Probabilistic Methods in Combinatorics: Homework Assignment Number 4

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Solutions will be collected in class on Tuesday, June 20, 2017.
This deadline (June 20) is strict

1. Show that for any $\epsilon > 0$ there is a $C = C(\epsilon)$ such that every set S of at least $\epsilon 4^n$ vectors in Z_4^n contains four vectors so that the Hamming distance between any pair of them is at least $n - C\sqrt{n}$.

   Hint: use an appropriate martingale to show that more than 3/4 of the vectors are within distance $C\sqrt{n}/2$ of S

2. Let H be a graph with m edges and maximum degree at most 10. Let U be a random set of vertices of H obtained by picking each vertex, randomly and independently, with probability $p = \frac{1}{\log m}$. Show that the probability that U is independent is $(1-p^2)^{m(1-o(1))}$ (where the $o(1)$ term tends to zero as m tends to infinity.)

3. Prove that there exists n_0 so that for every $n > n_0$ there is a bipartite graph G with classes of vertices A and B, where $|A| = |B| = n$, in which the degree of every vertex $a \in A$ is exactly $\lfloor n^{0.4} \rfloor$ and for every two subsets $X \subset A$ and $Y \subset B$ so that $|X| = |Y| \geq n^{0.9}$, the induced subgraph of G on $X \cup Y$ contains a cycle of length 4.

4. Find a threshold function for the following property of a graph G on n vertices: every set of at least $n/2$ vertices of G contains a (not necessarily induced) cycle of length 5. (Recall that, by definition, $t(n)$ is such a threshold function if when $p(n) = o(t(n))$, then with high probability, $G(n, p(n))$ does not satisfy the property, and if $t(n) = o(p(n))$ then with high probability $G(n, p(n))$ satisfies it).

5. Prove that there exists a constant d_0 so that for every $d > d_0$, every d-regular graph contains a spanning subgraph with minimum degree at least 10 and girth at least 10.