BROWNIAN MOTION HOMEWORK ASSIGNMENT 13

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(i) In this exercise we prove the special case of the Burkholder-Davis-Gundy inequalities which we used in the proof of existence and continuity of Brownian local time. Let B be a standard one-dimensional Brownian motion and let $(\mathcal{F}(t))$, $t \geqslant 0$, be its (completed) filtration.

(a) Let $p > 0$ be an integer. Prove that there exists $C_p, c_p > 0$ such that the following holds. Let $k > 0$, $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \infty$ and random variables (X_j), $1 \leqslant j \leqslant k$, satisfy $X_j \in \mathcal{F}(t_j)$ and $\mathbb{P}(|X_j| \leqslant 1) = 1$. Defining

$$\begin{aligned} X &:= \sum_{j=1}^{k} X_j (B(t_{j+1}) - B(t_j)), \\ Y &:= \sum_{j=1}^{k} X_j^2 (B(t_{j+1}) - B(t_j))^2, \end{aligned} \tag{1}$$

we have

$$c_p \mathbb{E}(Y^p) \leqslant \mathbb{E}(X^{2p}) \leqslant C_p \mathbb{E}(Y^p).$$

(b) Let $q > 0$ be an integer and $t > 0$. Prove that there exists $C_{q,t} > 0$ such that for any X, Y as in the previous part, with $t_{k+1} = t$, we have

$$\max(\mathbb{E}(X^{2q}), \mathbb{E}(Y^q)) \leqslant C_{q,t}.$$

(c) Let p, C_p, c_p be as in the first part. Let $-\infty \leqslant a < b \leqslant \infty$. Fix $t > 0$ and define the two random variables

$$\begin{aligned} Z &:= \int_0^t 1_{(a,b)}(B(s)) dB(s), \\ W &:= \int_0^t 1_{(a,b)}(B(s)) ds. \end{aligned}$$

Prove that

$$c_p \mathbb{E}(W^p) \leqslant \mathbb{E}(Z^{2p}) \leqslant C_p \mathbb{E}(W^p).$$

Remark: Theorem 7.12 in the Brownian motion book is stated for continuous functions f. However, examination of the proof shows it suffices that f is bounded and $f \circ B$ is almost surely Riemann-integrable on the interval $[0, t]$. You may assume this fact without redoing the proof of Theorem 7.12.

(ii) Let $h : \mathbb{R} \to [0, \infty)$ be continuous with compact support and let $a \in \mathbb{R}$ and $t > 0$. Prove that, almost surely,

$$\int_{-\infty}^{\infty} h(a) \left(\int_0^t 1_{(a,\infty)}(B(s)) dB(s) \right) da = \int_0^t \left(\int_{-\infty}^{\infty} h(a) 1_{(a,\infty)}(B(s)) da \right) dB(s).$$

Hint: Approximate the ordinary integral by a Riemann sum. To ensure that this is possible, recall the definition (Tanaka's formula) and continuity of the local time process.

(iii) Let B be a standard Brownian motion and $L_t(a)$ be its local time process. Denote also the level sets $S_a := \{t \colon B(t) = a\}$.

(a) Prove that for any $a \in \mathbb{R}$, almost surely, $L_t(a)$ is increasing as a function of t and for any open interval $I \subset S_a^c$ we have that $L_t(a)$ is constant when $t \in I$.

(b) Prove that, almost surely, $L_t(0) > 0$ for all $t > 0$.

(c) Prove that for any $a \in \mathbb{R}$, almost surely, for every $t \in S_a \setminus \{0\}$ and every $q < t < r$ we have $L_q(a) < L_r(a)$.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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