SPIN

Soon after quantum mechanics was developed (1924-25), there appeared phenomena in atomic spectroscopy which could not be explained.

- Multiplets (several lines very close together) were discovered in the spectra of many atoms, including hydrogen. This is called the fine structure.
- The spectral lines shift in magnetic fields (Zeeman effect). The shift in a field B_z could be expected to be $mg_l\beta B_z$, where $g_l = 1$, m is the L_z eigenvalue, $\beta = e\hbar/(2\mu c)$, e and μ are the charge and mass of the electron, and c is the speed of light. This was indeed found in some cases, but not always (normal and anomalous Zeeman effect).
- The Stern-Gerlach experiment (1922) shot silver atoms through an inhomogeneous magnetic field. This atom is paramagnetic, meaning it has a magnetic moment. The interaction energy between this moment and the field depends on the angle between them. Classically, we would expect a bell (Gaussian) curve of atoms on the collecting screen, with the largest density at the midpoint. In reality, two narrow bands were observed, which were explained by two orientations of the magnetic moment. This moment is proportional to the angular momentum and has the same direction. Quantum mechanics showed that angular momentum is quantized, explaining the discrete orientations of the moment. However, since L has $2l+1$ orientations, two orientations give $l = \frac{1}{2}$, which does not correspond to any angular momentum known classically.

Goudsmit and Uhlenbeck proposed in 1925 the existence of an internal degree of freedom called spin, which has all the properties of angular momentum. The spin has no classical analog, or, in other words, we cannot draw a picture describing it in the real world. It is definitely NOT connected to a rotation of the electron about some axis. It has some unusual properties. In non-relativistic quantum mechanics, based on the Schrödinger equation, it is an ad hoc addition required to explain certain experimental phenomena. In relativistic quantum mechanics, based on the Dirac equation, the spin comes out naturally with all its properties.

Theory

Three operators are defined, $\hat{S}_x, \hat{S}_y, \hat{S}_z$, with angular momentum commutation relations. The associated magnetic moment is

$$\vec{\hat{M}}_S = -\frac{g_s\beta}{\hbar}\vec{\hat{S}}\ .$$

g_s is chosen to fit experiment (for classical angular momentum $g = 1$).

Since the $\hat{S}$ operators obey angular momentum commutation relations, they commute with S^2. S^2 and S_z will therefore have common eigenfunctions χ_{sm_s}, with the usual form of eigenvalues:

$$S^2\chi_{sm_s} = s(s+1)\hbar^2\chi_{sm_s}$$

$$S_z\chi_{sm_s} = m_s\hbar\chi_{sm_s}.$$

Since m_s has two values, $s = \frac{1}{2}$ and $m_s = \pm\frac{1}{2}$. The magnetic moment is $-(g_s\beta/\hbar)(m_s\hbar) = \pm g_s\beta/2$. The experimental value magnetic moment is $\pm\beta$, which gives $g_s = 2$, as obtained also from the Dirac equation. This is different from the usual angular momentum, which has $g = 1$. The exact experimental value is $g = 2.00232$. The small correction is due to the quantization of the electromagnetic field, and is calculated from quantum electrodynamics.

The spin operators and functions are not in spatial coordinates. They may be represented in an abstract 2-dimensional spin space. They are usually defined in terms of the Pauli matrices, $S_i = \frac{1}{2}\hbar\sigma_i,\ i = x, y, z$, where

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenfunctions are

$$\chi_{\frac{1}{2}\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{\frac{1}{2}-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are just the well known α and β functions. Defining $S^{\pm} = S_x \pm iS_y$, we obtain

$$S^+\alpha = 0 \quad S^+\beta = \alpha$$

$$S^-\alpha = \beta \quad S^-\beta = 0$$