Lecture 5

Random walks in Random Environment for $d \geq 2$

Walking on $\mathbb{Z}^d$, $d \geq 2$, nearest neighbor walk $P$-environment measure. Generally 11D.

Uniformly elliptic: $\exists \epsilon > 0, P(w(x,e) > \epsilon) = 1 \forall e \in \mathbb{E}$

Elliptic: $P(w(x,e) > 0) = 1$

$P^x$ - quenched RW - starting at $x$ and walking in $\omega$.

$P^x$ - annealed measure - also average over $\omega$.

**Question 1: Recurrence/Transience**

We study directional transience.

I.e., fix $l \in \mathbb{S}^{d-1}$. Study $X_n \cdot l$

Claim: $P^0(\limsup_n X_n \cdot l \text{ is finite}) = 0$ for any unif. elliptic $\omega$

Saw this last time.

So we have 3 options:

1. $\lim_{n \to \infty} X_n \cdot l = +\infty$
   
   Call this event $\mathcal{A}_l$ - directional transience in direction $l$.

2. $\lim_{n \to \infty} X_n \cdot l = -\infty$

   Call this event $\mathcal{A}_{-l}$

3. $\limsup_{n \to \infty} X_n \cdot l = +\infty$

   Call this $\mathcal{O}_l$ - projection to $l$ is neighborhood recurrent.

   $\liminf_{n \to \infty} X_n \cdot l = -\infty$
By Claim: \( P^x(A \cup U A \cup U O e) = 1 \)

\[ \implies P^x(A \cup U A \cup U O e) = 1 \]

Important open question: Is \( P^0(\mathcal{A} e) \in \{0, 1\} \)?

What we know is only:

1. **Thm.** (Kalikow 1981, IID, Uniformly elliptic. Extended to IID elliptic by Merkl-Zerner 2001)
   \( P^0(O e) \in \{0, 1\} \). This is equivalent to \( |P^0(A \cup U A \cup e)| \in \{0, 1\} \)

2. In \( d = 2 \),
   **Thm.** (Merkl-Zerner 2001, IID elliptic): \( P^0(\mathcal{A} e) \in \{0, 1\} \)
   They also showed that there exists a stationary and ergodic \( \mathcal{P} \) which is elliptic for which \( P^0(\mathcal{A} e) - P(\mathcal{A} e) = \frac{1}{2} \)

We proceed to prove Kalikow's 0-1 law

In the IID unif. elliptic case:

Take \( l = (1, 0, 0, \ldots) \) for notational simplicity.

Assume that \( P^0(\mathcal{A} e) > 0 \) (or similarly \( P^0(A \cup e) > 0 \))

**Goal:** \( P^0(\mathcal{O} e) = 0 \)

Claim: \( P^0(\exists n \geq 1, X_n \cdot l < 0) < 1 \)

**Proof:** if the claim is not true, then
\[ P^0(\exists n \geq 1, X_n \cdot l < 0) = 1 \]

This implies that for any \( x \in \mathbb{Z}^d \), \( P^x(\exists n \geq 1, X_n \cdot l < 0) = 1 \)

and this implies that \( P^0(\lim_{n \to \infty} X_n \cdot l < 0) = 1 \)

\[ \text{never visits points with negative x-coord.} \]
and this contradicts the assumption that $P^0(\xi e) > 0$.

To continue, denote $P = P^0(\exists n \geq 1, X_n \cdot l < 0) = P^0(\exists n \geq 1, X_n \cdot l < X \cdot l)$ for any $x \in \mathbb{Z}^d$ by trans-inv.

Idea: Define "regeneration times" - times at which the walk first enters a half space it never visited before.

At each regeneration time, have probability $p$ to go back left of that point and these are "bounded above by independent".

Let $s_0 = 0$ and inductively, for each $k$,

\[ R_k := \min \{ n > s_k : X_n \cdot l < X_{s_k} \cdot l \} \quad k \geq 0 \]

\[ s_k := \min \{ n > R_{k-1} : X_n \cdot l > \max_{m \leq R_{k-1}} X_m \cdot l \} \quad k \geq 1 \]

By definition, $P^0(R_0 < \infty) = p$.

\[ \mathbb{P}(R_0 < \infty, s_1 < \infty, R_1 < \infty) = \sum_{x \in \mathbb{Z}^d} P^0(R_0 < \infty, s_1 < \infty, R_1 < \infty, X_{s_1} = x) \]
\* = \mathbb{E}_p \left[ P^0_{\omega}(R_0 < \infty, S_1 < \infty, R_1 < \infty, X_{s_1} = z) \right] 
\xrightarrow{\text{Strong Markov property}} \mathbb{E}_p \left[ P^0_{\omega}(R_0 < \infty, S_1 < \infty, X_{s_1} = z) \mid P^2_{\omega}(\exists n \geq 1 \mid X_n \cdot t < 2 \cdot l) \right] 
\text{a func. only of} \quad (w_x)_x \colon x \cdot l \leq 2 \cdot l 
= \mathbb{E}_p \left[ P^0_{\omega}(R_0 < \infty, S_1 < \infty, X_{s_1} = z) \right] \mathbb{E}_p \left[ P^2_{\omega}(\exists n \geq 1 \mid X_n \cdot l < 2 \cdot l) \right] 
\text{a func. only of} \quad (w_x)_x \colon x \cdot l \geq 2 \cdot l 
= P \cdot P^0_{\omega}(R_0 < \infty, S_1 < \infty) = P \cdot \sum_{z \in \mathbb{Z}^d} P^0_{\omega}(R_0 < \infty, S_1 < \infty, X_{s_1} = z) 
= P \cdot P^0_{\omega}(R_0 < \infty, S_1 < \infty) \leq P \cdot P^0_{\omega}(R_0 < \infty) = P^2 

\text{By the same argument,} 
P^0_{\omega}(R_0 < \infty, S_1 < \infty, \ldots, R_k < \infty) \leq P^{k+1} 

\Rightarrow P^0_{\omega}(\text{all } R_k \text{ and } S_k \text{ are finite}) = 0 

On \Omega \setminus 0, \text{ all } R_k \text{ and } S_k \text{ are finite so that} 
P^0_{\omega}(\Omega \setminus 0) = 0, \text{ as we wanted to prove} 

\text{We now show the Merkl-Zerner thm.:} 
\text{Planarity will be used to create intersections between random walk trajectories}
Recall: Lévy’s upward thm:
if \((F_n)_{n=0}^{\infty}\) is a filtration and \(A\) an event then,
\[
P(A | F_n) \xrightarrow{n \to \infty} P(A | \sigma (\bigcup_{n} F_n)) \text{ a.s.}
\]

We use this for the RW and the event \(A_e\). (we assume for simplicity that \(L^0(1,0)\))
\[
\mathcal{F}_n = \sigma (X_n, \ldots, X_0). \text{ Then, for each } \omega \text{ and any } x \in \mathbb{Z}^d
\]
\[
P_{\omega}^{X_n}(A_e) = \mathbb{P}_\omega (A_e | \mathcal{F}_n) \xrightarrow{n \to \infty} \mathbb{P}_\omega (A_e | \sigma (X_n, X_{n-1}, \ldots)) = 1 \text{ if } A_e
\]

**Idea of the proof**
Assume, to get a contradiction, that \(P^0(A_e) > 0\) and \(P^0(A_\bar{e}) > 0\). Recalling that then \(P^0(A_e \cup A_\bar{e}) = 1\) by Kolmogorov’s 0-1 law.

Start one random walker at \((-L, 0)\) and another one at \((L, y_L)\).

After the first \(L\) steps, both walkers are nearly certain if \(A_e\) happens for them or not.

But since they’re still in independent environments then these decisions are indep. and it is possible (since \(P^0(A_e) > 0, P^0(A_\bar{e}) > 0\)) that the one started at \((L, y_L)\) will satisfy \(A_e\) and the one started at \((-L, 0)\) will satisfy \(A_\bar{e}\).

On this event, it is very unlikely that they intersect.

since: \(P_{\omega}^{X_L^e}(A_e) \approx 1\) \(P_{\omega}^{X_L^e}(A_\bar{e}) \approx 0\) from \(\Box\)
Lastly, using planarity, it is shown that they intersect with a uniformly positive prob. when \((y_L)\) is well chosen.

We sketch the proof (not in every detail).
First, let us obtain aversion of \(\bigcirc\) with a uniform rate of convergence.

**Claim 1:** \(\forall \varepsilon > 0 \ \exists M_{\varepsilon} > 0 \ \text{s.t.} \ \forall x \in \mathbb{Z}^d,\)

\[
P_x^x(\|P_n^x(A_e) - 1_{A_e}\| < \varepsilon \ \forall n > M_{\varepsilon}) > 1-\varepsilon
\]

**Proof:**

The proof is the same for all \(x\), so we prove for \(x = 0\).

By \(\bigcirc\) we know that \(\|P_n^0(A_e) - 1_{A_e}\| < \varepsilon\) occurs for all \(n > N_{\varepsilon}(\omega, x_0)\) where \(N_{\varepsilon}(\omega, x_0) < \infty\).

Now, take \(M_{\varepsilon}\) so large s.t. \(P(\{N_{\varepsilon} > M_{\varepsilon}\} < \varepsilon\)

We now consider a RW \((X^1_n)\) starting at \((-L,0)\)
and another RW \((X^2_n)\) starting at \((L,y_L)\) for some \(y_L\) to be chosen later.

Call the annealed measure of both RWs by \(P_l\)

**Claim 2:** for any \((y_L)_L > 1\),

\[
P_l(\lim_{n \to \infty} X^1_n : l = +\infty, \lim_{n \to \infty} X^2_n : l = -\infty) \xrightarrow{L \to \infty} P^0(A_e) \cdot P^0(A-e)
\]

first sample \(w\)
then run the 2 walks in \(w\)
**proof:** If either \( P^0(A_e) = 0 \) or \( P^0(A_t) = 0 \) then, \( \forall L, \)

\[
\lim_{L \to \infty} P_L \left( \lim_{n \to \infty} X_n^1 = -\infty, \lim_{n \to \infty} X_n^2 = -\infty \right) = 0
\]

and there is nothing to prove. Assume then that both \( P^0(A_e) > 0 \) and \( P^0(A_t) > 0 \).

By Kalikow's 0-1 law we have \( P^0(A_e \cup A_t) = 1 \). It is simple to see that:

\[
\lim_{L \to \infty} P_L \left( \lim_{n \to \infty} X_n^1 = -\infty, \exists n \geq 1, X_n^1 \geq 0 \right) = 0
\]

\[
\lim_{L \to \infty} P_L \left( \lim_{n \to \infty} X_n^2 = -\infty, \exists n \geq 1, X_n^2 < 0 \right) = 0
\]

Hence, \( \forall \varepsilon > 0 \) \( \exists L \) s.t. \( \forall L > L \)

\[
|P_L \left( \lim_{n \to \infty} X_n^1 = -\infty, \lim_{n \to \infty} X_n^2 = -\infty \right) - P \left( \exists n \geq 1, X_n^1 \geq 0 \right) | \leq \varepsilon
\]

Lastly, \( P_L \left( \exists n \geq 1, X_n^1 \geq 0 \right) = P \left[ P_{\omega} \left( \exists n \geq 1, X_n^1 \geq 0 \right) \right] \)

\[
\uparrow
\]

\( = P \left( \exists n \geq 1, X_n^1 \geq 0 \right) \left[ P \left( \exists m \geq 1, X_m^2 < 0 \right) \right] \xrightarrow{L \to \infty} P^0(A_t) \cdot P^0(A_e) \text{ by } \odot
\]

**Claim 3:** for any \((Y)\)

\[
\lim_{L \to \infty} P_L \left( \lim_{n \to \infty} X_n^1 = -\infty, \lim_{n \to \infty} X_n^2 = -\infty, \text{ the walk trajectories intersect} \right) = 0
\]

**proof:** It suffices to prove that:

\[
\lim_{L \to \infty} P_L \left( \lim_{n \to \infty} X_n^1 = -\infty, \lim_{n \to \infty} X_n^2 = -\infty, \text{ at some } x \text{ with } x \neq \infty \right) = 0
\]

The intersections necessarily happens after more than \( L \) steps of \( X_n^2 \).

Fix \( \varepsilon > 0 \). Taking \( L > M_\varepsilon \), we have that

\[
|P_W^{X_n^2}(A_{-\varepsilon}) - 1 - A_{-\varepsilon} | \leq \varepsilon
\]

for all \( n \geq L \) with \( P \)-prob. at least \( 1 - \varepsilon \). In particular, \((**)\) occurs at the intersection point. Define a stopping time

\[
T = \min \{ n \geq 0 : P_W^{X_n^2}(A_{\varepsilon}) \leq \varepsilon \}
\]

on the event of intersection and when \((**)\) happens and

\[
\lim_{n \to \infty} X_n^2 = -\infty \text{ then } T = \infty.
\]

Note that \( P_W^{(L_0)} \left( \lim_{n \to \infty} X_n^1 = D, T < \infty \right) \leq \varepsilon \). By strong Markov property at \( T. \)
In conclusion, 
\[ P_L \left( \lim_{n \to \infty} X_n^1 \cdot t = \infty, \lim_{n \to \infty} X_n^2 \cdot t = -\infty, \text{trajectories intersect at } x \text{ with } x \cdot t \leq 0 \right) \leq P_L \left( \lim_{n \to \infty} X_n^1 \cdot t = \infty, \lim_{n \to \infty} X_n^2 \cdot t = -\infty, \text{(III holds, trajectories intersect at } x \text{ with } x \cdot t \leq 0 ) \right) + \varepsilon \leq P_L \left( \lim_{n \to \infty} X_n^1 \cdot t = \infty, T < \infty \right) + \varepsilon \leq 2 \varepsilon. \]

which concludes the proof as \( \varepsilon \) is arbitrary.

It remains to "make the walks intersect" to get a contradiction

claim 4: For a specific \((y_L)\)
\[ P_L \left( \lim_{n \to \infty} X_n^1 \cdot t = +\infty, \lim_{n \to \infty} X_n^2 \cdot t = -\infty, \text{no intersection} \right) \leq \frac{1}{2} P^0(Ae) P^0(A-e) \]

proof:
Choose \(y_L\) as the \(P_L\)-median of the \(y\)-coord for the last visit of \(X_n^1\) to the line \(X = L\), conditioned on \(\lim_{n \to \infty} X_n^1 \cdot t = +\infty\).

By def, we have \(\text{prob.} \geq \frac{1}{2}\) to have the last visit with \(y\)-coord \(y_L\) and \(\text{prob.} \geq \frac{1}{2}\) for \(y\)-coord \(\leq y_L\).

Homework exercise to prove that the claim holds with this choice of \(y_L\).

Putting claims 2,3,4 together we conclude that \(P^0(Ae) \cdot P^0(A-e) = 0\) as required.