Advanced topics in probability lecture 13 - random-field Potts model

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Random-field Potts model with q states:

$\sigma : \Lambda \to \{1, \dots, q\}$, $H^{\eta}_{\Lambda}(\sigma) = -\sum_{\substack{u \sim v \\ u,v \in \Lambda}} 1_{\sigma_u = \sigma_v} - \lambda \sum_{v \in \Lambda} \sum_{j=1}^{q} \eta_{v,j} 1_{\sigma_v = j}$

$\Lambda \subseteq \mathbb{Z}^d$ finite

$\lambda > 0$ is the disorder strength.

$(\eta_{v,j})_{\substack{v \in \Lambda \\ 1 \le j \le q}}$ is the disorder, in this talk IID $N(0,1)$.

Usually we put boundary cond. fixing the value of σ to some τ on the boundary of Λ.

In this talk, for simplicity, we discuss only the ground state $\sigma^{\Lambda,\tau,\eta}$, which is the configuration minimizing $H^{\eta}_{\Lambda}(\sigma)$ subject to the bdry. cond. τ. (it is unique since η has a continuous dist.)

Goal for today: 1) In two dimensions the ground state is disordered.

2) In $d \ge 3$, the ground state is ordered when λ is small (when λ is large it is disordered) ← maybe exercise.

Two dimensions

Thm. (Dario-Harel-Peled 2021, following a non-quantitative version by Aizenman-Wehr 1989):

When $d=2$, $\forall \lambda > 0$ $\exists C > 0$ s.t. $\forall L \ge 2$ $\forall 1 \le j \le q$,

$$\mathbb{E}\left[\sup_{\tau_1,\tau_2} \frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} 1_{\sigma_v^{\Lambda_L,\tau_1,\eta} = j} - 1_{\sigma_v^{\Lambda,\tau_2,\eta} = j}\right] \le \frac{C}{(\operatorname{Log}\operatorname{Log} L)^{1/2}}.$$

over η

bdry. cond.

$\Lambda_L = \{-L, \dots, L\}^2$

The maximal possible percentage difference due to bdry cond.

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$
The maximal possible percentage difference in the spins taking state j in Λ_L due to bdry. cond.

Remarks: 1) It would be very interesting to improve the quantitative bound. The truth, known for $q=2$, should be Ce^{-cL}. Same methods may improve to $\frac{C}{(\log\log L)^{1/2}}$ but it's not clear how to improve beyond.

τ_1 $\boxed{\text{maximize \#vertices with state } j}$ Λ_L $\qquad$ τ_2 $\boxed{\text{minimize \#vert. with state } j}$ Λ_L

with same disorder

2) By symmetry of the states and the disorder,

$$\mathbb{E}\left[\sup_{\tau}\left|\frac{1}{|\Lambda_L|}\sum_{v\in\Lambda_L} 1_{\sigma_v^{\Lambda_L,\tau_1,\eta}=j}-\frac{1}{q}\right|\right]\le\frac{C}{(\log\log L)^{1/4}}.$$

3) We conjecture a much stronger fact, that the ground state is determined at every vertex far from the boundary solely by η (not by τ) with high probability.

Conj.: $$\mathbb{E}\left[\sup_{\tau_1,\tau_2}\left|1_{\sigma_0^{\Lambda_L,\tau_1,\eta}=j}-1_{\sigma_0^{\Lambda_L,\tau_2,\eta}=j}\right|\right]\xrightarrow[L\to\infty]{}0.$$

(unique ground state at infinite volume).

Proof of thm.: Fix $j=1$ throughout the proof.

Call a set $\Lambda\subseteq\mathbb{Z}^2$ finite, ε-fluctuative, if

$$\sup_{\tau_1,\tau_2}\frac{\lambda}{|\Lambda|}\sum_{v\in\Lambda}1_{\sigma_v^{\Lambda,\tau_1,\eta}=1}-1_{\sigma_v^{\Lambda,\tau_2,\eta}=1}\le\varepsilon$$

At most ε percentage difference in the number of spins equal to 1, due to bdry. cond.

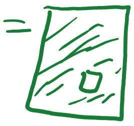
The thm. bounds the average of the percentage diff. when $\Lambda=\Lambda_L$.

The thm. bounds the average of the percentage diff. when $\Lambda = \Lambda_L$.

Whether Λ is ε-fluc. depends only on η in Λ.
edge bdry. of Λ

Main lemma: $\forall c_1 > 0\ \exists c_2 > 0$ s.t. if $|\partial\Lambda| \le c_1\sqrt{|\Lambda|}$ then for each $\delta > 0$,

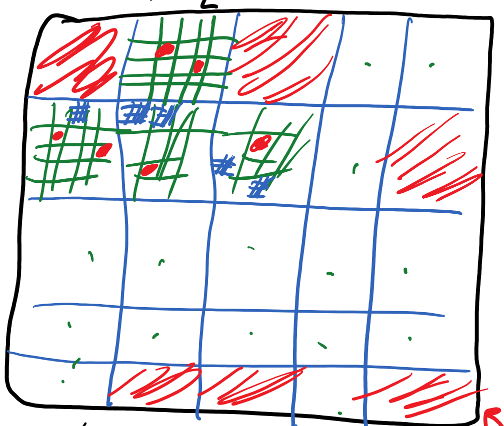
$$P(\Lambda \text{ is } 2\delta\text{-fluctuative}) \ge e^{-\frac{c_2}{\delta^4}}.$$

(lower bound doesn't depend on the size of Λ) e.g., $\Lambda =$  or $\Lambda =$ 

The thm. will follow by "boosting" the prob. bound in the main lemma.

Tool: Mandelbrot percolation:

Λ_L



Fix a large abs. const $c > 0$,

Set $\delta := \frac{c}{(\log\log L)^{1/4}}$ and $K = \frac{5}{\delta}$.

Partition Λ_L into K squares of equal side length (ignoring issues with divisibility), each of these divide into K squares and so on until reaching squares of side length 1.

red sq. are "taken"

A square is taken if it is 4δ-fluctuative and the squares containing it are not 4δ-fluc.,

Define the set of "bad" $v \in \Lambda_L$ to be

$$B := \{v \in \Lambda_L : v \text{ is not in a taken sq.}\}.$$

Then, the quantity in the thm. satisfies

$$\sup_{\tau_1,\tau_2} \frac{\lambda}{|\Lambda_L|} \sum_{v\in\Lambda_L} 1_{\sigma_v^{\Lambda_L,\tau_1,\eta}=1} - 1_{\sigma_v^{\Lambda_L,\tau_2,\eta}=1} \le 4\delta + \frac{\lambda|B|}{|\Lambda_L|}$$

In particular, the expectation of the LHS

In particular, the expectation of the LHS is at most $4\delta + \frac{\lambda}{|\Delta_L|}\,|E|B|$.

Thus we only need to show (to get the thm.) that $\frac{|E|B|}{|\Delta_L|} << \frac{1}{(\log\log L)^{1/4}}$.

It's enough to show, for each fixed $V \in \Delta_L$,

$$\mathbb{P}(V \in B) << \frac{1}{(\log\log L)^{1/4}}.$$

Fix $V \in \Delta_L$. Write $\Delta_0(V) \supseteq \Delta_1(V) \supseteq \ldots$ for the set of squares of the partition which contain V.

$$\{V \in B\} = \bigcap_{\ell} \{\Delta_\ell(V) \text{ is not taken}\}$$

$$= \bigcap_{\ell} \{\underbrace{\Delta_\ell(V) \text{ is not } 4\delta\text{-fluctuative}}\}$$

$\subseteq \{\Delta_\ell(V) \setminus \Delta_{\ell+1}(V) \text{ is not } 2\delta\text{-fluctuative}\}$

since $\frac{|\Delta_{\ell+1}(V)|}{|\Delta_\ell(V)|} \le 2\delta$

Δ is ε-fluc. if $\sup_{\tau_1,\tau_2} \frac{1}{|\Delta|}\sum_{v\in\Delta} \mathbb{1}_{v}^{\Delta,\tau_1,v} = 1 \ldots \tau_2 \le \varepsilon$

$$\subseteq \bigcap_{\ell} \underbrace{\{\Delta_\ell(V) \setminus \Delta_{\ell+1}(V) \text{ is not } 2\delta\text{-fluct.}\}}$$

These events are independent since the annuli are disjoint.

by main lemma

$$\Longrightarrow \mathbb{P}(V \in B) \le \prod_{\ell} \left(1 - e^{-\frac{c_2}{\delta^4}}\right)$$

$$= \left(1 - \underbrace{e^{-\frac{c_2}{\delta^4}}}\right)^{\frac{\log L}{\log(c/\delta)}}$$

Δ_L

divide into K squares recursively. There are $\approx \frac{\log(L)}{\log(\frac{c}{\delta})}$ scales

$\delta = \frac{c}{(\log\log L)^{1/4}}$

$e^{-c\log\log L} = \frac{1}{(\log L)^c}$

$< e^{-c\sqrt{\log L}}$ for the right choice

$$\le e^{-c\sqrt{\operatorname{Log} L}} \quad \text{for the right choice of const.}$$

$$\ll \frac{1}{(\operatorname{Log}\operatorname{Log} L)^{1/4}} \quad \text{as we wanted}$$

We now prove the main lemma:

edge bdry. of Λ

Main lemma: $\forall c_1 > 0 \ \exists c_2 > 0$ s.t. if $|\partial\Lambda| \le c_1\sqrt{|\Lambda|}$ then for each $\delta > 0$,

$$\mathbb{P}(\Lambda \text{ is } 2\delta\text{-fluctuative}) \ge e^{-\frac{c_2}{\delta^4}}.$$

Idea of proof: The analysis uses the convexity properties of the ground state energy in η.

Define $F^{\Lambda,\tau,\eta} = -\frac{1}{|\Lambda|} H(\sigma^{\Lambda,\tau,\eta})$

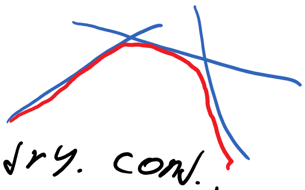
(minus the averaged ground state energy).

1) $F^{\Lambda,\tau,\eta}$ is a convex fcn. of η.

Recall $H(\sigma) = -\sum_{u\sim v} \mathbb{1}_{\sigma_u=\sigma_v} - \lambda \sum_v \sum_j \eta_{v,j} \mathbb{1}_{\sigma_v=j}$

is a linear fcn. of η for each fixed σ.

Hence $H(\sigma^{\Lambda,\tau,\eta})$ is a minimum of linear fcns. of η and hence concave.



2) $F^{\Lambda,\tau,\eta}$ is not strongly affected by bdry. cond.,

$$|F^{\Lambda,\tau_1,\eta} - F^{\Lambda,\tau_2,\eta}| \le \frac{|\partial\Lambda|}{|\Lambda|} \quad \text{a.s.}$$

Since

$$\left|H\left(\boxed{\begin{matrix}\text{ground}\\ \text{state}\\ \text{of bdry.}\\ \text{cond. } \tau_1\end{matrix}}\ \tau_1\right) - H\left(\boxed{\begin{matrix}\text{ground}\\ \text{state}\\ \text{of bdry.}\\ \text{cond. } \tau_2\end{matrix}}\ \tau_2\right)\right| \le |\partial\Lambda|$$

3) Derivative of $F^{\Lambda,\tau,\eta}$ with respect to η:

3) Derivative of $F^{\Lambda,\tau,\eta}$ with respect to η:

$$\frac{d}{d\eta_{v,j}} F^{\Lambda,\tau,\eta} = \frac{\lambda}{|\Lambda|} \mathbb{1}_{\sigma_v^{\Lambda,\tau,\eta} = j} \quad \text{a.s.}$$

Decompose η according to the average of $\eta_{v,1}$ over v and all the orthogonal degrees of freedom:

$$\hat{\eta}_\Lambda = \frac{1}{|\Lambda|} \sum_v \eta_{v,1}.$$

$$\eta^\perp_{v,j} = \begin{cases} \eta_{v,j} & j \neq 1 \\ \eta_{v,j} - \hat{\eta}_\Lambda & j = 1 \end{cases}$$

a Change of basis to η.

By Gaussianity, $\hat{\eta}_\Lambda$ is indep. of the other Gaussians, $\hat{\eta}_\Lambda \sim N(0, \frac{1}{|\Lambda|})$.

Henceforth we condition on $\eta^\perp$, working only with the randomness of $\hat{\eta}$.

$$\frac{d}{d\hat{\eta}_\Lambda} F^{\Lambda,\tau,\eta} = \frac{\lambda}{|\Lambda|} \sum_{v\in\Lambda} \mathbb{1}_{\sigma_v^{\Lambda,\tau,\eta} = 1}$$

Which is the sum appearing in the main lemma.

Note also that $\left|\frac{d}{d\hat{\eta}_\Lambda} F^{\Lambda,\tau,\eta}\right| \leq \lambda$.

So $F^{\Lambda,\tau,\eta}$ is a λ-Lipschitz fcn. of $\hat{\eta}_\Lambda$.

Rephrasing the main lemma:

$\forall C_1 > 0\ \exists C_2 > 0$ s.t. if $|\partial\Lambda| \leq C_1\sqrt{|\Lambda|}$, $\forall \delta > 0$,

$$\mathbb{P}\left(\sup_{\tau_1,\tau_2} \frac{d}{d\hat{\eta}_\Lambda} F^{\Lambda,\tau_1,\eta} - \frac{d}{d\hat{\eta}_\Lambda} F^{\Lambda,\tau_2,\eta} < 2\delta\right) \geq e^{-\frac{C_2}{\delta^4}}.$$

Claim (exercise): Let $g:\mathbb{R}\to\mathbb{R}$ be a convex and λ-Lipschitz. Define a neighborhood of g,

$$N_r(g) := \{h:\mathbb{R}\to\mathbb{R} \text{ convex } \lambda\text{-Lip.} : \sup_{x\in\mathbb{R}} |g(x)-h(x)| \le r\}.$$

Then for each $r,\delta>0$,

Lebesgue measure $\to$ $$\mathrm{Leb}(\{x\in\mathbb{R} : \exists h\in N_r(g),\ |h'(x)-g'(x)|\ge\delta\}) \le \frac{C\lambda r}{\delta^2}$$

for an abs. const. $C>0$.

Remark: 1) It is known that if a seq. of convex fcns. conv. pointwise to a limiting fcn., then at differentiability points of the limit, also the derivatives converge.
The claim quantifies such a statement.
2) Convex Lip. fcns. may not be differentiable everywhere, but left and right limits of derivatives suffice for us, and we don't get into this point.

Proof of main lemma from Claim:

Fix a bdry. cond. τ_0. Define a convex λ-Lip. fcn. by $g^\tau(x) = F^{\Lambda,\tau,x}$ where x is the value of η_Λ (all other η values are frozen by the conditioning).
Note by the small effect of bdry. cond.,

$$g^\tau \in N_r(g^{\tau_0}) \text{ for } r = \frac{|\partial\Lambda|}{|\Lambda|}.$$

By the claim,

$$\mathrm{Leb}\left(\left\{x\in\mathbb{R} : \exists\tau \text{ bdry. cond.},\ \left|\frac{d}{d\hat\eta_\Lambda}F^{\Lambda,\tau,\hat\eta_\Lambda}\Big|_{\hat\eta_\Lambda=x} - \frac{d}{d\hat\eta_\Lambda}F^{\Lambda,\tau_0,\hat\eta_\Lambda}\Big|_{\hat\eta_\Lambda=x}\right| > \delta\right\}\right)$$

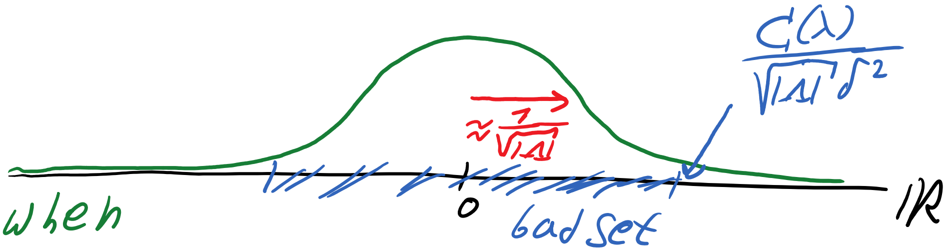
$$\le \frac{C\lambda r}{\delta^2} = \frac{C\lambda|\partial\Lambda|}{|\Lambda|\delta^2} \le \frac{C'\lambda}{\sqrt{|\Lambda|}\,\delta^2}.$$

$$\le \frac{4\lambda}{\delta^2} = \frac{4\lambda|\partial\Delta|}{|\Delta|\delta^2} \le \frac{C_1\lambda}{\sqrt{|\Delta|}\,\delta^2}.$$

$|\partial\Delta| \le C_1\sqrt{|\Delta|}$

For the event in the lemma not to hold, we need $\hat{\eta}_\Delta$ to fall in the set whose Leb. measure we just estimated.

$\hat{\eta}_\Delta \sim N(0, \frac{1}{|\Delta|})$



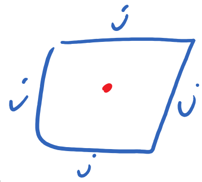
worse case is when the bad set is a symmetric interval around 0

The prob. for $\hat{\eta}_\Delta$ to land outside of the bad set is $\ge e^{-\frac{C(\lambda)}{\delta^4}}$.

Three and higher dimensions

Thm. (Ding–Zhuang 2021): When $d \ge 3$, λ small,

$$\mathbb{P}(\sigma_0^{\Lambda_L, i, \eta} \neq i) \le e^{-\frac{c}{\lambda^2}}$$



where $\sigma^{\Lambda_L, i, \eta}$ is the ground state in $\Lambda_L = \{-L, \ldots, L\}^d$ has bdry. cond. $\tau \equiv i$.

Remark: The argument extends to low, pos. temp.

History: For the random-field Ising model such a result was known, by Imbrie 1985 at zero temp. and Bricmont–Kupiainen 1988 at low, pos. temp. However, the proof was very long.

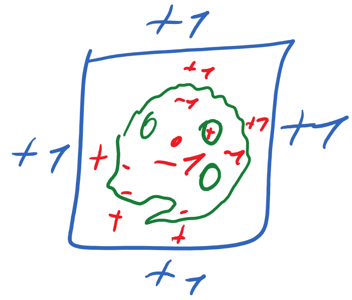
Found a very interesting

was very long.
Ding-Zhuang found a very interesting short proof, along the lines of the Peierls argument.

Idea of proof in the Ising case: $\sigma : \Lambda_L \to \{-1, 1\}$,

$$H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$$

Suppose $\sigma_0^{\Lambda_L, +1, \eta} = -1$.



Define
$\mathcal{C}$ = connected comp. of -1's containing 0 in $\sigma^{\Lambda_L, +1, \eta}$.

Let $\overline{\mathcal{C}}$ be the same set "without the holes".

I.e., $\overline{\mathcal{C}}$ is the complement of the infinite conected comp. of $\mathcal{C}^c$.

So, $\overline{\mathcal{C}}$ is a set satisfying:
1) $0 \in \overline{\mathcal{C}}$, 2) $\mathcal{C}$ and $\overline{\mathcal{C}}$ are connected 3) on the bdry. of $\overline{\mathcal{C}}$ we have -1 inside and $+1$ outside.

Usual Peierls argument (without η)
Flip the sign of $\sigma^{\Lambda_L, +1}$ on $\overline{\mathcal{C}}$ and lower the energy by $2|\partial \overline{\mathcal{C}}|$
edge boundary.
However,
In the presence of disorder we do not know whether the energy lowers by this operation, due to the

lowers by this operation, due to the $-\lambda \sum_{v} \eta_v \sigma_v$ term

NO quenched discrete symmetry!

To get around this: Flip $\sigma^{\Lambda,+1,\eta}$ on $\overline{\mathcal{C}}$ to create $\sigma^{\overline{\mathcal{C}}}$ and at the same time flip the sign of η on $\overline{\mathcal{C}}$ to create $\eta^{\overline{\mathcal{C}}}$. $\eta_v^{\overline{\mathcal{C}}} = \begin{cases} \eta_v & v \notin \overline{\mathcal{C}} \\ -\eta_v & v \in \overline{\mathcal{C}}. \end{cases}$

Then $H^{\eta}(\sigma^{\Lambda,+1,\eta}) - H^{\eta^{\overline{\mathcal{C}}}}(\eta^{\overline{\mathcal{C}}}) = 2|\partial \overline{\mathcal{C}}|$

An averaged discrete symmetry!

We learn that there is another disorder $\eta^{\overline{\mathcal{C}}}$, for which the ground state energy is lower than our initial ground state energy by $2|\partial \overline{\mathcal{C}}|$.

How to use this?

Main lemma (Ding-Zhuang 2021): In $d \geq 3$, λ small

$$\mathbb{P}\left(\exists\, \overline{\mathcal{C}},\ \begin{matrix} 0 \in \overline{\mathcal{C}} \\ \overline{\mathcal{C}} \text{ conn.} \\ \overline{\mathcal{C}}^c \text{ conn.} \end{matrix},\ \left| \begin{matrix}\text{ground} \\ \text{state} \\ \text{energy} \\ \text{of } \eta\end{matrix} - \begin{matrix}\text{ground} \\ \text{state} \\ \text{energy} \\ \text{of } \eta^{\overline{\mathcal{C}}}\end{matrix} \right| \geq |\partial \overline{\mathcal{C}}| \right) \leq e^{-\frac{c}{\lambda^2}}.$$

The thm. is an immediate consequence.

Tools in the main lemma:

1) For each $A \subseteq \mathbb{Z}^d$ finite,

1) For each $A \subseteq \mathbb{Z}^d$ finite,

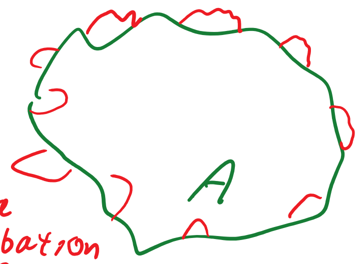
$$P\left(\left|\begin{matrix}\text{ground}\\ \text{state}\\ \text{energy}\\ \text{of } \eta\end{matrix} - \begin{matrix}\text{ground}\\ \text{state}\\ \text{energy}\\ \text{of } \eta^A\end{matrix}\right| \geq t\sqrt{|A|}\right) \leq C e^{-ct^2}$$

sub-Gaussian tail bound.

A consequence of a concentration inequality for Lipschitz fcns. of independent Gaussians.

2) Cannot conclude by a direct union bound as there are too many $\bar{C}$.

Use a "chaining argument" (a smart union bound) to conclude (Fisher-Fröhlich-Spencer 1984)



A' is a perturbation of A.

The diff. of ground state energies of η^A and $\eta^{A'}$ is small since the sets are close. Use such estimates in the chaining.