Random Walks and Brownian Motion Exercise 2

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The exercise needs to be handed in by May 23'rd in class.

In all of the following, unless otherwise indicated, we assume $(S_n)_{n=0}^{\infty}$ is a random walk in $\mathbb{R}^d$ with $S_n := \sum_{i=0}^{n} X_i$ (the walk may or may not be on $\mathbb{Z}^d$). We write $\mathbb{P}^x$ or $\mathbb{E}^x$ to indicate that $X_0 = x$ a.s., $\mathbb{P}^\pi$ or $\mathbb{E}^\pi$ to indicate that $X_0 \sim \pi$ and $\mathbb{P}$ or $\mathbb{E}$ to indicate that $X_0 = 0$ a.s..

1. In this question we investigate finer properties of harmonic functions on $\mathbb{Z}^d$.

   (a) Let h be a space-time harmonic function on $\mathbb{Z}^d$ and fix $x, y \in \mathbb{Z}^d$. Show that under any coupling of the SRWs started at x and y, if we let $T := \min\{n \geq 0 \mid S_n^x = S_n^y\}$ ($T = \infty$ on the event that no such n exists) then for any n we have

   $$|h(x,0) - h(y,0)| \leq \mathbb{E}\left(|h(S_n^x, n) - h(S_n^y, n)| \, 1_{(T>n)}\right).$$

   (b) Say that $x \in \mathbb{Z}^d$ is *even* if the sum of the coordinates of x is even. Prove that the tail sigma-field $\mathcal{T}$ for SRW on $\mathbb{Z}^d$ is generated mod 0 by the event $\{X_0 \text{ is even}\}$. In other words, show that for any initial distribution π and any event $A \in \mathcal{T}$, either $\mathbb{P}^\pi(A) \in \{0,1\}$ or $\mathbb{P}^\pi(A \triangle \{X_0 \text{ is even}\}) \in \{0,1\}$ (where $A \triangle B := (A \cup B) \setminus (A \cap B)$).

   (c) For 1D SRW, let $T_a := \min(n \geq 0 \mid S_n = a)$ and prove that for any $a \in \mathbb{Z}$ we have $\sup_n \mathbb{E}(|S_n| 1_{(T_a > n)}) < \infty$.
   Hint: In lecture 1, we proved using the reflection principle that if $M_n := \max(S_k \mid 0 \leq k \leq n)$ then for any $m, k \geq 0$ we have $\mathbb{P}(M_n \geq m,\ S_n = m - k) = \mathbb{P}(S_n = m + k)$.

   (d) Show that there are no non-constant sublinear harmonic functions on $\mathbb{Z}^d$. In other words, show that if h is harmonic on $\mathbb{Z}^d$ and satisfies $\lim_{x\to\infty} \frac{h(x)}{|x|} = 0$ then h is constant.
   Hint: Use part (a) for an appropriate coupling and part (c).

2. Let G be a finitely generated group. Prove that the (standard) Cayley graph of the lamplighter group $\mathbb{Z}_2 \wr G$ is Liouville (for SRW on it) if and only if G is recurrent.

Hint: To show that recurrence implies Liouville, use the Kesten-Spitzer-Whitman theorem and one of our characterizations of the Liouville property.

3. For SRW in $\mathbb{Z}^d$ and $N \geq 0$, let $T_N := \min\{n \geq 0 \mid |S_n| \geq N\}$ where $|x|$ denotes the Euclidean norm of x. Show that for any $N \geq 1$ and any $x \in \mathbb{Z}^d$ with $|x| < N$ we have $N^2 - |x|^2 \leq \mathbb{E}^x T_N < (N+1)^2 - |x|^2$.

4. For $d \geq 1$, let S^1 and S^2 be two independent SRW on $\mathbb{Z}^d$ and $I = \{(t_1, t_2) \mid S^1_{t_1} = S^2_{t_2}\}$ be the set of pairs of times at which they intersect. Show that $|I|$ has larger than exponential tail. More precisely, show that there exist $C = C_d > 0$ and $\alpha > 0$ such that for all $t > 0$ we have

$$\mathbb{P}(|I| \geq t) \geq \exp\left(-Ct^{1-\alpha}\right).$$

5. Let $T \sim \mathrm{Geom}(p)$ and let S_n be a (general) RW in $\mathbb{R}^d$ independent of T. Define R to be the number of distinct points visited by S_n by time T, $R := |\{S_n \mid 0 \leq n < T\}|$. Prove that

$$\mathbb{E}R = \mathbb{E}T\mathbb{P}(S_n \neq 0 \text{ for } 0 < n < T) = \frac{1}{p}\mathbb{P}(S_n \neq 0 \text{ for } 0 < n < T).$$

Hint: Concatenate together countably many independent copies of the killed walk (or in other words, instead of killing the walk at time T, restart an independent copy of it). Imitate the proof of the Kesten-Spitzer-Whitman theorem and use the LLN to prove the result.

6. Recall that for SRW in $\mathbb{Z}^d$, the potential kernel a is defined by $a(x) := \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) - \mathbb{P}(S_n = x)$.

   (a) Prove that a is well-defined for all $d \geq 1$ (that is, the series converges) and satisfies $\Delta a(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$.

   (b) Show that $a(x) = |x|$ in one dimension.

   (c) Show that $a(x) \sim \frac{2}{\pi} \ln |x|$ as $x \to \infty$ in two dimensions.

7. For a sequence of integers $\{a_n\}_{n \geq 1}$, define $A := \{(0, 0, a_n) \mid n \geq 1)\} \subseteq \mathbb{Z}^3$. Let $\{S_n\}_{n \geq 0}$ be SRW on $\mathbb{Z}^3$.

   (a) Prove that for any $\{a_n\}$, $\mathbb{P}(S_n \in A \text{ infinitely often}) \in \{0, 1\}$.

   (b) Prove that for $a_n := n^2$, $\mathbb{P}(S_n \in A \text{ infinitely often}) = 0$.

   (c) Prove that for $a_n := \lfloor n \log n \rfloor$, $\mathbb{P}(S_n \in A \text{ infinitely often}) = 1$.
   Hint: One method to show that the probability is positive is to apply the Benjamini-Pemantle-Peres bound. To show that the asymptotic capacity of A is positive, construct an explicit sequence of measures on A. Consider measures giving mass proportional to $|x|^{-1}$ for points x in their support.