Probabilistic Methods in Combinatorics: Homework Assignment Number 1

Noga Alon

Solutions will be collected in class on Tuesday, April 4, 2017.

1. Let p be a prime number and let $A \subset Z_p$ be a set of $|A| < p^{2/3}$ residues modulo p. Show that there are elements $x, y \in Z_p$ such that for $A + x = \{(a + x) \ (mod\ p) : a \in A\}$ and $A + y = \{(a + y) \ (mod\ p) : a \in A\}$, the three sets $A, A + x$ and $A + y$ do not have a common intersection, that is $A \cap (A + x) \cap (A + y) = \emptyset$.

2. The (multi-colored) Ramsey number $r_j(k)$ is the smallest integer r so that in any coloring by j colors of the edges of the complete graph on r vertices there is a monochromatic copy of K_k.

   (i) Prove that if $\binom{n}{k} 3^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of the complete graph on n vertices by 3 colors without a monochromatic copy of K_k and conclude that for $k > 4$, $r_3(k) > 3^{k/2}$.

   (ii) Prove that for all $k > 4$, $r_8(k) > 4^k$ (which is much bigger than $8^{k/2}$).

3. Let $\{(A_i, B_i)_{1 \leq i \leq h}\}$ be a collection of pairs of subsets of the integers so that $|A_i| + |B_i| = n$ and $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq h$ and for every $1 \leq i < j \leq h$ $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$. Prove that $h \leq 2^n$.

4. (i) Show that the vertices of any tournament $T = (V, E)$ in which all outdegrees are at least 10 can be colored by 2 colors so that every vertex has at least one outneighbor of each color.

   (ii) Prove that for any integer k there is a directed graph (with no parallel edged) in which every outdegree is at least k and yet there is no 2-coloring of the vertex set so that each vertex has at least one outneighbor of each color.

5. Let $G = (V, E)$ be a directed graph with $n > 1$ vertices and $\lceil n \log_2 n \rceil$ directed edges. Prove that there is a tournament on n vertices containing no subgraph isomorphic to G.

6. (*) Bonus Question

   Prove that there is an absolute constant $c > 0$ so that the following holds. Let $G = (V, E)$ be a graph with chromatic number $k > 4$. Let $G' = (V, E')$ be a random spanning subgraph of G obtained by retaining each edge of G, randomly and independently with probability $1/2$. Then the probability that the chromatic number of G' is at most 2 is smaller than 2^{-ck^2}.

   Hint: Show first that this holds for $k = 5$.