Superconcentrators of depth 2 and 3; odd levels help (rarely)

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Abstract

It is shown that the minimum possible number of edges in an n- superconcentrator of depth 3 is $\Theta(n \log\log n)$, whereas the minimum possible number of edges in an n-superconcentrator of depth 2 is $\Omega(n(\log n)^{3/2})$ (and is $O(n(\log n)^2)$).

1 Introduction

An *n-superconcentrator* is a directed acyclic graph S with the following properties.
(i) There are two disjoint subsets of vertices of S, U (called the set of *inputs*) and V (called the set of *outputs*), each of cardinality n, where the indegree of each vertex in U is 0 and the outdegree of each vertex in V is 0.
(ii) For every two subsets $X \subset U$ and $Y \subset V$, where $1 \leq |X| = |Y| \leq n$ there are $|X|$-vertex disjoint paths of S from X to Y.

The *depth* of a superconcentrator is the maximum length of a directed path in it, and its *size* is the number of its edges. It is sometimes convenient to assume that the vertices of a depth-d superconcentrator are partitioned into $d+1$ levels, where the inputs form the first level, the outputs form the last one, and all the edges are directed from level i to level $i+1$. For fixed d this assumption does not change the minimum possible size by more than a constant factor.

Superconcentrators have been the subject of intensive study, as they are relevant to lower bounds as well as to the construction of certain networks with high connectivity properties. Pippenger [6] showed that there are n-superconcentrators of depth 2 and size $O(n \log^2 n)$, and showed that they must have size at least $\Omega(n \log n)$. The minimum possible size of n-superconcentrators of any depth $d \geq 4$ has been determined up to a constant factor in [4] (for all even values of $d \geq 4$) and in [7] (for all odd values of $d \geq 5$.) In particular, it follows from these results that for every even $d \geq 4$, the minimum possible size of an n-superconcentrator of depth d is equal, up to a constant factor, to that of an n-superconcentrator of depth $d+1$. In other words, in all these cases the extra odd level does not help in reducing the size.

In the present paper we determine the minimum possible size of an n-superconcentrator of depth 3 up to a constant factor. This size is $\Theta(n \log \log n)$ showing that for $d = 2$ the extra odd level does yield a saving in the size. In addition, we improve the lower bound of Pippenger for the minimum size of depth 2 n-superconcentrators and show that it is $\Omega(n(\log n)^{3/2})$.

2 The lower bound for depth 2

We need two lemmas. The first one is the following known bound concerning Zarankiewicz problem (cf., [3], Theorem VI.2.5).

Lemma 2.1 *Let $k_r(n)$ denote the minimum integer k such that every bipartite graph with n vertices in each vertex class and with at least k edges contains a complete bipartite subgraph with r vertices in each vertex class. Then*

$$k_r(n) \leq (r-1)^{1/r} n^{2-1/r} + \frac{1}{2}(r-1)n.$$

The second lemma is the following somewhat technical result proved in [7].

Lemma 2.2 *There exists an absolute positive constant $\delta > 0$ such that the following holds. For every sequence of s reals $c_1 \geq c_2 \geq \ldots \geq c_s \geq 0$ and for every $1 \leq p \leq m \leq s$, if the inequality*

$$\sum_{i=r}^{s} c_i^2 \geq 1/r$$

holds for all r, $p \le r \le m$, *then*

$$\sum_{i=1}^{s} c_i \ge \delta(\log m - \log p).$$

Theorem 2.3 *Depth* 2 *n-superconcentrators have size* $\Omega(n(\log n)^{3/2})$.

Proof We assume, whenever it is needed, that n is sufficiently large. Let $v_1, v_2, \ldots, v_s$ be the vertices in the middle level of a given n-superconcentrator S, and let d_i be the degree of v_i, where $d_1 \ge d_2 \ge \ldots \ge d_s$. We may assume that all edges are incident with vertices in the middle level. Let r be any integer which does not exceed s. Let G_r be the bipartite graph whose classes of vertices are the set of n inputs of S and the set of n outputs of S in which there is an edge between an input x and an output y iff x is connected with y in S through a vertex v_i with $i \ge r$.

Obviously, for every set X of inputs and every set Y of outputs with $|X| = |Y| = r$ there is at least one edge in G_r between X and Y, since otherwise there are no r vertex disjoint paths in S between X and Y. It thus follows from Lemma 2.1 that the number of *non* edges in G_r is smaller than

$$k_r(n) \le (r-1)^{1/r} n^{2-1/r} + \frac{1}{2}(r-1)n = n^2 e^{-(ln\ n - ln\ (r-1))/r} + \frac{1}{2}(r-1)n.$$

We restrict our attention only to integers r between, say, $n^{1/3}$ and $n^{1/2}$. In this range, the right hand side of the last inequality is at most

$$n^2(1 - (ln\ n - ln\ (r-1))/2r) + \frac{1}{2}(r-1)n.$$

It follows that G_r contains at least

$$n^2(ln\ n - ln(r-1))/2r - \frac{1}{2}(r-1)n \ge \frac{1}{16} n^2 ln\ n/r$$

edges, where in the last inequality we used the fact that n is large and that $n^{1/3} \le r \le n^{1/2}$.

Observe, next, that the number of edges of G_r is at most

$$\sum_{i=r}^{s} x_i(d_i - x_i) \le \frac{1}{4}\sum_{i=r}^{s} d_i^2,$$

where x_i is the number of inputs adjacent to v_i (and hence $d_i - x_i$ is the number of outputs adjacent to v_i). Therefore, for every integer r, $n^{1/3} \le r \le n^{1/2}$

$$\sum_{i=r}^{s} d_i^2 \ge \frac{1}{4} n^2 ln\ n/r.$$

Define $c_i = \frac{2d_i}{n\sqrt{ln\ n}}$. Then for every r as above

$$\sum_{i=r}^{s} c_i^2 \geq 1/r.$$

Therefore, by Lemma 2.2, $\sum_{i=1}^{s} c_i \geq \Omega(\log n)$, and hence the number of edges of S, which is $\sum_{i=1}^{s} d_i$, is at least $\Omega(n(\log n)^{3/2})$, completing the proof. □

3 The lower bound for depth 3

Theorem 3.1 *Depth 3 n-superconcentrators have size $\Omega(n \log\log n)$.*

Proof Let C be a depth 3 n-superconcentrator. The levels of C will be denoted V_0 (the inputs), V_1, V_2 and V_3 (the outputs). Let H be the set of edges of the superconcentrator. All the edges are directed from the i-th level to the $i+1$-st. Let D_i be the set of vertices of V_1 with indegree m satisfying

$$n^{2^{-i}} > m \geq n^{2^{-i-1}},$$

and the vertices of V_2 whose outdegree m satisfies the same inequality. Let $t = \log_2 \log_2 n - \log_2 \log_2 \log_2 \log_2 n - 2$. Thus, for $i \leq t$,

$$n^{2^{-i-2}} \geq \log_2 \log_2 n.$$

Assume that the number of all edges of the superconcentrator is less than $\frac{1}{16} n \log_2 \log_2 n$. We shall show that for every $i = 0, 1, \ldots, t-2$,

$$|\{(u,v) \in H;\ (u \in V_0 \textit{ and } v \in D_i \cup D_{i+1} \cup D_{i+2})$$

Since the sets D_i are disjoint this will prove the bound. Suppose the condition fails for i. Let U_0^i be the set of inputs connected with some vertex in $D_i \cup D_{i+1} \cup D_{i+2}$ and let U_3^i be the set of outputs connected with some vertex in $D_i \cup D_{i+1} \cup D_{i+2}$. By the assumption $|U_0^i|, |U_3^i| < \frac{1}{4}n$. Define

$$k = \lceil n^{1-2^{-i-1}} \rceil.$$

Clearly

$$n^{1-2^{-i-1}} \leq k \leq 2n^{1-2^{-i-1}}.$$

Let X be a random subset of cardinality k of the inputs, and let Y be a random subset of cardinality k of the outputs. Let z be the random variable "*the number of vertex disjoint paths connecting* $X \setminus U_0^i$ *with* $Y \setminus U_3^i$". Since the number of vertex disjoint paths connecting X with Y is at least k, we have

$$z(X,Y) \geq k - max(|U_0^i \cap X|, |U_3^i \cap Y|).$$

Therefore, the expected value $E(z)$ of z satisfies

$$E(z) \geq k - E(|U_0^i \cap X|) - E(|U_3^i \cap Y|)$$

$$\geq k - |U_0^i|k/n - |U_3^i|k/n \geq k - 2\frac{1}{4}n\frac{k}{n} \geq k/2.$$

Suppose that

$$|D_0 \cup \ldots \cup D_{i-1}| \geq k/4.$$

Since each vertex in this union has indegree or outdegree at least $n^{2^{-i}}$, we would have, in this case, at least

$$\frac{k}{4}n^{2^{-i}} \geq \frac{1}{4}n^{1-2^{-i-1}}n^{2^{-i}} \geq \frac{1}{4}n^{1+2^{-i-1}} \geq \frac{1}{4}n \log_2 \log_2 n$$

edges. Hence we may assume that $|D_0 \cup \ldots \cup D_{i-1}| \geq k/4$. Let z' be the random variable "*the number of vertex disjoint paths connecting* X *with* Y *which do not contain vertices from* $D_0 \cup \ldots \cup D_{i+2}$." Then $z' \geq z - k/4$, and hence $E(z') \geq k/4$. Let $(u,v) \in H$ be an edge, where $u \in V_1$, $v \in V_2$, u, v not in $D_0 \cup \ldots \cup D_{i+2}$. Since the indegree of u and the outdegree of v are less than $n^{2^{-i-3}}$, we have

$$Prob(\ there\ is\ a\ path\ from\ X\ to\ Y\ through\ (u,v)\ )$$

$$= Prob(\ there\ is\ p \in X\ (p,u) \in H) \cdot Prob(\ there\ is\ q \in Y\ (v,q) \in H)$$

$$< (n^{2^{-i-3}}k/n)^2 \leq 4n^{2 \cdot 2^{-i-3} + 2\cdot(-2^{-i-1})} = 4n^{2^{-i-2}-2^{-i}}.$$

Hence

$$E(z') \leq |H| 4n^{2^{-i-2}-2^{-i}}.$$

Comparing it with $k/4$, the lower bound for $E(z')$, we get

$$|H| \geq \frac{1}{16}k \cdot n^{-2^{-i-2}+2^{-i}} \geq \frac{1}{16}n^{1-2^{-i-1}-2^{-i-2}+2^{-i}} = \frac{1}{16}n^{1+2^{-i-2}},$$

which is, by the assumption about i, at least $\frac{1}{16}\log_2 \log_2 n$. This contradicts our assumption and hence completes the proof. □

4 The upper bound for depth 3

The upper bound is proved by a (probabilistic) construction. We need the following two lemmas, proved by applying simple probabilistic arguments. The first lemma deals with graphs known as expanders and the second one with graphs usually called concentrators.

Lemma 4.1 *For every two integers $m \geq a \geq 1$ there is a bipartite graph H with classes of vertices L, M, where $|L| = |M| = m$, and with at most $3m\lceil \frac{m}{a} \log \frac{em}{a} \rceil$ edges, so that for any $X \subset L$ and $Y \subset M$ with $|X| = |Y| = a$ there is an edge of H joining a member of X with a member of Y.*

Proof Let L and M be two disjoint sets of vertices, each of cardinality m. For each vertex $v \in L$ choose, randomly and independently, $d = 3\lceil \frac{m}{a} \log \frac{em}{a} \rceil$ (not necassarily distinct) neighbors in M. To complete the proof it suffices to show that the expected number of pairs of sets $X \subset L$ and $Y \subset M$, with $|X| = |Y| = a$ and with no edge between X and Y, is smaller than 1. This is indeed the case, since the above expectation is

$$\binom{m}{a}^2 (1 - a/m)^{ad} < (em/a)^{2a} e^{-3a \log(em/a)} < 1,$$

as needed. □

Lemma 4.2 *For every three integers n, m, p, where $n \geq m \geq 2p$, there is a bipartite graph F with classes of vertices C and D, where $|C| = n$, $|D| = m$, and with at most*

$$16n\lceil \frac{\log(en/p)}{\log(em/p)} \rceil$$

edges, so that every set X of $i \leq p$ vertices in C has at least $i + 1$ neighbors in D.

Proof Define $d = 16\lceil \frac{\log(en/p)}{\log(em/p)} \rceil$ and let C and D be two disjoint sets of vertices, where $|C| = n$, $|D| = m$. For each vertex $v \in C$ choose, randomly and independently, d (not necessarily distinct) neighbors in D. Let F be the random bipartite graph obtained in this manner. The probability that a fixed subset $X \subset C$ of cardinality i has at most i neighbors in D is at most

$$\binom{m}{i} (i/m)^{id},$$

since there are at most $\binom{m}{i}$ ways of choosing a set Z of cardinality i containing all the neighbors of the members of X, and the probability that indeed all these neighbors lie in Z is at most

$(|Z|/m)^{|X|d}$. It follows that the expected number E of subsets $X \subset C$ of cardinality at most p that have at most $|X|$ neighbors in D is at most

$$E \le \sum_{i=1}^{p} \binom{n}{i}\binom{m}{i}(i/m)^{id} \le \sum_{i=1}^{p}(en/i)^i(em/i)^i(i/m)^{id}$$

$$\le \sum_{i=1}^{p} e^{i(\log(en/i)+\log(em/i)-d\log(m/i))}.$$

Since the function $g(i) = \frac{\log(en/i)}{\log(em/i)}$ is an increasing function of i for $1 \le i \le p$ and since $m/i \ge 2$ implies that $\log(m/i) \ge \frac{1}{4}\log(em/i)$ we conclude that

$$d\log(m/i) \ge 16\frac{\log(en/i)}{\log(em/i)}\frac{1}{4}\log(em/i) = 4\log(en/i).$$

Therefore,

$$E \le \sum_{i=1}^{p} e^{i(2\log(en/i)-4\log(en/i))} \le \sum_{i=1}^{p} e^{i(-2\log(2e)} < 1.$$

This completes the proof. $\Box$

Corollary 4.3 *Let n be an integer and let $r \ge 100$ be a real, $n \ge r$. Then there is a depth-3 directed acyclic graph $G = G_r$ with classes of vertices V_0, V_1, V_2, V_3, in which all edges are directed from V_i to V_{i+1}, $(0 \le i \le 3)$, with the following properties.*
(i) $|V_0| = |V_3| = n$.
(ii) $|V_1| = |V_2| = \lfloor n/r^{2/3} \rfloor$.
(iii) The number of edges of G is $O(n)$.
(iv) For every $A \subset V_0$ and $B \subset V_3$ with $|A| = |B| \le n/r$ there are $A' \subset A$ and $B' \subset B$ such that $|A \setminus A'| = |B \setminus B'| \le n/r^{7/6}$ and such that there are $|A'|$-vertex disjoint (directed) paths in G between A' and B'. Moreover, in case $n/r^{2/3} \le \sqrt{n}$ such paths exist for $A' = A$, $B' = B$.

Proof Construct G as follows. The subgraph of G induced on $V_0 \cup V_1$ is obtained from the graph in Lemma 4.2 with $n, m = \lfloor n/r^{2/3} \rfloor$ and $p = \lfloor n/r \rfloor (\le m/2)$ by taking $V_0 = C, V_1 = D$ and by directing all edges from V_0 to V_1. Symmetrically, the subgraph of G induced on $V_3 \cup V_2$ is obtained from Lemma 4.2 by taking the same parameters as above, $C = V_3$ and $D = V_2$ and by directing all edges from V_2 to V_3.

Put $m = \lfloor n/r^{2/3} \rfloor$. In case $m \le \sqrt{n}$ let the induced subgraph of G on $V_1 \cup V_2$ be the complete directed bipartite graph in which each vertex in V_1 is connected by a directed edge to each vertex in

V_2. Otherwise, this induced subgraph is obtained from the graph in Lemma 4.1 with $m, a = \lfloor n/r^{7/6} \rfloor$ by taking $V_1 = L, V_2 = M$ and by directing all edges from V_1 to V_2.

Notice that by the two lemmas above, the number of edges of G is indeed $O(n)$, as needed. Suppose $A \subset V_0$, $B \subset V_3$, and $|A| = |B| \leq n/r$. By the properties of the construction in Lemma 4.2 and by Hall's Theorem it follows that there are $A^* \subset V_1$ and $B^* \subset V_2$, such that $|A^*| = |B^*| = |A|$ $(= |B|)$ and there are two matchings in G joining the vertices of A with these of A^* and the vertices of B with these of B^*. In case $m \leq \sqrt{n}$ there is a matching of G between the two sets A^* and B^*, providing, together with the previous two matchings, vertex disjoint paths between the vertices of A and those of B, as needed. Otherwise, we have to show that there is a matching of size at least $|A^*| - \lfloor n/r^{7/6} \rfloor$ between A^* and B^*. However, such a matching certainly exists by the property of the graph in Lemma 4.1. In fact, it can even be constructed by adding, one by one, edges to such a matching as long as there is an edge joining a yet unmatched vertex of A^* with a yet unmatched vertex of B^*. Since in the construction of Lemma 4.1 there is an edge between any two sets of cardinality a in the two vertex classes, this procedure cannot be terminated before it matches all vertices of A^* but at most $a - 1$ of them with vertices in B^*. This completes the proof of the lemma. □

Theorem 4.4 *There are depth-3 n-superconcentrators of size $O(n \log \log n)$.*

Proof Let U_0 and U_3 be two disjoint classes of vertices, $|U_0| = |U_3| = n$. We construct a superconcentrator of depth 3 whose inputs are the members of U_0 and whose outputs are these of U_3. Put $r_1 = 100$ and $r_{i+1} = r_i^{7/6}$ for $i \geq 1$, and let l be the first i such that $n/r_i^{2/3} \leq \sqrt{n}$. Clearly $l = O(\log \log n)$.

For each $i, 1 \leq i \leq l$, let G_i be the graph of Corollary 4.3 with $r = r_i$. Let G be the graph obtained from the disjoint union of all these graphs G_i by identifying U_0 and U_3 with the first and last layers, respectively, of all these l graphs. Let G' be the graph obtained from G by adding to it a set of $O(n)$ directed edges from U_0 to U_3, such that there is at least one edge between any two subsets of cardinality at least $n/100$ of these two classes. (Such a set of edges exists by Lemma 4.1 with $m = n, a = \lfloor n/100 \rfloor$.)

Clearly G' is a depth-3 directed acyclic graph with $O(n \log \log n)$ edges. To complete the proof we show that G' is a superconcentrator. Suppose, thus, that $A \subset U_0$, $B \subset U_3$ are two subsets,

$|A| = |B|$. By applying the direct edges, if necessary, we can match members of A with members of B until we are left with at most $n/100$ vertices in each of these classes. Next, we can use the edges of G_1 to obtain additional vertex-disjoint paths until the sizes of the unmatched subsets of A and B are reduced to at most $n/r_1^{7/6} = n/r_2$. Continuing in this manner, the edges of each G_i are used in order to reduce the sizes of the remaining unmatched vertices to at most n/r_{i+1}, where in the last step (when the edges of G_l are used) all the remaining vertices are matched by vertex disjoint paths. This completes the proof. □

5 Concluding remarks

The results in the present paper, together with these in [4] and [7] determine, up to a constant factor, the smallest possible size of an n-superconcentrator of depth d for all $d \geq 3$. Since this size is trivially n^2 for $d = 1$, the only remaining case is depth 2, first studied in [6]. In [2] it is shown that size $\Omega(n \log n)$ is required in depth 2 even if we only assume that for a single value of k, where $n^\epsilon \leq k \leq n^{1-\epsilon}$, (for any fixed $\epsilon > 0$), there are at least $\log k$ vertex disjoint paths between any two sets of k inputs and k outputs. Therefore it is not surprising that the $\Omega(n \log n)$ lower bound of [6] is not tight for depth 2 superconcentrators, as shown in Section 2. We suspect that the $O(n \log^2 n)$ upper bound proved in [6] is closer to the correct value of the minimum possible size for depth 2 than the $\Omega(n(\log n)^{3/2})$ lower bound, proved here.

Our proof of the $O(n \log \log n)$ upper bound for the size in depth 3 is not constructive. Although Lemma 4.1 can be replaced (with some insignificant loss in the constants) by an appropriate construction using some of the known explicit expanders, it is much more difficult to obtain an explicit version of Lemma 4.2. In fact, it seems difficult to obtain an explicit construction of size $O(n^{1+\epsilon})$ and depth 3 even for a fixed (small) $\epsilon > 0$. See [5], [1] for the (modest) known explicit constructions for bounded depth, small size superconcentrators.

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