Graph Theory 0366-3267

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Fall Semester 2011

Homework 2

Due: Dec. 21, 2011

1. Prove that in a connected graph G every two paths of maximum length share a vertex.

2. Let Q_k be the k-dimensional cube defined as follows: $V(Q_k) = \{0,1\}^k$, $(\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k)) \in E(Q_k)$ iff $\mathbf{x}$ and $\mathbf{y}$ differ in exactly one coordinate. Prove: $\kappa(Q_k) = \kappa'(Q_k) = k$.

3. Let $k \geq 2$. Prove that every k-connected graph on at least $2k$ vertices contains a cycle of length at least $2k$.

4. Let d be a positive integer. Prove that every $2d$-regular connected graph G with an even number of edges contains a spanning d-regular subgraph.

5. Let G be the graph whose vertices are the $4n$ squares of the 4-by-n "chessboard", where two vertices are adjacent if and only if a knight can jump between the corresponding squares. (Formally, the set of vertices is: $\{(i,j) : 1 \leq i \leq 4, 1 \leq j \leq n\}$, and $((i,j),(i',j'))$ is an edge in G if and only if either $|i - i'| = 1$ and $|j - j'| = 2$, or $|i - i'| = 2$ and $|j - j'| = 1$.) Does G contain a Hamilton cycle? Prove your claim!

6. Let $t(n, H_n)$ be the maximum number of edges in a graph G on n vertices, not containing a Hamilton cycle H_n. Prove: $t(n, H_n) = \binom{n-1}{2} + 1$. (You need to prove both lower and upper bounds for $t(n, H_n)$.)

7. Let G be a graph of connectivity $\kappa(G)$ with independence number $\alpha(G)$. Assume $\kappa(G) \geq \alpha(G) - 1$. Prove that G contains a Hamilton path.