BROWNIAN MOTION HOMEWORK ASSIGNMENT 10

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(i) Solve exercise 7.1 from the Brownian motion book (you may use Lemma 1.41 as needed. See the beginning of Section 1.4 for the definition of the Dirichlet space $D[0, 1]$).

(ii) Solve exercise 7.2 from the Brownian motion book.

(iii) Let $(\mathcal{F}(t))$, $t \geqslant 0$, be a complete filtration (i.e., $\mathcal{F}(t)$ contains all sets of measure 0 for each t). A *local martingale* $(M(t))$, $t \geqslant 0$, is an adapted (to $(\mathcal{F}(t))$) stochastic process for which there exists a sequence (T_n) of stopping times satisfying

(a) (T_n) are almost surely increasing to infinity: $\mathbb{P}(T_n \leqslant T_{n+1}) = 1$ and $\mathbb{P}(T_n \to \infty) = 1$.

(b) For each n, $(M(t \wedge T_n))$, $t \geqslant 0$, is a martingale.

In the next class we will see examples of local martingales which are not martingales. This exercise explores some basic properties of local martingales.

Let $(M(t))$, $t \geqslant 0$, be a continuous local martingale. That is, a local martingale whose sample paths are almost surely continuous.

(a) Prove that if $\mathbb{E}|M(0)| < \infty$ and M is bounded from below in the sense that there exists some $C < \infty$ for which $\mathbb{P}(M(t) \geqslant -C) = 1$ for all t then M is a supermartingale. Remark: in particular, if M is bounded both from below and from above then M is a martingale.

(b) Suppose there exists a sequence (a_n) such that for any n,

$$\mathbb{P}\left(\sup_{0 \leqslant s < \infty} |M(s \wedge T_n)| \leqslant a_n\right) = 1. \tag{1}$$

Prove that for any fixed $t \geqslant 0$, the sequence $(M(t \wedge T_n))$ (indexed by n) is a discrete time martingale.

(c) Suppose that, in addition to (1),

$$\sup_n \mathbb{E}\left(M(t \wedge T_n)^2\right) < \infty \quad \text{for all } t. \tag{2}$$

Prove that M is a martingale.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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