BROWNIAN MOTION HOMEWORK ASSIGNMENT 2

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Some reminders: $\mathbb{N} = \{1, 2, \ldots\}$. The space $\mathbb{R}^{\mathbb{N}}$ is equipped with the product topology and the (Borel) sigma algebra generated from it. The sigma algebra on $\mathbb{R}^{\mathbb{N}}$ is the smallest one making all the coordinate functions measurable. A measurable subset A of $\mathbb{R}^{\mathbb{N}}$ is called cylindrical if there exists some n and measurable subset B of $\mathbb{R}^n$ such that

$$1_A(x_1, x_2, \ldots,) = 1_B(x_1, x_2, \ldots, x_n).$$

A function $\pi : \mathbb{N} \to \mathbb{N}$ is called a *finite permutation* if it is one-to-one and onto and there exists some n_0 such that $\pi(n) = n$ for all $n \geqslant n_0$. Given a function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ and a finite permutation π we let $\pi(f)$ be the function $\pi(f) : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\pi(f)(x_1, x_2, \ldots,) := f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

A measurable function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is called *exchangeable* if $\pi(f) = f$ for all finite permutations π. A measurable subset of $\mathbb{R}^{\mathbb{N}}$ is called exchangeable if its indicator function is exchangeable. The set of all such exchangeable sets forms a sub sigma-algebra of the measurable sets of $\mathbb{R}^{\mathbb{N}}$.

(i) In this question we prove:

Theorem 0.1. *(Hewitt-Savage $0-1$ law) Let $X_1, X_2, \ldots$ be an IID sequence of random variables and let A be an exchangeable subset of $\mathbb{R}^{\mathbb{N}}$. Then*

$$\mathbb{P}((X_1, X_2, \ldots) \in A) \in \{0, 1\}.$$

(a) Let μ be a (Borel) probability measure on $\mathbb{R}^{\mathbb{N}}$ and let A be a measurable subset of $\mathbb{R}^{\mathbb{N}}$. Prove that for any $\varepsilon > 0$ there exists a cylindrical subset B of $\mathbb{R}^{\mathbb{N}}$ such that

$$\mu(|1_A - 1_B|) < \varepsilon. \tag{1}$$

Here, by μ of a measurable function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ we mean $\int f(x) d\mu(x)$.
Hint: Prove that the set of such approximable subsets forms a sigma algebra.

(b) Let μ be the measure on $\mathbb{R}^{\mathbb{N}}$ given by $(X_1, X_2, \ldots)$ (where the (X_i) are an IID sequence). Let A be an exchangeable subset of $\mathbb{R}^{\mathbb{N}}$, let $\varepsilon > 0$ and let B be a cylindrical subset satisfying (1). Deduce that for any finite permutation π we have

$$\mu(|\pi(1_B) - 1_B|) < 2\varepsilon.$$

(c) In the setting of the previous part, deduce that $\mu(B)(1 - \mu(B)) < \varepsilon$.

(d) Deduce the Hewitt-Savage $0-1$ law.

(ii) Solve exercise 1.6 from the Brownian motion book.

(iii) Solve exercise 1.11 from the Brownian motion book.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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