about its differntiability. We then go on to scaling and time-inversion invariance of BM, and we explore a few examples of their applications. Finaly, we introduce Markov property of BM and Blumenthal's 0-1 law.

Tags for today's lecture: Holder, scaling invariance, time inversion, Markov, Blumenthal 0-1.

1 Previously, on RW and BM

A Brownian Motion starting at $x \in \mathbb{R}$ is a random continuous function on $[0, \infty)$ such that:

1. $B(0) = x$ a.s.
2. Independent increments: $\forall n, \forall 0 = t_0 < t_1 < t_2 < ... < t_n$ the random variables $\{B(t_i) - B(t_{i-1})\}_{i=1}^{n}$ are independent.
3. $\forall 0 \leq t, 0 < h \;\; B(t+h) - B(t) \sim N(0, h)$

Last time we discussed the modulus of continuum of BM:

1. $\exists C > 0$ s.t. a.s. for every sufficiently small h and every $0 \leq t \leq 1-h$ we have $|B(t+h) - B(t)| \leq C\sqrt{h \log(\frac{1}{h})}$
2. $\forall c < \sqrt{2}$ a.s. $\forall \varepsilon > 0 \, \exists 0 < h < \varepsilon$ and $\exists t \in [0, 1-h] \, s.t. \;\; |B(t+h) - B(t)| \geq c\sqrt{h \log(\frac{1}{h})}$

2 Hölder Continuity

Defenition: A function $f : [0, \infty\} \to \mathbb{R}$ is called locally α-*Hölder continuous at* x if there exist $\varepsilon > 0$ and $c > 0$ s.t. $|f(x) - f(y)| \leq c|x-y|^\alpha$ for every $y \, s.t. \, |x-y| < \varepsilon$. α is called the Hölder exponent.

Corollary: $\forall \alpha < \frac{1}{2}$ a.s. BM is locally α-Hölder continuous at every $x \in [0, \infty)$

Proof: The previous results imply this for $B[0,1]$. Now, $B[1,2]$ is a new BM run up to time 1 started at $B(1)$. Hence this also holds for $B[k,k+1]$ for any of the countably many k's. (The right endpoints also satisfy the Hölder estimate for $y < x$, because $B[0,1)$ has the same distribution as the BM run backwards: $\{B(1-t) - B(1) \,|\, 0 \le t \le 1\}$) ■

Property 2 above shows that a.s. there is $t \in [0,1]$ where the BM is not $\frac{1}{2}$-HC. We will not show this, but a.s. for any $\alpha > \frac{1}{2}$ BM is nowhere α-HC. There do exist random points, called *slow times*, where the BM is $\frac{1}{2}$-HC, but they are very rare.

Notice that differentiability imply 1-HC. Next we show the weaker claim that the BM is not differentiable.

Theorem (Paley, Wiener, Zygmund, 1993): a.s. BM is nowhere differentiable.

Moreover, for every t, either $D^*B(t) = \infty$ or $D_*B(t) = \infty$ or both, where

$$D^*f(t) = \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}$$

$$D_*f(t) = \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}$$

Proof: It suffices to prove this for $t \in [0,1]$.

Assume that there exists $t_0 \in [0,1]$ with

$$-\infty < D_*B(t) \le D^*B(t) < \infty$$

In other words,

$$\limsup_{h\downarrow 0} \frac{|B(t_0+h) - B(t_0)|}{h} < \infty$$

Hence, for some (random) M, by the continuity of BM,

$$\sup_{h\in[0,1]} \frac{|B(t_0+h) - B(t_0)|}{h} \le M \;\;\bigstar$$

We need to show that for any $M \in \mathbb{N}$, the probability that there exists a t_0 satisfying $\bigstar$ is 0.

Fix an $M \in \mathbb{N}$ and also $4 \le n \in \mathbb{N}$. Suppose there exists a $t_0 \in [\frac{k-1}{n}, \frac{k}{n}]$ for some $1 \le k < n$. Then, for every $1 \le j \le n-1$ we have:

$$\left|B(\frac{k+j}{n}) - B(\frac{k-1+j}{n})\right| \underset{tri.\,ineq.}{\le} \left|B(\frac{k+j}{n}) - B(t_0)\right| + \left|B(\frac{k-1+j}{n}) - B(t_0)\right| \underset{\bigstar}{\le} M\frac{2j+1}{n}$$

Now, define $\Omega_{n,k} = \left\{ \left| B(\frac{k+j}{n}) - B(\frac{k-1+j}{n}) \right| \leq M\frac{2j+1}{n} \;\; j = 1, 2, 3 \right\}$

$P(\Omega_{n,k}) = \prod_{j=1}^{3} P(\left| B(\frac{k+j}{n}) - B(\frac{k-1+j}{n}) \right| \leq M\frac{2j+1}{n}) = \prod P(|B(1)| \leq M\frac{2j+1}{\sqrt{n}}) \leq (\frac{14M}{\sqrt{n}})^3$

as $\left| B(\frac{k+j}{n}) - B(\frac{k-1+j}{n}) \right| \sim N(0, \frac{1}{n}) \sim \frac{B(1)}{\sqrt{n}}$.

Hence, $P(\bigcup_{k=1}^{n} \Omega_{n,k}) \leq n \left(\frac{14M}{\sqrt{n}} \right)^3 \to 0$.

But if there exists $t_0 \in [0, 1]$ satisfying $\bigstar$ then for every $n \geq 4$, $\bigcup_{k=1}^{n} \Omega_{n,k}$ holds. Hence no such $t_0$exists with prob. 1. ■

3 Distributional prob. of the BM Process

3.1 Scaling Invariance (or Brownian scaling)

Note: Standard BM$\leftrightarrow x = 0$

Theorem: If B is a SBM and $a > 0$, then the process $\{X(t)|t \geq 0\}$ given by $X(t) = \frac{1}{a}B(a^2 t)$ is a SBM.

Proof: It is straighforward that X is a.s. a continuous function, and X has the correct finite dimension distribution.■

Application: For $a, b > 0$ let $T(a, b) = \min \{t \,|\, B(t) \in \{-a, b\}\}$

Letting $X(t) = \frac{1}{a}B(a^2 t)$ we have:

$$\mathbf{E}T(a, b) = a^2 \mathbf{E}min \left\{ t \geq 0 | \, X(t) \in \{-1, \frac{b}{a}\} \right\} = a^2 \mathbf{E}T(1, \frac{b}{a})$$

and furthermore, $\mathbf{E}T(a, a) = a^2 \mathbf{E}T(1, 1)$.

Similarly, $P(B(T(a, b)) = b)$ is a function of $\frac{b}{a}$ ($= P(X(T(1, \frac{b}{a})) = \frac{b}{a}$).

3.2 Time Inversion Invariance

Theorem: If B is a SBM then the process $\{X(T) \,|\, t \geq 0\}$ given by

$$X(t) = \begin{cases} tB(\frac{1}{t}) & t > 0 \\ 0 & t = 0 \end{cases}$$

is a SBM.

Proof: It is straightforward that $X(t)$ is continuous on $(0,\infty)$. for any $t_1,...,t_n$ $(X(t_1),...,X(t_n))$ is a Gaussian vector, and $\mathbf{E}X(t_i) = 0$, and one can check that the covariance of BM is preserved $\text{cov}(\text{X}(\text{t}_i), X(t_j)) = min(t_i, t_j)$ so the finite dim. dist. are preserved. Hence it remains only to verify continuity at 0. Since $\{X(t)\}_{t\in\mathbb{Q}} \stackrel{d}{=} \{B(t)\}_{t\in\mathbb{Q}}$ then a.s. $\lim_{\substack{t\downarrow 0\\ t\in\mathbb{Q}}} X(t) = 0$. Since the rationals are dense and X is continuous on $[0,\infty)$, we deduce $\lim_{t\downarrow 0} X(t) = 0$ a.s.■

Remark: The *Ornstein-Uhlenbeck process:* $\{Y(t)\,|\,t\in\mathbb{R}\}$ is defined by $Y(t) = e^{-t}B(e^{2t})$ using a SBM B. This process is a stationary Markov process (and is the limit of a RW with drift towards the origin proportional to its location). The time inversion is equivalent to saying that Y is a reversible process $\{Y(t)\,|\,t\in\mathbb{R}\} \stackrel{d}{=} \{Y(-t)\,|\,t\in\mathbb{R}\}$.

3.3 Applications to basic properties of BM

1) Law of Large Numbers:$\lim_{t\to 0}\frac{B(t)}{t} = 0$ a.s.

Proof: Define $X(t)$ as the time inversion of B, and note $\lim_{t\to 0}\frac{B(t)}{t} = \lim_{t\to 0} X(\frac{1}{t}) = 0\,a.s.$ ■

2) a.s. $\limsup_{t\to\infty}\frac{B(t)}{\sqrt{t}} = \infty$, $\liminf_{t\to\infty}\frac{B(t)}{\sqrt{t}} = -\infty$

Proof: It is sufficient to prove this as $t\to\infty$ along the integers. Note $\left\{\limsup_{t\to\infty}\frac{B(t)}{\sqrt{t}} = \infty\right\} = \bigcap_{M=1}^{\infty}\left\{\limsup_{n\to\infty}\frac{B(n)}{\sqrt{n}} > M\right\}$. Fix M, we will show that $P(B(n) > M\sqrt{n}\ i.o.) = 1$. Noticing $\{B(n)\,|\,n\geq 0\}$ is a RW with $N(0,1)$ increments, we deduce from the Hewitt-Savage 0-1 law that $P(B(n) > M\sqrt{n}\ i.o.) \in \{0,1\}$.

For any sequence of events A_n, we have $P(A_n\,occurs\,i.o.) \geq \limsup_{n\to\infty} P(A_n)$ by a form of Fatou's lemma, or by writing:

$$\{A_n\,occurs\,i.o.\} = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} B_n$$

where $B_n$is a decreasing sequence ofevents, so $P(\bigcap_{n=1}^{\infty} B_n) = \lim P(B_n)$ by monotone convergence. But $P(B_n) \geq \limsup P(A_k)$.

Hence, we deduce the result from noting $\limsup P(B(n) > M\sqrt{n}) = P(B(1) > M) > 0$■

3) a.s. $\limsup_{h\downarrow 0} \frac{B(h)}{\sqrt{h}} = \infty, \liminf_{h\downarrow 0} \frac{B(h)}{\sqrt{h}} = -\infty$

Proof: Same as 2), by time inversion.

4) Letting $\tau = \inf\{t > 0 \,|\, B(t) \geq 0\}$ and $\sigma = \inf\{t > 0 \,|\, B(t) \leq 0\}$then $P(\tau = 0) = P(\sigma = 0) = 1$. (Follows from 3)

5) a.s BM has no interval of monotonicity.

Proof: If BM has an interval of monotonicity then it also has one with rational endpoints. Fix $q_1, q_2 \in \mathbb{Q}\ a < q_1 < q_2 < b$. We will show $P([q_1, q_2]\, is\, an\, int.\, of\, mon.) = 0$. Dividing $[q_1, q_2]$ into n distinct subintervals, by the independent increments property $P(increments\, on\, each\, sub\, interval\, have\, same\, sign) = 2 \cdot 2^{-n} \to 0$ ■

4 Markov Property and Blumenthal's 0-1 Law

Defenition: A d-dimensional BM is a process of the form $\{(B_1(t), B_2(t), ..., B_d(t)) \,|\, t \geq 0\}$ where the B_i are independent 1D BM.

Defenition: Two continuous (sample path continuous) stochastic processes X, Y are called *independent* if for every $t_1, ..., t_n, s_1, ..., s_n \in \mathbb{R}$ we have that $(X(t_1), ..., X(t_n))$ is independent of $(Y(s_1), ..., Y(s_n))$.

Remark: Since the processes have continuous paths, if follows that any event measurable with respect to X is independent of any event measurable with respect to Y. This remains true if you have just right-continuity or continuity in probability.

Theorem (Markov property of BM)

If $\{B(t) \,|\, t \geq 0\}$ is a BM, then for any $s > 0$ $\{B(t + s) - B(s) \,|\, t \geq 0\}$ is a SBM independent of $\{B(t) \,|\, 0 \leq t \leq s\}$.

Proof: The independence follows from the independent increments property. Continuity and finite dim. dist. are easily checked.

1) A *Filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$ is a sequence of σ-algebra, satisfying $\mathcal{F}(t) \subseteq \mathcal{F}(s)\, \forall t \leq s$.

2) A Filtered probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}(t)\}_{t\geq 0}$ is a probability space with a filtration s.t. $\mathcal{F}(t) \subseteq \mathcal{F}$ for all $t \geq 0$.

3) A process X is called adapted to the filtration $\{\mathcal{F}(t)\}$ of $\sigma(\{X(s)|0 \leq s \leq t\}) \subseteq \mathcal{F}(t)$.

4) For BM, we let $\mathcal{F}^0(t) = \sigma(\{B(s) \,|\, 0 \leq s \leq t\})$. We also define $\mathcal{F}^+(t) = \bigcap_{\varepsilon>0} \mathcal{F}^0(t + \varepsilon)$.

We have $\mathcal{F}^0(t) \subseteq \mathcal{F}^+(t)$. Both are filtrations and B is adapted to both.

Theorem: (Markov property for $\mathcal{F}^+$)

If B is a BM and $s \geq 0$ then $\{B(t+s) - B(s) \,|\, t \geq 0\}$ is a SBM independent of $\mathcal{F}^+(s)$. That is, any event in $\sigma(\{B(t+s) - B(s)\})$ is independent of any event in $\mathcal{F}^+(s)$.

Proof: It suffices ti show that any event depending only on finitely many coordinates $\{(B(t_1+s) - B(s), ..., B(t_n+s) - B(s))\}$is independent of $\mathcal{F}^+(s)$, by continuity of B. Any event of this form is independent if $\mathcal{F}^0(s+\varepsilon)$ for small enough ε. Hence, independence of $\mathcal{F}^+(s)$.$\blacksquare$

Corollary: (Blumenthal's 0-1 law)

The *germ σ-algebra* $\mathcal{F}^+(0)$ is trivial (all events have 0-1 probability) for a BM.

Proof: For a SBM, $\mathcal{F}^+(0) \subseteq \sigma(\{B(t)\})$ but is independent of it by the previous theorem. for any different starting point, the result follows since we can apply the transformation $B \to B - x$ to B and $\mathcal{F}^+(0)$ and get a SBM and its germ σ-algebra. $\blacksquare$

Application: (Triviality of the tail σ-algebra)

The *tail σ-algebra* $\tau = \bigcap\limits_{t>0} \sigma(\{B(s)|s \geq t\})$. Then for a BM, τ is trivial.

Proof: Time inversion maps $\mathcal{F}^+(0)$ to τ. Hence the result follows, from the Blumenthal's 0-1 law.

Remark: For any $A \in \tau$ $P^x(A) \in \{0,1\}$ and is either 0 for all x or 1 for all x. But this is not the case for $\mathcal{F}^+(0)$.

References

[Brownian Motion] Peter Morters and Yuval Peres, 2008.