JAKIMOVSKI 90 Meeting

Approximation of Manifold-valued Functions Based on Refinements

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Based on a joint work with Nira Dyn.

Motivation – manifolds and approximation

"A manifold is a topological space that is locally Euclidean[1]" from Wolfram Web Resources.

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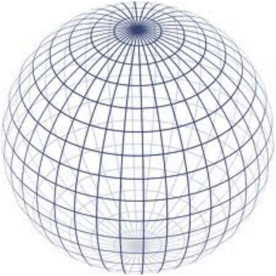

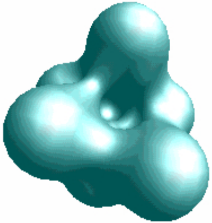

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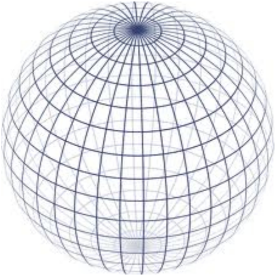

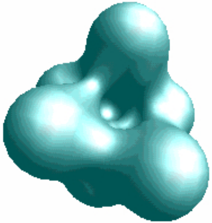

The research of manifolds is popular in geometry, physics and many more. It is also becomes popular in many applied math branches, just to name a few: geophysics, medical imaging, image analysis, CAGD, machine learning.

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Problem formulation

1. Estimate a mapping on a manifold of the form

$$f \colon \mathbb{R} \to \mathcal{M},$$

where $\mathcal{M}$ is the manifold. This function is a curve on $\mathcal{M}$.

2. We are given only a discrete set of samples, $\{f(t_i)\}_{i \in \mathbb{Z}}$.

3. One requires a smooth approximant, $\Gamma \colon \mathbb{R} \to \mathcal{M}$ where

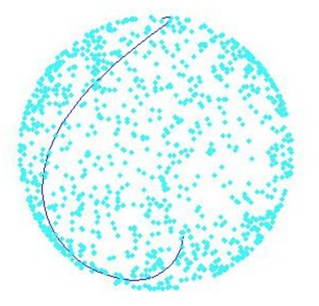
$$\Gamma(t) \approx f(t), \quad t \in \Omega,$$

in the sense of a metric on $\mathcal{M}$.

Examples of curves on manifolds

Example (1)

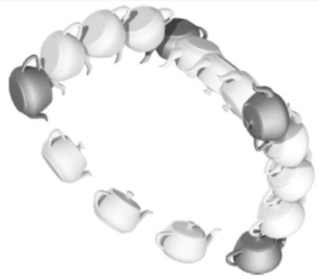
A trajectory on the sphere $S^2 \subset \mathbb{R}^3$.



Examples of curves on manifolds

Example (2)

A curve generated by the motion group.

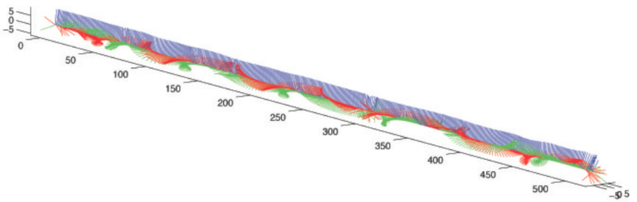


An interpolating curve of samples (the dark teapots)

Examples of manifold data

Example

The analysis of recorded flight data ("black box"). The data consist of pitch/roll/yaw and modelled as $\mathcal{SO}(3)$ raw data.



Many other examples of manifold data, e.g,

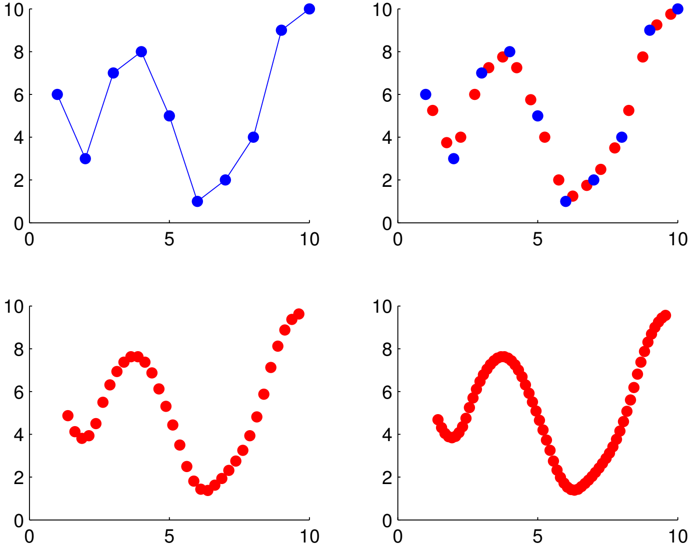
- Normals (points on the unit sphere)
- Positive definite matrices

Outline

1. Linear univariate subdivision schemes.
2. The adaptation of linear schemes to manifold data.
3. Geodesic averaging.
4. Geodesic refinements: local and global.

Linear univariate subdivision schemes

Linear univariate subdivision schemes



Linear univariate subdivision schemes

A linear univariate subdivision scheme refines repeatedly sequences of numbers (points) by replacing the current sequence $\mathbf{f} = \{f_i\}_{i\in\mathbb{Z}}$ by $\{(\mathcal{S}(\mathbf{f}))\}_{i\in\mathbb{Z}}$ according to the refinement rule

$$\mathcal{S}\left(\mathbf{f}\right)\big)_i = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j, \quad i \in \mathbb{Z},$$

assuming the mask $\mathbf{a} = \{a_i\}_{i\in\mathbb{Z}}$ has a finite (small) support. The symbol of $\mathcal{S}$ is the Laurent polynomial $a(z) = \sum_{j\in\mathbb{Z}} a_j z^j$.

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In binary refinement (each data point is replaced by two new ones) it is convenient to write

$$(S(\mathbf{f}))_{2i} = \sum_{j\in\mathbb{Z}} a_{2j} f_{i-j}$$

$$(S(\mathbf{f}))_{2i+1} = \sum_{j\in\mathbb{Z}} a_{2j+1} f_{i-j}.$$

Linear univariate subdivision schemes

A subdivision schemes is termed interpolatory if

$$(S(\mathbf{f}))_{2i} = f_i, \quad i \in \mathbb{Z}.$$

namely if $a_{2j} = \delta_{j,0}$, $j \in \mathbb{Z}$.

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Thus, $(S(\mathbf{f}))_i = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j$ is a weighted average of $\{f_j \mid a_{i-2j} \neq 0\}$, possibly with negative weights.

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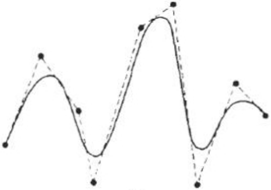
Consider as an example the cubic B-spline subdivision scheme.



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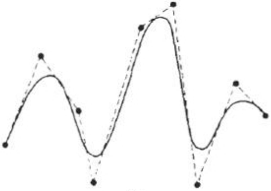

The curve is defined as

$$\Gamma(t) = \sum_i f(t_i) B(t-i),$$

where B is the cubic B-spline with integer knots ($B(t)$ is a cubic polynomial on each interval $(i, i+1)$, $i \in \mathbb{Z}$, $B(t) \in C^2(\mathbb{R})$).

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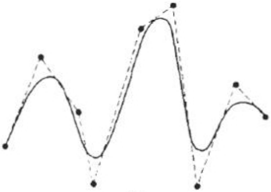

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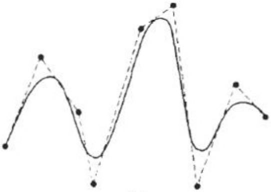

The refinement rules of the scheme are

$$(S(\mathbf{f}))_{2i} = \frac{1}{8} f_{i-1} + \frac{3}{4} f_i + \frac{1}{8} f_{i+1}$$
$$(S(\mathbf{f}))_{2i+1} = \frac{1}{2}(f_i + f_{i+1}).$$

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2. Subdivision schemes are highly local, which is valuable when approximating manifold-valued functions.
3. An interpolatory refinement leads to multiresolution analysis via a pyramid transform. Thus, an interpolatory subdivision scheme adapted to a manifold, induces an analysis tool for manifold data.

How to adapt linear subdivision schemes to manifolds?

An easy example

A subdivision scheme for positive numbers (Goldman et. al.): let $\{p_i\}_{i\in\mathbb{Z}}$ be a sequence of **positive** numbers.

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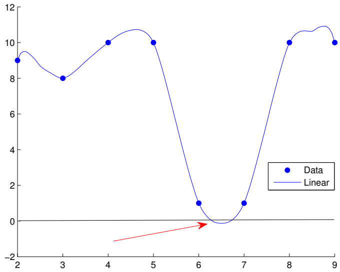

The problem - the linear $4-$pt scheme fails to stay positive in the above example

Two solutions

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Sol 1

1. Map the points to the real line by $p_i \mapsto \log(p_i)$, and apply a linear interpolatory subdivision scheme with limit $s(t)$, $t \in \mathbb{R}$.
2. Define the interpolant by $\Gamma(t) = \exp\left(s(t)\right)$. Obviously $\Gamma(t) > 0$.

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Sol 2

1. Take a linear interpolatory subdivision scheme. The insertion rule is a weighted average $w_1 p_1 + \ldots + w_k p_k$, $\sum_{i=1}^{k} w_i = 1$.
2. Use the geometric average instead of the arithmetic one, $\sum_{i=1}^{k} w_i p_i \to \prod_{i=1}^{k} p_i^{w_i}$.

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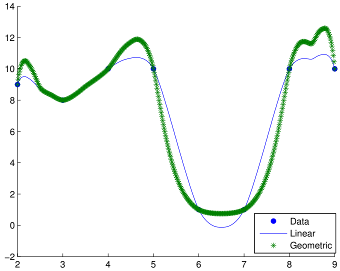
Due to commutativity of multiplication these two solutions are equivalent!

The commutative diagram

$$
\begin{tikzcd}
\mathbf{p_0} = \{p_i\}_{i\in\mathbb{Z}} \arrow[r, "\text{modified scheme}"] \arrow[d, "\log"'] & \Gamma(t) = \exp\left(s(t)\right) \\
\mathbf{a_0} = \{\log p_i\}_{i\in\mathbb{Z}} \arrow[r, "\text{linear scheme}"'] & s(t) \arrow[u, "\exp"']
\end{tikzcd}
$$

Figure: The $\log - \exp$ diagram

The adaptation result



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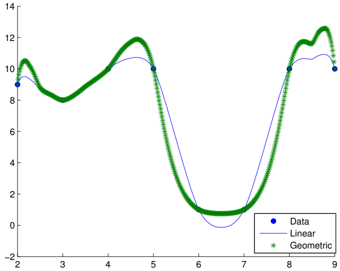

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What is the generalization to manifolds?

Adaptation methods – projection method

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1. Apply the refinement rule of a linear scheme in the Ambient space and project the result back to the manifold.
2. Is an extrinsic method: depends on the embedding of the manifold in a Euclidean space.

Adaptation methods – tangent space (Donoho et. al., Wallner et. al., Yu et. al.)

A similar idea of the exp-log, applies to general manifolds, and is efficient due to the locality of the subdivision schemes. The adaptation of the refinement rules consists of the following four steps:

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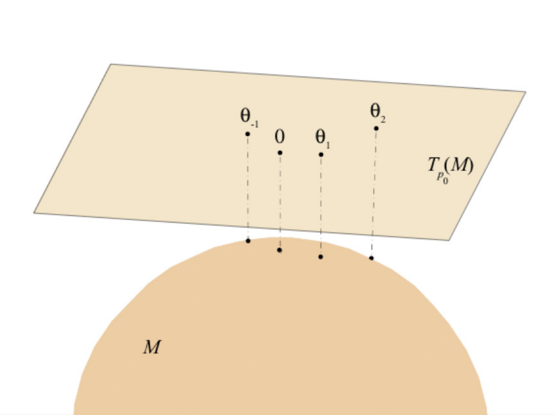
An inherent difficulty in this approach is the choice of the base point.

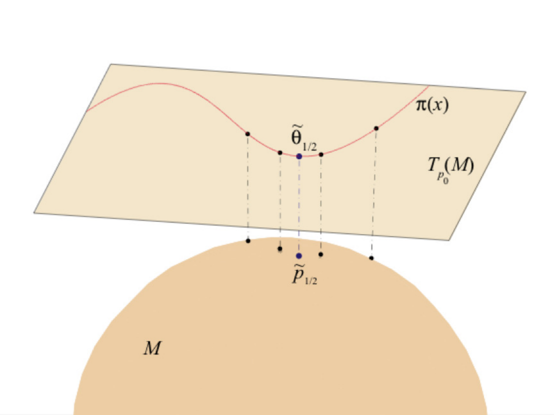
Adaptation methods – tangent space

Project into a tangent plane



Adaptation methods – tangent space

Calculation on the tangent plane



Adaptation methods – Riemannian center of mass (Grohs, Ebner)

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In Euclidean space, a weighted average $\sum_{j=1}^{n} w_j f_j$ for $f_j \in \mathbb{E}^d$ and $w_j \geq 0$, $j = 1, \ldots, n$, is equivalent to

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On manifolds we similarly define a weighted average by

$$\arg\min_{p \in \mathcal{M}} \sum_{j=1}^{n} w_j d^2(p, p_j).$$

This average is also termed **Karcher mean** in case of matrices or **Fréchet mean** in general metric spaces, and can be computed by iterations.
In this adaptation the linear average of a refinement rule is replaced by the Riemannian center of mass.

Adaptation methods – repeated binary averages (Dyn and Wallner)

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1. rewrite the refinement rule in terms of repeated binary averages form.
2. replace each arithmetic weighted average by a binary manifold average (details soon).

Binary manifold average

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2. (Symmetry) $M_t(p_1, p_2) = M_{1-t}(p_2, p_1)$.
3. (Interpolation) $M_0(p_1, p_2) = p_1$ and $M_1(p_1, p_2) = p_2$ (in analogy to $(1-t)f_1 + tf_2$).

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Note that even if we use the word "average" we in general do not restrict t to the interval $[0, 1]$.

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- In general metric spaces we define geodesics by the metric property

$$d(M_t(p_1, p_2), p_2) = (1 - t)d(p_1, p_2), \quad t \in [0, 1],$$

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Geodesic average

Geodesic induce natural averaging:

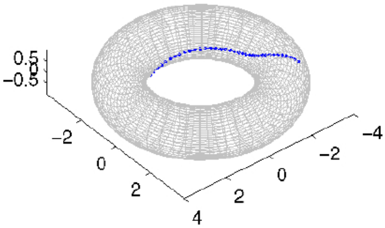
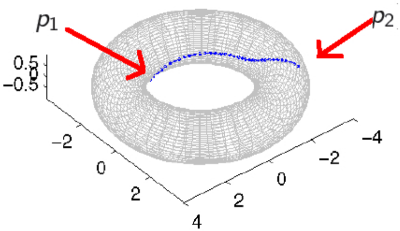


Figure: Geodesic on a torus

The geodesic average $M_t(p_1, p_2)$ is the point on the geodesic curve between p_1 and p_2 satisfying the metric property

Geodesic average

$M_{\frac{1}{2}}(p_1, p_2)$ is the midpoint of the geodesic curve between p_1 and p_2.



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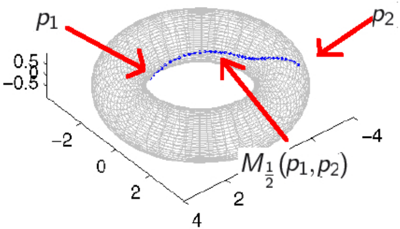

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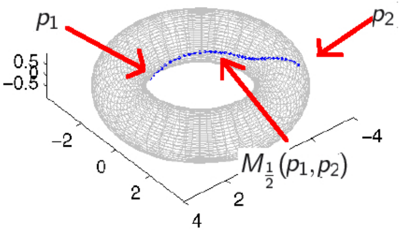

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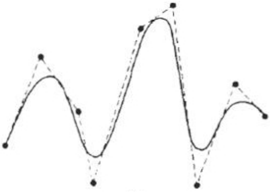
Note that $M_0(p_1, p_2) = p_1$ and $M_1(p_1, p_2) = p_2$ as required. The metric property ensures the symmetry of the average, namely

$$M_t(p_1, p_2) = M_{1-t}(p_2, p_1).$$

Recall the cubic B-spline scheme



The refinement rules of this scheme in terms of repeated binary averages are

$$
\begin{aligned}
(S(\mathbf{f}))_{2i} &= \frac{1}{8}f_{i-1} + \frac{3}{4}f_i + \frac{1}{8}f_{i+1} \\
&= \frac{1}{2}\left((\frac{1}{4}f_{i-1} + \frac{3}{4}f_i) + (\frac{3}{4}f_i + \frac{1}{4}f_{i+1}) \right) \\
(S(\mathbf{f}))_{2i+1} &= \frac{1}{2}(f_i + f_{i+1}).
\end{aligned}
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Cubic B-spline scheme – **local adaptation with geodesic averages**

Using the repeated binary averages form and the geodesic averages, we get the refinement rules

$$\mathcal{S}(\mathbf{p})_{2i} = M_{\frac{1}{2}}(M_{\frac{3}{4}}(p_{i-1}, p_i), M_{\frac{1}{4}}(p_i, p_{i+1}))$$
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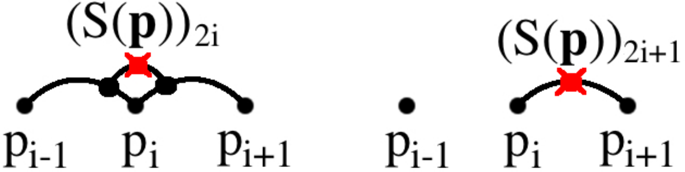

The arcs stand for geodesics.

Our adaptation method – weighted inductive means

Our adaptation method – weighted inductive means

Definition

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a finite sequence of manifold elements, and let $\mathbf{w} = (w_1, \ldots, w_n)$ be their associated real weights satisfying $\sum_{j=1}^{n} w_j = 1$. We further assume that $w_1 \geq w_2 \geq \ldots \geq w_n$. Then, the repeated geodesic average $\mathfrak{M}_n(\mathbf{p}, \mathbf{w})$ is defined recursively as,

$$\begin{cases} M_{w_2}(p_1, p_2) & \text{if } n = 2 \\ M_{w_n}(\mathfrak{M}_{n-1}\left((p_1, \ldots, p_{n-1}), \frac{1}{1-w_n}(w_1, \ldots, w_{n-1})\right), p_n) & \text{if } n > 2. \end{cases}$$

In case of symmetry,

$$w_j = w_{n-j+1}, \quad j = 1, \ldots, \ell \quad , \quad \ell = \lfloor n/2 \rfloor.$$

Then, for even n (for odd n we split the median weight) we define

$$\mathfrak{M}_n^S(\mathbf{p}, \mathbf{w}) = M_{1/2}(\mathfrak{M}_{n/2}(\mathbf{p}^1, \mathbf{w}^1), \mathfrak{M}_{n/2}(\mathbf{p}^2, \mathbf{w}^1).$$

Convergence

- Most popular method to show convergence of adapted linear subdivision schemes is by **proximity**.
- The main drawback is that the convergence via proximity is guaranteed only for "close enough data points". Namely, small enough $\delta(\{p_i\}) = \sup_i d(p_i, p_{i+1)}$.
- We aim to show that our schemes are convergent for **any initial data**.
- This is done by *contractivity*, that is the existence of $\mu < 1$ s.t., $\delta(\{\mathcal{S}\left(\{p_i\}\right)) \leq \mu\delta(\{p_i\})$.

Adaptation and convergence - example

The interpolatory 4−point scheme,

$$(S(\mathbf{f}))_{2i} = f_i, \qquad \text{and} \qquad (S(\mathbf{f}))_{2i+1} = -\omega(f_{i-1}+f_{i+2})+(\frac{1}{2}+\omega)(f_i+f_{i+1}).$$

With $\omega \in (0, \omega^*)$ and $\omega^* \approx 0.19273$ the unique solution of the cubic equation $32\omega^3 + 4\omega - 1 = 0$, the limits generated by the scheme are C^1. The case $\omega = \frac{1}{16}$ coincides with the cubics Dubuc-Deslauriers scheme.

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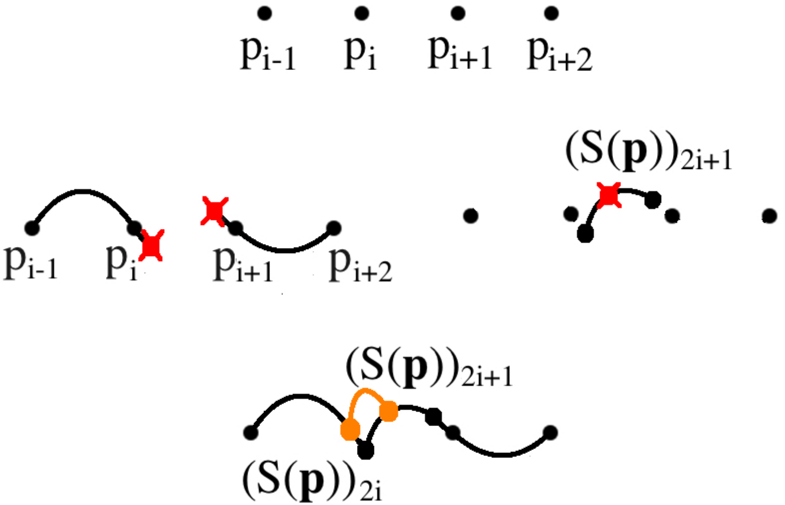
$$\mathcal{S}(\mathbf{p})_{2i} = p_i, \quad \text{and} \quad \mathcal{S}(\mathbf{p})_{2i+1} = \mathfrak{M}_4^S\left(\mathbf{p}, \mathbf{w}\right),$$

where

$$\mathfrak{M}_4^S\left(\mathbf{p}, \mathbf{w}\right) = M_{\frac{1}{2}}\left(M_{-2\omega}\big(p_i, p_{i-1}\big), M_{-2\omega}\big(p_{i+1}, p_{i+2}\big)\right),$$

with $\mathbf{w} = (\frac{1}{2}+\omega, -\omega, \frac{1}{2}+\omega, -\omega)$ and $\mathbf{p} = (p_i, p_{i-1}, p_{i+1}, p_{i+2})$.

Adaptation and convergence - example



Convergence - results

Theorem

The adapted version of the interpolatory 4-pt and 6-pt scheme, as well as the first four B-spline schemes (piecewise linear, quadratic corner cutting, cubic, and quintic) are converged from any (admissible) initial data.

Approximation order – in a nutshell

Having an intrinsic approximation order means there exists a constant C independent of the data $\mathbf{p}$ such that

$$d(\mathcal{S}(\mathbf{p})(t), f(t)) \leq C\left(\delta(\mathbf{p})\right)^n, \quad t \in \mathbb{R}.$$

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Theorem

Any convergent subdivision scheme that reproduces linear polynomials, which is adapted by the inductive geodesic average and having a contraction factor has a second approximation order.

Adaptation with repeated geodesic averages – summary

- The adaptation of linear subdivision schemes (and other operators) can be done, and in constructive manner, by the inductive averages.
- Reproducing linear polynomials is transform to geodesics reproduction. This is crucial for approximation order.
- We show the convergence, from any initial data, for several highly popular schemes.
- Nevertheless, the convergence technique is valid only for (relatively) small mask sizes. Namely, for general manifolds, without further assumptions, working with averaging many elements (or approximating the Kercher mean) is limited.

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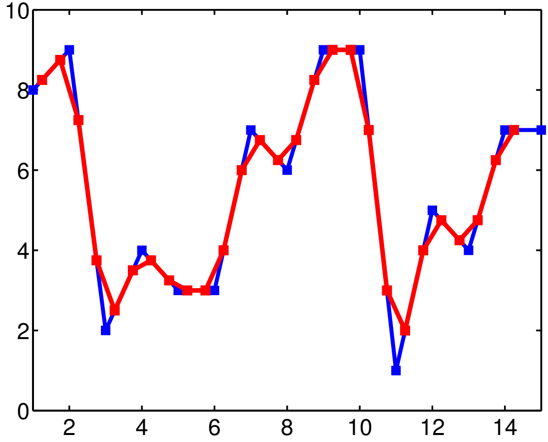
We look for a new approach to be considered!

Global refinement - classical origins

The quadratic B-spline subdivision scheme (corner-cutting)



First refinement

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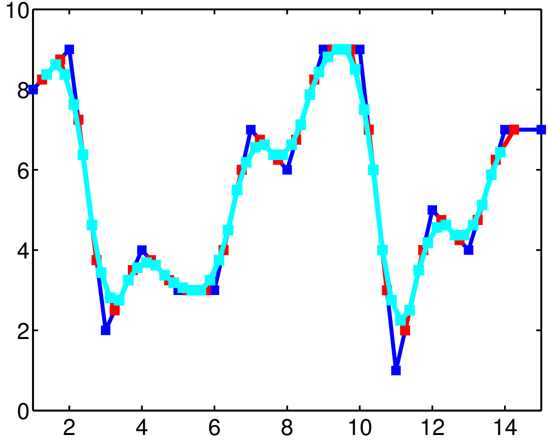

First two refinements

Global refinement - classical origins

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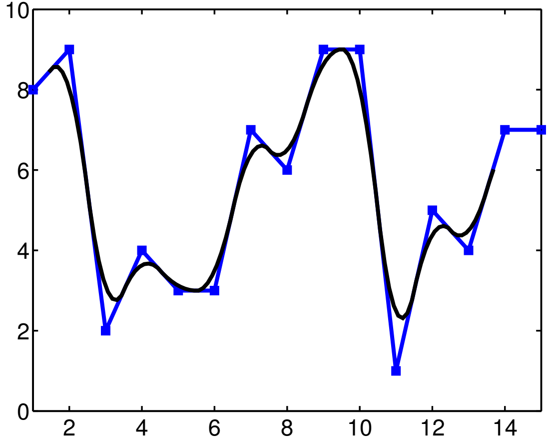

The limit

Global refinement - classical origins

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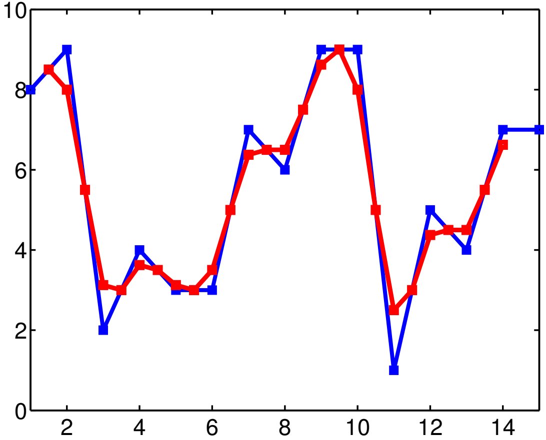

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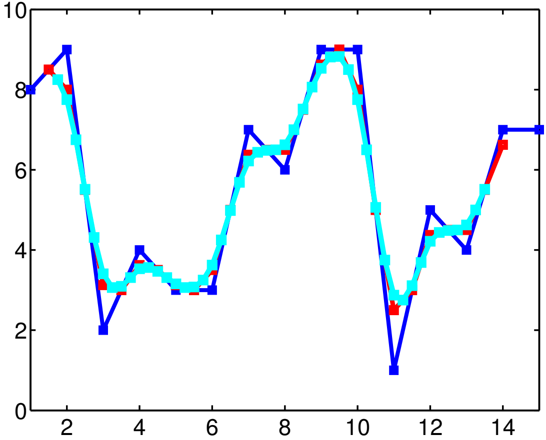

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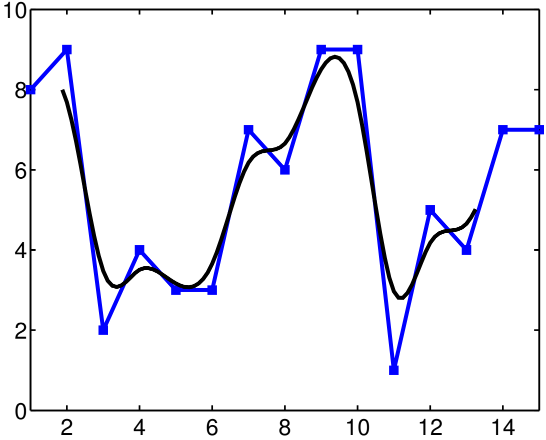

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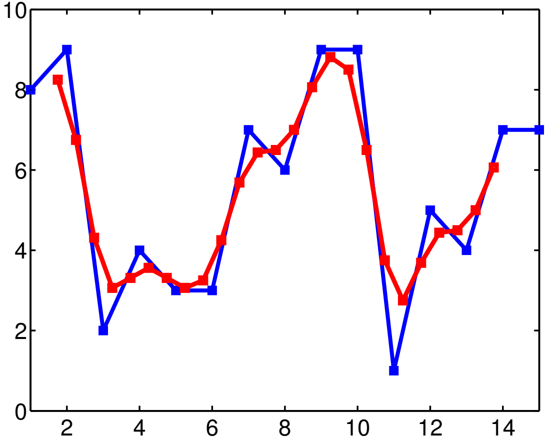
Global refinement - classical origins

The quartic B-spline subdivision scheme



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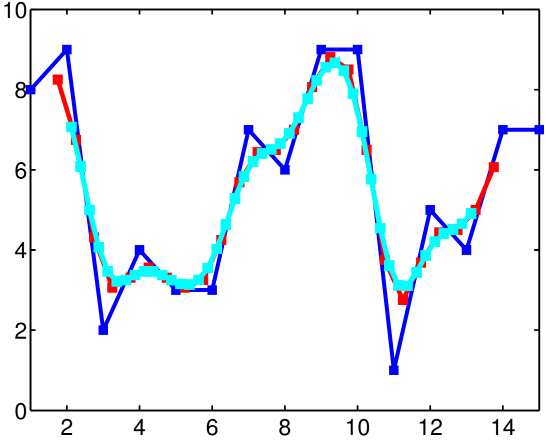

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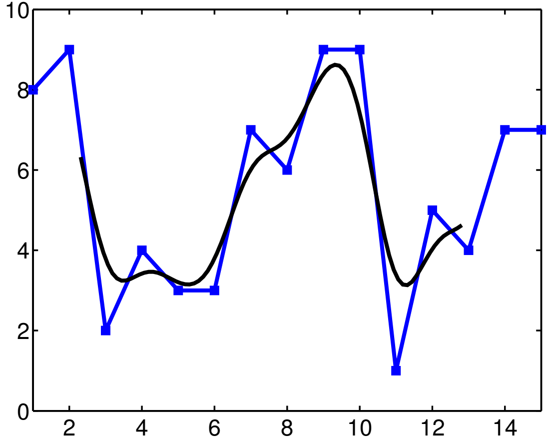

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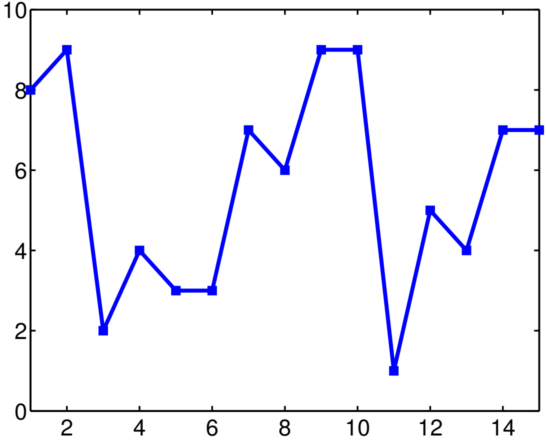

Linear (interpolation)

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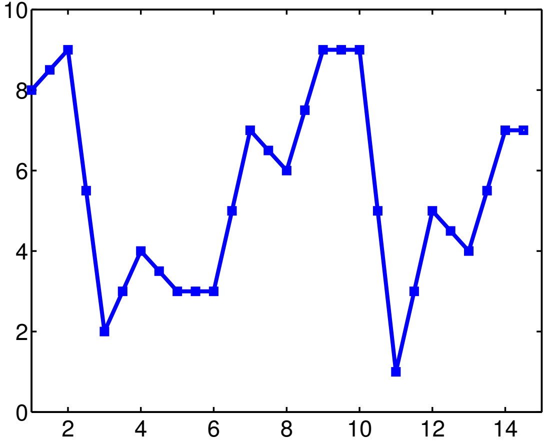

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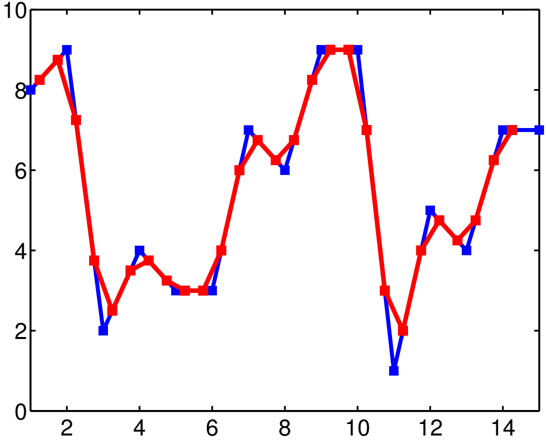

Quadratic (corner cutting)

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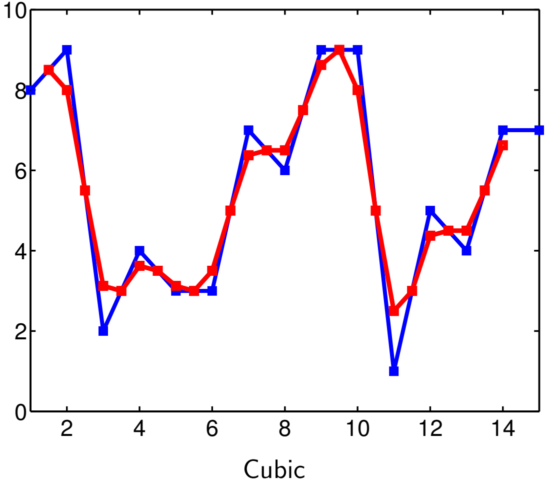

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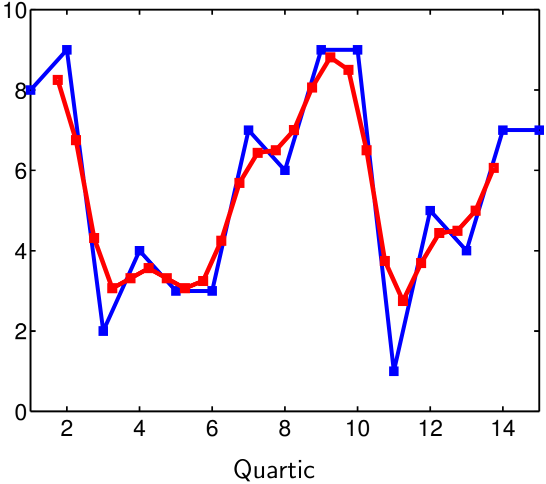

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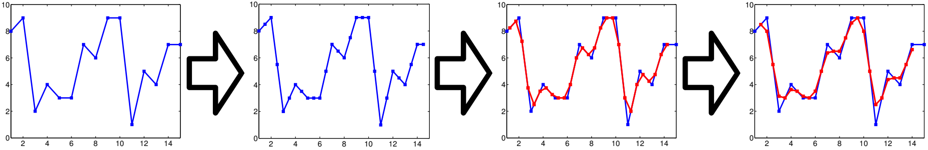

Initial data $\Rightarrow$ Doubling $\Rightarrow$ Averaging $\Rightarrow$ Averaging

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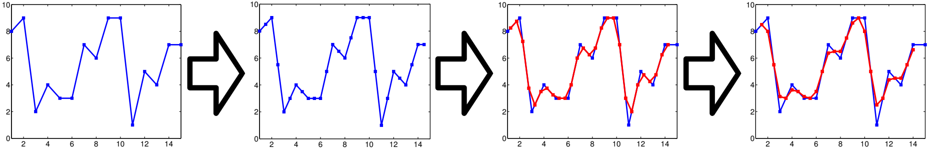

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This is the **Lane-Riesenfeld** algorithm.

The linear Lane-Riesenfeld algorithms – global refinements

(Lane and Riesenfeld, 1980)

The refinement step of an $m-$th degree B-spline subdivision scheme by several simple global averaging steps is also correspond to the factory of the symbol

$$a^{[m]}(z) = (1+z)\frac{(1+z)}{2}\left(\frac{1+z}{2}\right)^{m-1}.$$

The data: $m, \quad \mathbf{f} = \{f_i \in \mathbb{R} : i \in \mathbb{Z}\}$

for $i \in \mathbb{Z}$ do: $f_i^0 \leftarrow f_i \quad (\mathbf{f}^0 \leftarrow \mathbf{f})$

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For the B-spline case

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The adaptation of converging linear schemes

Assume $\mathcal{S}$ has a factorizable symbol over the reals and satisfies $a(-1) = 0$ and $a(1) = 2$. Namely, its symbol is

$$a(z) = z^{-s}(1+z)\frac{1+\alpha_1 z}{1+\alpha_1}\cdots\frac{1+\alpha_m z}{1+\alpha_m}$$

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2. for $j = 2, \ldots, m$ do:

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The adaptation: the averages in step 1 and 2 are replaced by $M_{\frac{1}{\alpha_1+1}}(p_i, p_{i+1})$ and $M_{\frac{1}{\alpha_j+1}}(p^{j-1}_i, p^{j-1}_{i-1})$, respectively.

Convergence results - $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, m$

Given the adaptation of a linear converging scheme with symbol

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Theorem

The convergence **for any initial manifold data** *is guaranteed when*

1. $\alpha_j > 0$, $j = 1, \ldots, m$ *(positive weights, typical for approximation).*
2. $\{\alpha_j \mid \alpha_j > 0\} \neq \emptyset$ *and* $\min_{\alpha_j>0} \max\{\frac{1}{1+\alpha_j}, \frac{\alpha_j}{1+\alpha_j}\} \prod_{i=2}^{m} \xi(\alpha_i) < 1$ *where*

$$\xi(\alpha) = \begin{cases} 1, & 0 < \alpha, \\ 1 + 2|\frac{\alpha}{1+\alpha}|, & -1 < \alpha < 0, \\ 1 + 2|\frac{1}{1+\alpha}|, & \alpha < -1. \end{cases}$$

(we allow negative weights, typical for interpolation)

General symbols

The symbol, which is a real polynomial, can be factorized into m_1 real linear factors (in addition to 1+z) and m_2 quadratic real factors. Then,

$$a(z) = z^{-s}(1+z)\left(\prod_{i=1}^{m_1}\frac{1+\alpha_i z}{1+\alpha_i}\right)\left(\prod_{i=m_1+1}^{m_1+m_2}\frac{1+2\,\mathrm{Re}(\alpha_i)z+|\alpha_i|^2z^2}{1+2\,\mathrm{Re}(\alpha_i)+|\alpha_i|^2}\right).$$

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- The quadratic factors stands for ternary average.
- We define a "pyramid" averaging for three elements, with optimal parameters.
- We extend the algorithm for symbols with complex roots.
- A general convergence result is derived.

Convergence results - symbols having single complex root

For this special case we have,

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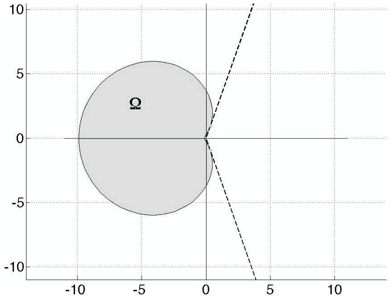
Theorem

Let $\mathcal{S}$ be a linear subdivision scheme, adapted globally such that $m_1 \geq 1$, $m_2 = 1$ and $\alpha_i > 0$, $1 \leq i \leq m_1$. Then, the adapted scheme converges from all admissible input data, whenever α_{m_1+1} is outside the domain Ω.



Thank you