Probabilistic Methods in Combinatorics: Homework Assignment Number 2

Noga Alon

Solutions will be collected in class on Tuesday, April 25, 2017.

1. Prove that there is an absolute constant $c > 0$ so that any 3-uniform hypergraph $H = (V, E)$ with n vertices, m edges and no isolated vertices contains an independent set (that is, a set of vertices containing no edge) of size at least $cn^{3/2}/m^{1/2}$.

2. (i). Show that for any two integers k and ℓ and for any real p, $0 < p < 1$, and any integer n, the Ramsey number $r(k, \ell)$ is at least
$$n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1-p)^{\binom{\ell}{2}}.$$
(ii). Apply the above to prove that the Ramsey number $r(4, k)$ satisfies $r(4, k) \geq c(k/\ln k)^{\alpha}$ for some absolute constant $c > 0$ and for the largest $\alpha > 0$ for which you can derive this inequality from the result in (i).

3. Prove that there is an absolute constant $c > 0$ so that the random graph $G = G(n, 100/\sqrt{n})$ contains, with high probability (that is, with probability that tends to 1 as n tends to infinity) a set of at least $cn^{3/2}$ pairwise edge disjoint triangles.

4. Let $S_1, S_2, \ldots, S_k$ be a collection of subsets of $\{1, 2, \ldots, n\}$. Prove that if n is sufficiently large and $k \leq 1.99 \frac{n}{\log_2 n}$ then there are two distinct subsets X, Y of $\{1, 2, \ldots, n\}$ so that $|X \cap S_i| = |Y \cap S_i|$ for all $1 \leq i \leq k$.

5. (i) Prove that there is an absolute constant $c > 0$ so that the following holds. For every prime p and every set $A \subset Z_p$, $|A| = k$, there is an $x \in Z_p$ so that the set $\{xa (mod\ p) : a \in A\}$ intersects every interval of length at least $c\frac{p}{\sqrt{k}}$ in Z_p.

   (ii) Conclude that there is $c' > 0$ so that if p is a prime which is $3 (mod\ 4)$ then any interval of length at least $c'\sqrt{p}$ in Z_p contains a quadratic residue.

6. Prove that there is an absolute constant $c > 0$ so that the following holds. For every prime p and every set $A \subset Z_p$, $|A| = k$, there is a polynomial $f(x)$ of degree at most 3 over Z_p so that the set $\{f(a) (mod\ p) : a \in A\}$ intersects every interval of length at least $c\frac{p}{k^{2/3}}$ in Z_p.