Graph Theory 0366-3267

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Homework 1
Due: Nov. 30, 2011

1. Prove that every simple graph with $n \geq 7$ vertices and at least $5n - 14$ edges contains a subgraph with minimum degree at least 6.

2. Prove that the number of graphs on n labeled vertices with all degrees even is $2^{\binom{n-1}{2}}$.

3. Prove that every graph $G = (V, E)$ with $|E| = m$ edges has a bipartition $V = V_1 \cup V_2$ such that the number of edges of G crossing between V_1 and V_2 is at least $m/2$.

4. **(a)** Let G be a graph with all degrees at least three. Prove that G contains a cycle with a chord.

(b) Let G be a graph on $n \geq 4$ vertices with $2n - 3$ edges. Prove that G contains a cycle with a chord.

5. Let $0 < d_1 \leq d_2 \leq \ldots \leq d_n$ be integers. Prove that there exists a tree with degrees $d_1, \ldots, d_n$ if and only if

$$d_1 + \ldots + d_n = 2n - 2\,.$$

6. Prove that every graph G with minimal degree d contains every tree on $d + 1$ vertices as a subgraph.

7. Let X be an n-element set and let $A_1, \ldots, A_n$ be distinct subsets of X. Prove that there exists an element $x \in X$ such that the subsets $A_1 \cup \{x\}, \ldots, A_n \cup \{x\}$ are distinct as well. (*Hint:* Define a graph G with vertex set $[n]$, where i, j are connected by an edge if the symmetric difference between A_i and A_j is a single element y; use y to label this edge. Prove that there is a forest in G containing exactly one edge with each label used. Use this to obtain the desired x.)

8. Compute the number of spanning trees in the complete bipartite graph $K_{m,n}$.