The Complexity of the Outer Face in Arrangements of Random Segments*

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ABSTRACT

We investigate the complexity of the outer face in arrangements of line segments of a fixed length ℓ in the plane, drawn uniformly at random within a square. We derive upper bounds on the expected complexity of the outer face, and establish a certain phase transition phenomenon during which the expected complexity of the outer face drops sharply as a function of the total number of segments. In particular we show that up till the phase transition the complexity of the outer face is almost linear in n, and that after the phase transition, the complexity of the outer face is roughly proportional to $\sqrt{n}$. Our study is motivated by the analysis of a practical point-location algorithm (so-called walk-along-a-line point-location algorithm) and indeed, it explains experimental observations of the behavior of the algorithm on arrangements of random segments.

Categories and Subject Descriptors

I.3.5 [**Computational Geometry and Object Modeling**]: Boundary representations; Curve, surface, solid, and object representations; G.3 [**PROBABILITY AND STA TISTICS**]: Probabilistic algorithms; F.1.2 [**Modes of Computation**]: Probabilistic computation; F.2.2 [**Nonnumerical Algorithms and Problems (E.2-5, G.2, H.2-3)**]: Computations on discrete structures; Geometrical problems and computations

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General Terms

Algorithms, Theory

1. INTRODUCTION

Given a finite collection $\mathcal{C}$ of line segments in the plane, the *arrangement* $\mathcal{A}(\mathcal{C})$ is the subdivision of the plane into vertices, edges and faces induced by $\mathcal{C}$. A *vertex* is either a segment endpoint or an intersection point of two segments; an *edge* is a maximal relatively open connected portion of a segment not meeting any vertex, and a *face* is a maximal open connected region of the plane not containing any vertex or edge. Arrangements are similarly defined for other types of curves in the plane or for (hyper)surfaces in higher dimensions. They have been intensively studied in combinatorial and computational geometry, and have numerous applications (see, for example, the surveys [1], [15], and the book [21]). For a more recent survey on using arrangements in practice see [13].

In this paper we investigate arrangements of *random* segments, and concentrate on the specific family of random arrangements defined as follows: Let $\mathcal{S}$ denote a unit square in the plane. Fix an integer n and a real positive length ℓ. Let $\mathcal{C}$ be a collection of n oriented line segments, all of length ℓ, where each $s \in \mathcal{C}$ is chosen independently by first choosing the location of an endpoint of s uniformly at random in $\mathcal{S}$, and then choosing an orientation for s uniformly at random on the unit circle Γ. Thus, a segment in $\mathcal{C}$ may protrude through the boundary of $\mathcal{S}$. For simplicity we call these segments *valid segments*, and denote the first and second endpoints of s as the *source* and *target* of s, respectively. Let $\mathbf{C}_\ell^n$ denote the probability space of all such collections $\mathcal{C}$. Note that $\mathbf{C}_\ell^n = (\mathcal{S} \times \Gamma)^n$, with the uniform (Lebesgue) measure.

An arrangement of a finite number of bounded curves has a unique unbounded face, which we call the *outer face*. We focus on the following combinatorial quantity associated with the outer face. Let $f(\mathcal{C})$ denote the number of segments in $\mathcal{C}$ that appear on the boundary of the outer face of $\mathcal{A}(\mathcal{C})$. Notice that a segment may contribute several edges to the boundary of the outer face, but we count it in $f(\mathcal{C})$ only once. Let $f(n,\ell)$ denote $\mathrm{E}[f(\mathcal{C})]$, where the expectation is over $\mathcal{C} \in \mathbf{C}_\ell^n$. We obtain sharp bounds for $f(n,\ell)$ and for several related quantities. (By the standard theory of planar arrangements (see, e.g., [21]), the upper bound on the actual complexity (number of edges) of the outer face is $O(f(\mathcal{C})\alpha(f(\mathcal{C})))$, where $\alpha(\cdot)$ is the inverse Ackermann function.)

Our study is motivated by an experimental inspection of *point-location* strategies in planar subdivisions. Point-location queries are a basic operation on arrangements [5, Chapter 6]: Given a query point q, we wish to efficiently report the *cell* (vertex, edge, or face) of the arrangement containing q. There are various classical point-location data structures that support queries in time that depends logarithmically on the *complexity* of the arrangement, namely the overall number of its vertices, edges, and faces; see for example [4], [11], [17], [18], and [19]. However, in practice one often prefers a mechanism that does not require extra preprocessing (beyond the initial construction of the arrangement, because, in spite of their optimal asymptotic performance, these structures tend to be cumbersome to construct in practice). A typical practical solution is to "walk" on the arrangement from some initial point in some known cell towards the query point. For example we start with a point vertically above the query point, in the outer face. If we let ρ denote the vertical ray emanating from the query point upwards, then a typical "walk" step is to compute a face edge that intersects the ray ρ closest to the query point and to advance toward the query point, by switching to the face on the other side of the edge, and keep iterating the search for an "exit edge" within this face. See, e.g., [7] for details of the special case of a walk point-location in triangulations.

While this technique may sound too wasteful from a theoretical point of view, its advantage is that the search over the edges of the current face is very simple to implement, requires no extra data structure, and runs very efficiently and robustly. Still, the efficiency depends on the number of edges the "walk" algorithm has to inspect, and its success depends on this number being small.

As a first approximation towards bounding the number of inspected edges, we consider only the first step, in which we search over the edges of the outer face, and seek a sharp bound on the expected number of these edges, under the random model that we assume. Notice that typically, an arrangement induced by many random segments within this model, has a single giant face (the outer face) and many other smaller sized faced.

Figure 1 depicts the number of segments that appear on the outer face of $\mathcal{A}(\mathcal{C})$, for a random $C \in \mathbf{C}_\ell^n$, where $\ell = 0.08$ is fixed and n is a parameter between 0 and 10000, averaged over ten runs for each n. For small values of n, $f(n,\ell)$ grows linearly; then there is a sharp drop, and then $f(n,\ell)$ starts growing again but at a much slower rate than before.

It is not difficult to give an informal interpretation of this phenomenon. However in this paper we strive to analyze this and related phenomena more precisely. Moreover, with the increase of interest in the experimental study of algorithms, we anticipate that analyses of the type given here could be helpful in guiding and understanding the practical performance of a variety of algorithms on arrangements and on other geometric structures, constructed on random input sets.

Related work.

Before stating our precise findings, we first review some related results. Bose and Devroye [2] proved that the number of triangles visited in the straight line walk algorithm, in the special case of a Delaunay triangulation of uniformly distributed points in a compact convex set in the plane, is $O(|pq|\sqrt{n})$, where $|pq|$ is the length of the segment we traverse and n is the number of points in the triangulation.

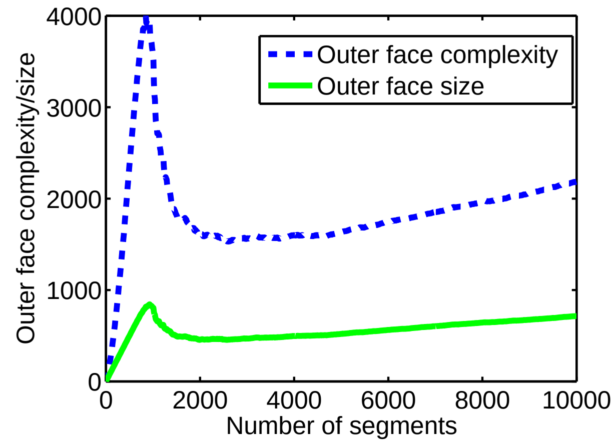

Figure 1: The average number of segments that appear on the outer face of $C \in \mathbf{C}_\ell^n$ (the "size" of the face, solid graph) and the complexity of the face (dashed graph), averaged over ten runs for each n, where $\ell = 0.08$ is fixed and n is a parameter between 0 and 10000.

Devroye et al. [9] presented a simple walk algorithm for point-location queries in Delaunay triangulations of n random points in the plane, with expected time $O(n^{1/3})$.

Devillers et al. [7] compared the run-time of common walking techniques for random point-location queries in general triangulations in $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^d$.

Flato et al. [12] described the `Arrangement` package[1] of CGAL (Computational Geometry Algorithms Library), with several point-location strategies that it implements. Our paper was motivated by the experimental results in that work.

Devroye and Toussaint [10] and later Golin et al. [14] showed a related result for the complexity of the convex hull of the intersection points of random lines in the plane. Both papers prove that an arrangement of n lines chosen at random from the plane (using natural models) has a vertex set whose convex hull has constant (expected) size (the worst-case bound is $\Theta(n)$).

Summary of results.

In this paper we prove the following theorems:

(i) The outer face complexity undergoes a rapid phase transition.

THEOREM 1.1. **[Sparse arrangements]** *For any $\epsilon > 0$ there exists some constant $c_1 = c_1(\epsilon) > 0$ such that for every $\ell > 0$ and $c_1\lceil\frac{1}{\ell^2}\rceil > n \geq 0$,*

$$f(n,\ell) \geq (1-\epsilon)n. \tag{1}$$

THEOREM 1.2. **[Critically dense arrangements]** *There exist constants $c_2 > 0$ and $C_2 > C_1 > 0$, such that, for each $0 < \ell < 1$ and $n = \lceil\frac{c_2}{\ell^2}\rceil$,*

$$C_1\sqrt{n} \leq f(n,\ell) \leq C_2\sqrt{n}. \tag{2}$$

[1]The `Arrangement` package and many more can be downloaded from `http://www.cgal.org`.

(ii) The outer face complexity is asymptotically a fractional power.

THEOREM 1.3. **[Asymptotic complexity]** *For any* $\ell > 0$, $f(n, \ell) = \Theta(n^{1/2+o(1)})$ *as* $n \to \infty$.

Outline of the paper.

We start with an analysis of the complexity of the outer face before the phase transition (Section 2), proving Theorem 1.1. We present a refined analysis of the phase transition in Section 3, proving Theorem 1.2. In Section 4 we prove Theorem 1.3, concerning the behavior beyond the phase transition. We conclude with experimental results in Section 5.

As a corollary, we can conclude that for sufficiently small segment length ℓ, the expected run-time of the "walk-along-a-line" point-location strategy, as a function of the number of segments, exposes analogue phase transition characteristics; We omit the details for lack of space. The details will appear in the full version of the paper.

2. SPARSE ARRANGEMENTS — BEFORE THE PHASE TRANSITION

In this section we prove Theorem 1.1. Recall that $\mathbf{C}_\ell^n$ denotes the probability space of n random line segments of fixed length ℓ in the unit square $\mathcal{S}$, as described in the Introduction, and that $f(n, \ell)$ denotes the expected number of segments in $\mathcal{C}$ that appear on the boundary of the outer face of $\mathcal{A}(\mathcal{C})$, over $\mathcal{C} \in \mathbf{C}_\ell^n$.

THEOREM 1.1. **[Sparse arrangements]** *For any* $\epsilon > 0$ *there exists some constant* $c_1 = c_1(\epsilon) > 0$ *such that for every* $\ell > 0$ *and* $c_1\lceil \frac{1}{\ell^2} \rceil > n \geq 0$,

$$f(n, \ell) \geq (1 - \epsilon)n. \quad (1)$$

PROOF. To prove the theorem we bound the expectation of the complementary quantity, namely the expected number of segments that do *not* show up on the boundary of the outer face, showing that this value is $\leq \epsilon n$.

Let $\epsilon > 0$, $\ell > 0$ be given, and assume that $n \leq \mu\lceil \frac{1}{\ell^2} \rceil$ where $0 < \mu = \mu(\epsilon) < 1$ will be determined later. If $\ell \geq 1$ then obviously $n = 0$ and the proof follows. Assume that $\ell < 1$. Consider a fixed set $\mathcal{C} \in \mathbf{C}_\ell^n$. We call a segment $s \in \mathcal{C}$ *internal* if s does not appear on the closure of the outer face of $\mathcal{A}(\mathcal{C})$. Observe that for each internal segment in $\mathcal{A}(\mathcal{C})$ there exists some circular sequence $< s_{i_1}, s_{i_2}, \ldots, s_{i_k} >$ of $k \geq 4$ distinct segments, such that each adjacent pair of segments $(s_{i_j}, s_{i_{j+1}})$ intersect, and such that s is disjoint from the closure of the outer face of $\mathcal{A}(\{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\})$. We call such a sequence a *closed chain* of length k and s is considered *internal* to this chain.

For a fixed tuple of distinct indices $(i_1, i_2, \ldots, i_k)$, and another index i_{k+1}, let $V_{(i_1,i_2,\ldots,i_k;i_{k+1})}$ be the following indicator random variable

$$V_{(i_1,i_2,\ldots,i_k;i_{k+1})} = \begin{cases} 1 & \text{if } < s_{i_1}, s_{i_2}, \ldots, s_{i_k} > \text{ is a closed chain and } s_{i_{k+1}} \text{ is internal to the chain,} \\ 0 & \text{otherwise.} \end{cases}$$

Put $W_k := \sum V_{(i_1,i_2,\ldots,i_k;i_{k+1})}$, where the sum extends over all tuples $(i_1, i_2, \ldots, i_k)$ and indices i_{k+1}, as above for k fixed. Notice that, since $< s_{i_1}, s_{i_2}, \ldots, s_{i_k} >$ is a closed chain, the source point of $s_{i_{j+1}}$ is within a disc of radius 2ℓ centered at the source point of s_{i_j}, for each $j < k$. Also, since $s_{i_{k+1}}$ is internal to the chain, the source point of $s_{i_{k+1}}$ is within a disc of radius $(k/2)\ell$ centered at the source of s_{i_1}. Thus

$$\mathrm{E}[W_k] \leq \frac{n!}{(n-k)!}(n-k)\left(\pi(2\ell)^2\right)^{k-1}\pi((k/2)\ell)^2.$$

Let E_k denote the expected number of segments that are bounded by some closed chain of length k. The expected number of internal segments is at most $\sum_{k=4}^{n-1} E_k$, and we have $E_k \leq \mathrm{E}[W_k]/2k$ (each closed chain contributes $2k$ different tuples to the sum W_k). Observe that this is a rather weak estimate, because a segment may be counted many times on the right-hand side, once for each enclosing chain, but only once on the left-hand side. Hence

$$\begin{aligned}
\sum_{k=4}^{n-1} E_k &\leq \sum_{k=4}^{n-1} \frac{n!}{(n-k)!2k}(n-k)\left(\pi(2\ell)^2\right)^{k-1}\pi((k/2)\ell)^2 \\
&\leq \sum_{k=4}^{n-1} \mu^k \left(2/\ell^2\right)^k/(2k) \cdot n(4\pi)^{k-1}\ell^{2(k-1)}\pi(k/2)^2\ell^2 \\
&\leq n\sum_{k=4}^{n-1} (8\pi\mu)^k k/32.
\end{aligned}$$

Notice that the last quantity is independent of ℓ.

If, say, $8\pi\mu \leq 2/5$ then the last sum is bounded by twice the first summand. Hence, the right-hand sum is at most $n\frac{2\cdot(8\pi\mu)^4 4}{32}$. Letting $c_1 = \min(2/5, \sqrt[4]{4\epsilon})/8\pi$ completes the proof. $\square$

Note that, as intuitively expected, c_1 decreases with ϵ.

3. THE PHASE TRANSITION

In this section we prove Theorem 1.2.

THEOREM 1.2. **[Critically dense arrangements]** *There exist constants* $c_2 > 0$ *and* $C_2 > C_1 > 0$, *such that, for each* $0 < \ell < 1$ *and* $n = \lceil \frac{c_2}{\ell^2} \rceil$,

$$C_1\sqrt{n} \leq f(n, \ell) \leq C_2\sqrt{n}. \quad (2)$$

We proceed with several definitions and lemmas. Recall that $\mathcal{S}$ denotes the unit square. Let $\mathcal{A} = \mathcal{A}(\mathcal{C})$, $\mathcal{C} \in \mathbf{C}_\ell^n$, be an arrangement of n random segments, as defined above. Given a parameter k, we let $G = G(k)$ denote the partition of $\mathcal{S}$ into a grid of $k \times k$ equal squares.

An *m-boundary sequence* in G is a sequence $(\zeta_1, \zeta_2, \ldots, \zeta_m)$ of distinct G-squares with ζ_1 incident to the boundary of $\mathcal{S}$ and with each pair of adjacent squares (ζ_i, ζ_{i+1}) sharing a common edge.

An *exposed* m-boundary sequence in G, relative to $\mathcal{A}$, is an m-boundary sequence in G with all the squares in the sequence intersecting the outer face of $\mathcal{A}$.

A square that belongs to at least one exposed m-boundary sequences is called an *exposed* square. The squares that are incident to the boundary edges of $\mathcal{S}$ are the *boundary* squares of G and the other squares are *internal*. A G-square ζ is said to be *well bounded* in an arrangement of segments if the arrangement induced by the segments with sources inside ζ, has an outer face disjoint from ζ. See Figure 2.

Roughly speaking, we argue as follows: Given $0 < \ell \leq 1$, let $n = \lceil \mu/\ell^2 \rceil$, where $\mu > 0$ is a parameter to be determined

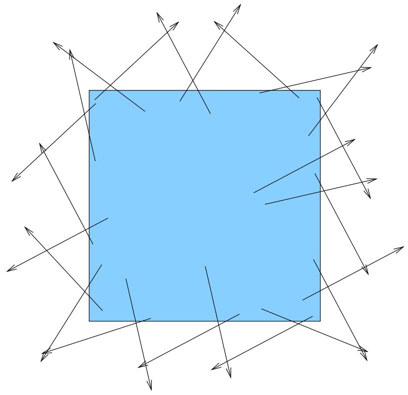

Figure 2: Example of a well bounded square — the solid segments (drawn as arrows emanating from their source points) completely separate the shaded square from the outer face.

later, and take $k = \max(\lfloor 1/\ell \rfloor, 1)$. Then, for a random arrangement $\mathcal{A} = \mathcal{A}(\mathcal{C})$, $\mathcal{C} \in \mathbf{C}_\ell^n$, and for a fixed square ζ in $G = G(k)$, the expected number of segments with sources in ζ is

$$\frac{n}{k^2} \geq \frac{\mu}{\ell^2} \cdot \ell^2 = \mu.$$

Therefore, by controlling the value of μ, we can ensure that, with high probability, a fixed grid square contains many segment sources, and therefore (using Lemma 3.1 below) is well bounded with high probability. That is, the subset of segments with sources within a fixed grid square induces an arrangement that, with high probability, completely separates the square from the outer face of $\mathcal{A}(\mathcal{C})$. This implies that only a small fraction of the squares of G are expected to intersect the outer face. Furthermore, with μ large enough, we can bound the probability that a fixed m-boundary sequence is exposed relative to $\mathcal{A}$ by a^m, for an appropriate constant $a < 1/3$. Since the overall number of m-boundary sequences is $O(k \cdot 3^m)$, it follows that the expected number of exposed m-boundary sequences, summed over all $m \geq 1$, is $O(k)$. This in turn provides a trivial upper bound of $O(k) = O(1/\ell) = O(\sqrt{n})$ on the expected number of exposed squares. Since any segment that meets such a square must have its source in a nearby square, and since the expected number of segments in any square is $O(1)$, one can deduce that the expected number of segments that contribute to the outer face is only $O(\sqrt{n})$.

In more detail, the proof proceeds as follows.

LEMMA 3.1. *For each $0 < p < 1$ there exists $\mu_1 = \mu_1(p)$, such that the following statement holds. Suppose $\ell > 0$, $k = \max(\lfloor 1/\ell \rfloor, 1)$ and let ζ be a $1/k \times 1/k$ square. Consider a random collection of μ_1 segments, each of length ℓ, where each segment is chosen by picking its source uniformly at random in ζ, and by choosing its orientation uniformly at random on the unit circle. Then, the probability that ζ is well bounded in the arrangement formed by these segments is at least p.*

PROOF. Since each of the segments is of length at least $1/(2k)$ and ζ is a $1/k \times 1/k$ square, we can arrange a subset of $\mu_0 = O(1)$ segments in some fixed pattern (e.g., $\mu_0 = 25$ as shown in Figure 2) to separate ζ from the outer face. Since a small perturbation of each segment still yields a configuration that keeps ζ disjoint from the outer face, we conclude that there is some (small) probability p_0 for ζ to be well bounded. Now a standard amplification argument implies that the probability of ζ to be well bounded is at least $1 - (1 - p_0)^m$, for $\mu_1 = m\mu_0$. This can be made $> p$ if we choose m (and μ_1) sufficiently large. $\square$

For each grid square ζ, define the indicator random variable

$$Z_\zeta = \begin{cases} 1, & \text{if the square } \zeta \text{ is not well bounded,} \\ 0, & \text{otherwise.} \end{cases}$$

By taking μ sufficiently large, with $n = \lceil \mu/\ell^2 \rceil$, as above, we can ensure, using Lemma 3.1, that $\mathrm{E}[Z_\zeta]$ is sufficiently small for each square ζ. Recall that if a grid square ζ is exposed then there exists some m-boundary sequence $(\zeta_1, \zeta_2, \ldots, \zeta_m = \zeta)$, for some $m \geq 1$, such that $Z_{\zeta_i} = 1$, for each $1 \leq i \leq m$.

LEMMA 3.2. *For any constant $0 \leq a < \frac{1}{3}$, the following statement holds:*
If for every m-boundary sequence $(\zeta_1, \ldots, \zeta_m)$

$$\Pr\left[\bigwedge_{1 \leq i \leq m} Z_{\zeta_i} = 1\right] \leq a^m, \tag{3}$$

then the expected number of exposed squares is at most $\frac{4ak}{1-3a}$.

PROOF. Denote by E_m the expected number of exposed squares that have a witness boundary sequence of length m. Let $Y_m < 4k \cdot 3^{m-1}$ denote the total number of boundary sequences of length m. By (3), the expected number of exposed squares is at most

$$\sum_{m=1}^{\infty} E_m \leq \sum_{m=1}^{\infty} Y_m a^m \leq \frac{4k}{3} \sum_{m=1}^{\infty} 3^m a^m = \frac{4ak}{1-3a}.$$

$\square$

LEMMA 3.3. *There exists a constant $b > 0$ so that if $0 < \ell \leq 1$ and $n > b/\ell^2$ then for every m-boundary sequence $(\zeta_1, \zeta_2, \ldots, \zeta_m)$, the inequality (3) holds with $a = 1/4$.*

PROOF. By Lemma 3.1 there exists some μ such that for a fixed grid square with μ random segments whose sources are chosen uniformly in it, the probability that the square is not well bounded is at most, say, $\frac{1}{256}$.

Suppose $b > 2\mu$ and $n \geq \lceil b/\ell^2 \rceil$. Each square ζ that satisfies $Z_\zeta = 1$ is classified as being either of type A, if ζ contains fewer than μ segment sources, or of type B, if ζ contains at least μ segment sources while not being well bounded.

Clearly, for any $m \geq 1$, if some fixed m-boundary sequence $(\zeta_1, \ldots, \zeta_m)$ is such that all its squares are exposed, then it contains either (at least) $m' = \lceil m/2 \rceil$ squares of type A, or (at least) m' squares of type B. We proceed to show that the probability of each of these two events is small. To this end, it is convenient to consider the following procedure for generating the random segments in the collection $\mathcal{C}$. Each segment $s \in \mathcal{C}$ is chosen, randomly and independently, in two steps. In the first step, select the $G(k)$-square containing the source of s, where all k^2 choices are equally likely. In the second step, choose the precise location of the source inside the selected grid-square, as well as the direction of the segment. Obviously this is equivalent to the original way of generating our random collection. This equivalent description is, however, more convenient for what follows.

The probability that there are at least m' squares of type A in our fixed m-boundary sequence can be bounded by examining the results of the random choices in the first step for all segments. Indeed, there are $\binom{m}{m'} < 2^m$ possible ways to choose m' squares among those of the sequence, and the probability that each of them contains less than μ sources of segments is at most the probability that in the union of all of them there are fewer than $m'\mu$ such sources. This is at most the probability that the value of a binomial random variable with parameters n and $P = m'/k^2$ is less than half its expectation. By Chernoff's Inequality this is bounded by

$$e^{-nP/8} \le e^{-bm/16},$$

which is less than $1/16^m$ provided, say, $b > 50$. Multiplying this estimate by the number of possible choices for the m' "uncrowded" squares of the sequence, we conclude that the probability of this event is smaller than $1/8^m$.

We next claim that the probability that there are at least m' squares of type B in our fixed m-boundary sequence is at most

$$\binom{m}{m'}\left(\frac{1}{256}\right)^{m/2} < \frac{1}{8^m}.$$

Indeed, there are $\binom{m}{m'}$ ways to select m' squares in the sequence. Fixing m' squares, consider our two-step choice of the segments in the collection $\mathcal{C}$. The probability that all these squares are of type B is at most the conditional probability that this happens, assuming that each of them contains at least μ sources of segments by the end of the first step. But this conditional probability depends only on the choices in the second step, and in the second step this is the intersection of m' mutually independent events, each having probability at most $1/256$, by Lemma 3.1 and the choice of μ. This proves the claim and completes the proof of the lemma, as $\frac{1}{8^m} + \frac{1}{8^m} \le \frac{1}{4^m}$ (with room to spare for all $m > 1$). $\square$

PROOF [**of Theorem 1.2**]. Take $k = \lfloor \frac{1}{\ell} \rfloor$, and construct the $k \times k$ grid partition $G = G(k)$ of $\mathcal{S}$. By Lemmas 3.2 and 3.3, there is a positive constant b, such that if $n \ge b/\ell^2$ then the expected number of exposed grid squares is at most $4k$. It is convenient to prove the theorem with $c_2 = b + 1$. We first prove the upper bound. Note that, since $\ell < 1$, $n = \lceil c_2/\ell^2 \rceil$ is at least $\lceil b/\ell^2 \rceil + 1$. For a fixed random segment s in our collection $\mathcal{C}$, the probability that s intersects the outer face is at most the probability that its source q lies within distance ℓ from a grid square ζ which is exposed in the random arrangement of all segments in $\mathcal{C}$ besides s, and then q lies in a square adjacent to ζ. Therefore, the probability that s intersects the outer face is at most

$$\frac{1}{k^2}\sum_{\zeta \in G(k)} \sum_{\zeta' \in G(k), \zeta' \cap \zeta \ne \emptyset} Pr\{\zeta' \text{ is exposed in } \mathcal{C} \setminus \{s\}\}$$

$$\le \frac{9}{k^2}\sum_{\zeta \in G(k)} Pr\{\zeta \text{ is exposed in } \mathcal{C} \setminus \{s\}\}.$$

The last sum is precisely the expected number of exposed squares in $\mathcal{C} \setminus \{s\}$ which, by Lemmas 3.2 and 3.3, is at most $4k$. It follows that the probability that s intersects the outer face is at most $36/k$. By linearity of expectation, the expected number of segments that intersect the outer face is

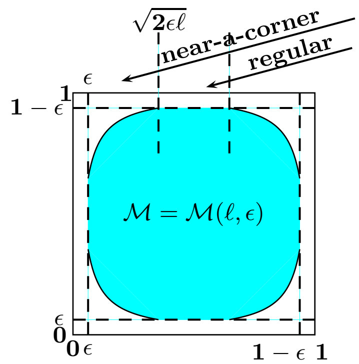

Figure 3: An illustration of $\mathcal{M}(\ell, \epsilon)$ for some $\ell, \epsilon > 0$. For example, the two dashed vertical lines at the top separate the region above the square into regular points (middle section) and near-a-corner points (side sections).

at most

$$\frac{36n}{k} = \left\lceil \frac{c_2}{\ell^2} \right\rceil \frac{36}{k} = O(k) = O(\sqrt{n}).$$

This establishes the upper bound.

The proof of the lower bound is simpler. The probability that the source of a fixed segment lies in a boundary grid square ζ, and no source of any other segment lies in any square intersecting ζ, is at least

$$\frac{4k-4}{k^2}\left(1 - \frac{6}{k^2}\right)^{n-1} = \Omega\left(\frac{1}{k}\right).$$

By linearity of expectation, the expected number of such segments is $\Omega(n/k) = \Omega(k) = \Omega(\sqrt{n})$. The lower bound follows, since any such segment intersects the outer face. $\square$

Note that the constants in the proof can be easily improved, but we make no attempts to optimize them here.

4. DENSE ARRANGEMENTS — BEYOND THE PHASE TRANSITION

In this section we prove Theorem 1.3.

THEOREM 1.3. [**Asymptotic complexity**] *For any $\ell > 0$, $f(n, \ell) = \Theta(n^{1/2+o(1)})$ as $n \to \infty$.*

Outline of the proof.

We estimate the probability, over $\mathcal{C} \in \mathbf{C}^n_\ell$, that a small disc external to $\mathcal{S}$ is well bounded in $\mathcal{A}(\mathcal{C})$, as a function of n, as n tends to infinity. Using this estimation we bound the probability over $\mathcal{C} \in \mathbf{C}^n_\ell$ that a certain narrow shell of intersecting discs which surrounds $\mathcal{S}$ is well bounded within $\mathcal{A}(\mathcal{C})$. Also, by taking the radii of the discs sufficiently small, the expected number of segments that are internal to this virtual strip and contribute to the outer face is $o(n^{1/2})$. On the other hand, we estimate the number of segments that intersect this strip by $O(n^{1/2+o(1)})$. Summing up these two expectations gives the required result. The major complication here is the analysis of the situation near the corners of the square.

We begin with several definitions and lemmas. Define $m(t) := \min(t, 1-t)$. Given $\ell > 0$, $\epsilon > 0$, let $\mathcal{M} = \mathcal{M}(\ell, \epsilon)$

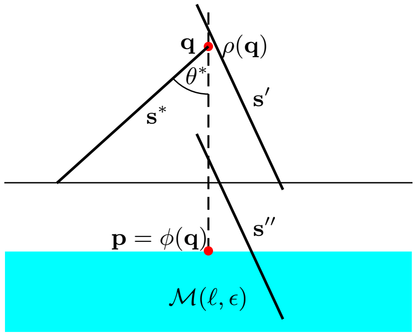

Figure 4: Bounding a regular point from one side. The segments s^* and s' bound q from the left and from the right, respectively; the segment s'' does not bound q at all. Note that the segment s^* is incident to both q and the boundary of $\mathcal{S}$, and forms an extremal angle with respect to pq.

denote the subset of $\mathcal{S}$ consisting of the points (x, y) that satisfy the following inequalities: $m(x) \geq \epsilon$, $m(y) \geq \epsilon$ and $m(x)m(y) \geq \sqrt{2\ell}\,\epsilon^{3/2}$. See Figure 3 for an illustration.

For each point $q \in \mathbb{R}^2$, let $\phi(q)$ denote the projection of q onto $\mathcal{M}$, namely the point in $\mathcal{M}$ that is closest to q. We say that a point q within distance less than or equal to ℓ from $\mathcal{S}$ is *regular* (resp., *near-a-corner* or *non-regular*) if $\phi(q)$ lies on one of the *straight* (resp., *hyperbolic*) arcs of $\partial\mathcal{M}$.

Notice that if $0 < \epsilon < \epsilon_0 = \min(1/(8\ell), 1/2)$, then $\mathcal{M}$ has positive area, $\partial\mathcal{M}$ is smooth, except for 8 points, and both regular and non-regular points exist.

Recall that a valid segment is a segment of length ℓ with source in $\mathcal{S}$. Given a point q outside $\mathcal{S}$, define the ray $\rho(q)$ as the ray normal to $\partial\mathcal{M}$ through q, external to $\mathcal{M}$, with origin at $\phi(q)$. A point q is *bounded from the right* by a valid segment s, if the source of s lies to the right of $\rho(q)$, and s intersects $\rho(q)$ further away than q from $\phi(q)$. More generally, a subset of points is *bounded from the right* by s, if s bounds each of the points in the set from the right. These and other definitions extend by symmetry to bounding from to the left. A subset of points is *well-bounded* within $\mathcal{S}$ if each of the points in the subset is well-bounded within $\mathcal{S}$. See Figure 4 for an illustration.

LEMMA 4.1. *For any $\ell > 0$, $0 < \epsilon < \min(\ell/4, 1/(8\ell))$, $n \geq 15000\ell/\epsilon^3$, and for any point q at distance ℓ from $\mathcal{M} = \mathcal{M}(\ell, \epsilon)$, let $D = D_q(\delta)$ denote the disc of radius $\delta = \epsilon^{3/2}/(60\sqrt{\ell})$ centered at q. Then*

$$\Pr\Big[D \textit{ is well bounded in } \mathcal{A}(\mathcal{C} \cup \partial\mathcal{S})\Big] \geq 1/2,$$

over $\mathcal{C} \in \mathbf{C}_\ell^n$.

PROOF. We split the proof into two parts, depending on whether: **(i)** q is a regular point, or **(ii)** q is a near-a-corner point.

Part (i): Assume that ℓ, ϵ, n, and δ satisfy the above conditions. Let q be a regular point, and let $D = D_q(\delta)$ denote the disc of radius δ centered at q. Recall that $\mathcal{C}$ is drawn uniformly at random from $\mathbf{C}_\ell^n$. Since $\epsilon < \min(\ell/4, 1/(8\ell))$, $\mathcal{M}$ has positive area. Fix a Euclidean coordinate system by setting $p = \phi(q)$ as the origin and the normal ray $\rho(q)$ through q as the positive y-axis; hence, $y(q) = \ell$. Let T denote the line $y = \ell + \epsilon/2$, and r the intersection point of T with the y-axis. See Figure 5 for an illustration. It is useful to extend the definition of bounding a point as follows: A point q is *strongly bounded* from the right-hand side by a valid segment s, if q is bounded from the right-hand side by s, and s intersects T to the left of r. As before, a set of points is *strongly bounded* from the right-hand side by s, if s strongly bounds each of the points in the set from the right-hand side. Notice that $\mathcal{S}$ is well bounded in $\mathcal{A}(\partial\mathcal{S})$. Therefore in order to show that D is bounded within $\mathcal{A}(\mathcal{C} \cup \partial\mathcal{S})$, it suffices, as is easily checked, to find two valid segments $s_1, s_2 \in \mathcal{C}$, which strongly bound D from the right-hand and the left-hand sides, respectively.

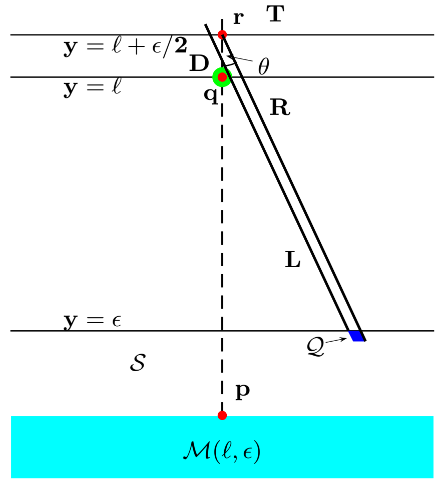

Figure 5: Bounding a regular point. L and R are the left uppermost and right lowest segments in $F^+(\theta)$.

We claim that the probability that a valid segment s strongly bounds D from the right-hand side is greater than $\epsilon^3/(5000\ell)$. Let θ^* denote the maximal angle attained between the positive y-axis and a valid segment of length ℓ that bounds q from the right- (resp., left-)hand side. It can easily be verified that

$$\begin{aligned} \ell\cos\theta^* &= \ell - \epsilon \\ \ell\sin\theta^* &= \sqrt{2\ell\epsilon - \epsilon^2}. \end{aligned}$$

For any θ in the range $[0, \theta^*]$, let $F^+(\theta)$ (*resp.*, $F^-(\theta)$) denote the locus of valid segments, which form an angle θ with the positive y-axis, and strongly bound D from the right- (*resp.*, left-)hand side, namely these segments bound each point in D from the appropriate side and intersect T on the opposite side of r.

For each fixed θ in the range $[\theta^*/3, \theta^*/2]$, the locus of source points (x, y) of segments in $F^+(\theta)$ forms a parallelogram $\mathcal{Q}$, given by the inequalities:

$y \leq \epsilon$ (the source point of s belongs to $\mathcal{S}$);
$y \geq (\ell + \epsilon/2) - \ell\cos\theta$ (the target point of s is above T);
$y \leq (\ell + \epsilon/2) - x\cot\theta$ (s intersects T to the left of r); and
$y \geq (\ell + \delta/\sin\theta) - x\cot\theta$ (q is bounded from the right-hand side by s).

See Figure 5 for an illustration. Notice that if q is a regular point, then $\mathcal{Q}$ is completely contained in $\mathcal{S}$. Since (i) $\theta^*/3 \le \theta \le \theta^*/2 < \pi/4$, (ii) $0 < \epsilon < \min(\ell/4, 1/2)$ and (iii) for any $0 \le \theta \le \pi/4$, $\frac{1}{4}\tan\theta \le \tan(\theta/3)$, we obtain

$$\sqrt{\frac{2\epsilon}{\ell}} \le \sqrt{\frac{2\epsilon}{\ell-\epsilon}} \le \frac{\sqrt{2\ell\epsilon-\epsilon^2}}{\ell-\epsilon} = \tan\theta^* \le \frac{\sqrt{2\ell(\ell/4)}}{\ell-\ell/4} < 1,$$

$$\text{and} \qquad \frac{1}{4}\sqrt{\frac{2\epsilon}{\ell}} \le \frac{1}{4}\tan\theta^* \le \tan(\theta^*/3) \le \tan\theta.$$

Similarly, since $1-\cos(\theta/2) \le (1-\cos\theta)/3$, for any $0 \le \theta \le 2\pi/3$, we have

$$\ell - \tfrac{\epsilon}{3} \le \ell\cos(\theta^*/2) \le \ell\cos\theta,$$

$$\sqrt{\tfrac{\epsilon}{\ell}} \le \tfrac{\sqrt{2\epsilon\ell-\epsilon^2}}{\ell} = \sin\theta^* \le \theta^*,$$

$$\text{and} \qquad \frac{1}{4}\sqrt{\frac{2\epsilon}{\ell}} \le \frac{\sqrt{2\ell\epsilon-\epsilon^2}}{3\ell} = \frac{1}{3}\sin\theta^* \le \sin\theta \le \theta.$$

$$\text{Since} \qquad \frac{\delta}{\sin\theta} \le \frac{\epsilon^{3/2}}{8\sqrt{2\ell}} \Big/ \left(\frac{1}{4}\sqrt{\frac{2\epsilon}{\ell}}\right) = \epsilon/4,$$

we conclude that $\mathcal{Q}$ is non-empty, with area greater than or equal to

$$(\epsilon - (\ell + \frac{\epsilon}{2} - \ell\cos\theta))(\frac{\epsilon}{4}\tan\theta) \quad > \quad \frac{\epsilon^{5/2}}{100\sqrt{\ell}}.$$

Since this holds for each θ in $(\theta^*/3, \theta^*/2)$, the probability p for D being bounded from the left- (*resp.*, right-)hand side is at least $\frac{\theta^*}{6\cdot 2\pi}$ times the above quantity, that is

$$p \ge \frac{\theta^*}{12\pi}\cdot\frac{\epsilon^{5/2}}{100\sqrt{\ell}} \ge \frac{\sqrt{\epsilon/\ell}}{12\pi}\cdot\frac{\epsilon^{5/2}}{100\sqrt{\ell}} \ge \frac{\epsilon^3}{5000\ell}.$$

Since $(1-x)^{(1/x)} < 1/e$ for any $0 < x < 1$, and since $\mathcal{C}$ was drawn uniformly at random from $\mathbf{C}_\ell^n$, with $n \ge 15000\ell/\epsilon^3$, it follows that the probability that D is not well bounded is at most $2(1-p)^n \le 2(1-p)^{3/p} < 2/e^3 < 1/2$, proving the first part of the lemma.

Part (ii): Suppose that ℓ, n, ϵ, $\mathcal{M}$, δ and q satisfy the conditions stated in the lemma, with q being a near-a-corner point. Let $D = D_q(\delta)$ denote the disc of radius δ centered at q. Assume, without loss of generality, that q is near the origin o, and set $p = (x_p, y_p)$ as the projection $\phi(q)$ onto $\mathcal{M}$. We assume further, without loss of generality, that $\epsilon \le x_p \le y_p \le \sqrt{2\epsilon\ell}$. As before, fix a Euclidean coordinate system by setting p as the origin and the normal ray $\rho(q)$ through q as the positive y^p-axis. Let T denote the line $y^p = \ell + \epsilon/4$ and r the intersection point of T with the y^p-axis. See Figure 6 for an illustration. A point q is *strongly bounded* from the right-hand side by a valid segment s, if q is bounded from the right-hand side by s, and s intersects T to the left of r. Let N denote the segment pq and set $0 < \xi < \pi/2$ as the angle between N and the x-axis. It follows by standard properties of hyperbolas that ξ is also the angle between the segment op and the positive y-axis, i.e.

$$(\sin\xi, \cos\xi) = \frac{(x_p, y_p)}{\sqrt{{x_p}^2 + {y_p}^2}}.$$

Let $\mathbf{CH}(p_1 \dots p_k)$ denote the convex hull of the point set $\{p_1, \dots, p_k\}$. Set $\theta^* = \min(\theta^{*+}, \theta^{*-})$ where θ^{*+} (*resp.*, θ^{*-})

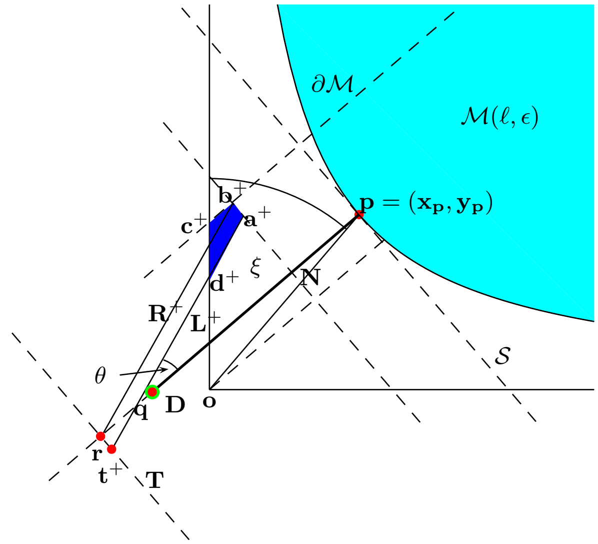

Figure 6: Strongly bounding a near-a-corner point from the right-hand side. The dark regions represent the loci of source points in $F^+(\theta)$.

denotes the maximal angle attained between a valid segment of length ℓ that bounds q from the right- (*resp.*, left-)hand side, and the positive y^p-axis. We claim that by the definition of $\mathcal{S}$ and $\mathcal{M}$, the circle of radius $\sqrt{{x_p}^2 + {y_p}^2}$ centered at p contains both the origin and the intersection points of the x^p-axis with $\partial\mathcal{S}$. If $\xi = 45^o$, then $\theta^* = \theta^{*+} = \theta^{*-}$, and since $x_p \le y_p$ then the origin is to the left of the positive y^p-axis. Thus by moving the origin along this circle toward the positive y^p-axis, θ^{*+} increases while θ^{*-} decreases, proving that $\theta^* = \theta^{*+} \le \theta^{*-}$, that is

$$\ell\cos(\xi + \theta^*) = \ell\cos\xi - x_p.$$

Since $0 < \xi < \xi + \theta^* < \pi/2$ and (by smoothness of the cosine function) $\ell\cos(\xi + \theta^*) = \ell(\cos\xi - \sin(\xi + \lambda\theta^*)\theta^*)$, for some $0 \le \lambda \le 1$, then

$$\ell(\cos\xi - \sin(\xi + \theta^*)\theta^*) \le \ell\cos\xi - x_p \le \ell(\cos\xi - \sin\xi\cdot\theta^*). \quad (4)$$

Thus

$$\theta^* \le x_p/(\ell\sin\xi) = \sqrt{{x_p}^2 + {y_p}^2}/\ell < \sqrt{2(2\ell\epsilon)}/\ell = 2\sqrt{\frac{\epsilon}{\ell}}. \quad (5)$$

$$\text{On the other hand, since} \qquad \frac{1}{2}\sqrt{\frac{\epsilon}{\ell}} = \frac{\epsilon}{\sqrt{2(2\ell\epsilon)}} \le \sin\xi < \frac{\pi}{2},$$

$$\text{we have} \qquad \sin(\xi + \theta^*) < 5\sin\xi,$$

and using Inequality (4) again we obtain that

$$\frac{2}{5}\sqrt{\frac{\epsilon}{\ell}} \le \frac{\sqrt{{x_p}^2 + {y_p}^2}}{5\ell} \le \frac{x_p}{5\ell\sin\xi} \le \frac{x_p}{\ell\sin(\xi + \theta^*)} \le \theta^*. \quad (6)$$

For any θ in the range $[0, \theta^*)$, let $F^+(\theta)$ (*resp.*, $F^-(\theta)$) denote the locus of valid segments, that form angle θ with the positive y^p-axis, and strongly bound D from the right- (*resp.*, left-) hand side, namely bound D from the appropriate side and intersect T on the opposite side of r. For each fixed θ in the range $[\theta^*/3, \theta^*/2]$, the locus of source points of segments in $F^+(\theta)$ (*resp.*, $F^-(\theta)$) forms a convex region $\mathbf{CH}(a^+b^+c^+o^*d^+)$ (*resp.*, $\mathbf{CH}(a^-b^-c^-o^*d^-)$), given by the following constraints:

1. the source of s is inside $\mathcal{S}$;
2. s intersects T;
3. the segment s intersects T on the left- (resp., right-) hand side of r; and
4. s bounds D from the right- (resp., left-)hand side.

Notice that, The segment $L^+ = a^+t^+$ (*resp.*, $L^- = a^-t^-$) is the closest segment to p in $F^+(\theta)$ (*resp.*, $F^-(\theta)$) with both segments $L^\pm$ tangent to D and $|L^\pm| = \ell$; The segment $R^+ = b^+r$ (*resp.*, $R^- = b^-r$) is contained in $F^+(\theta)$ (*resp.*, $F^-(\theta)$) with both segments $R^\pm$ parallel to $L^\pm$ respectively, $|R^+| = |R^-| = \ell$ and both a^+b^+ and a^-b^- parallel to the x^p-axis; Both points c^+ and c^- are contained in $\partial\mathcal{S}$ and both segments b^+c^+ and b^-c^- are parallel to the y^p-axis; The point d^+ (*resp.*, d^-) coincides with the intersection point of the segment L^+ (*resp.*, L^-) and $\partial\mathcal{S}$; and The point o^* is an optional extra corner for the $F^+(\theta)$ (*resp.*, $F^-(\theta)$) convex region, that is added at the origin if c^+ and d^+ (*resp.*, c^- and d^-) lie on different sides of $\mathcal{S}$.

Since $|a^\pm b^\pm| = |t^+r| = |t^-r| = (\epsilon/4)\tan\theta - \delta$, it follows, using Inequality (6), and the definitions of θ and δ, that

$$\left|a^\pm b^\pm\right| = \frac{\epsilon}{4}\tan\theta - \delta \geq \frac{\epsilon}{4}\cdot\frac{2}{15}\sqrt{\frac{\epsilon}{\ell}} - \frac{\epsilon^{3/2}}{60\sqrt{\ell}} \geq \frac{\epsilon}{60}\sqrt{\frac{\epsilon}{\ell}}.$$

It is easy to verify that if $\theta^*/3 < \theta < \theta^*/2$, then
$\ell\sin\xi(\theta^*/2) \leq \ell(\cos(\xi+\theta^*) + \sin\xi(\theta^*/2)) - (\ell\cos\xi - x_p)$
$\leq \ell(\cos(\xi+\theta^*) + \sin(\xi+\lambda\theta^*)\theta^*/2) - (\ell\cos\xi - x_p)$
$\leq \ell\cos(\xi+\theta^*/2) - (\ell\cos\xi - x_p) \leq \ell\cos(\xi+\theta) - (\ell\cos\xi - x_p)$,
for some $1/2 \leq \lambda \leq 1$ and

$$\sqrt{\frac{\epsilon}{2\ell}} = \frac{\epsilon}{\sqrt{2\epsilon\ell}} \leq \frac{x_p}{y_p} \leq \tan\xi.$$

If we let (u, v) denote the endpoint of N when rotated around q by $\theta^*/2$ degrees counterclockwise, then it follows from the definition of θ^* and simple geometric considerations that

$$|b^+c^+| \geq x_p - u - |qr| \geq \epsilon/2 - \epsilon/4 = \epsilon/4.$$

Hence, for each θ within this range, the probability that a valid segment strongly bounds D from the right-hand side, is greater or equal to the area of the right angle triangle $\mathbf{CH}(a^+b^+c^+)$, that is $\frac{\epsilon}{60}\sqrt{\frac{\epsilon}{\ell}}\frac{\epsilon}{8} = \frac{\epsilon^{5/2}}{500\ell}$.

For the analogous result concerning $F^-(\theta)$, note that $\mathbf{CH}(a^-b^-c^-o^*d^-)$ contains the triangle $\mathbf{CH}(a^-b^-w)$ where $w = o^*$ if c^- and d^- lie on different sides of $\mathcal{S}$, or with $w = c^-$ otherwise. In both cases w is further away than c^+ (*resp.*, c^-) from a^+b^+ (*resp.*, a^-b^-), hence we have the same lower bound of $area(\mathbf{CH}(a^+b^+c^+))$.

The probability, in $\mathbf{C}^n_\ell$, that q is bounded from any particular side by a valid segment, is greater than or equal to $\frac{\theta^*}{6\cdot 2\pi}$ times the lower bound on the area of $\mathbf{CH}(a^+b^+c^+)$, namely it is greater than

$$\frac{\theta^*}{12\pi}\cdot\frac{\epsilon^{5/2}}{500\ell} \geq \frac{2}{5}\frac{\sqrt{\epsilon/\ell}}{12\pi}\frac{\epsilon^{5/2}}{500\ell} \geq \frac{\epsilon^3}{7500\ell}.$$

Since $\mathcal{C} \in \mathbf{C}^n_\ell$ are drawn independently and $n \geq 15000\ell/\epsilon^3$, the second part of the lemma follows as in Part (i). $\square$

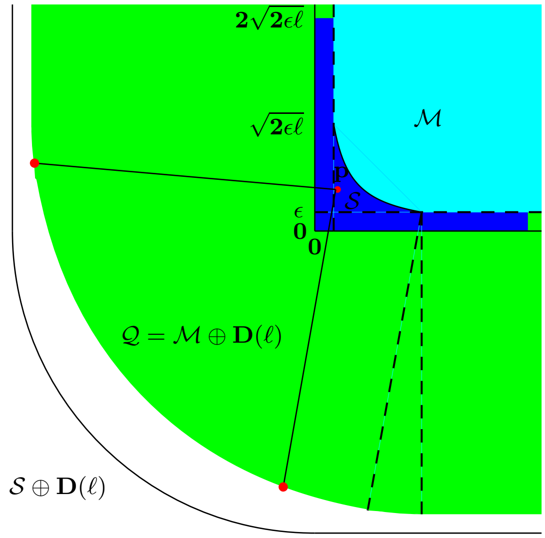

Figure 7: The Minkowski sum $\mathcal{Q}$ of $\mathcal{M}$ with a disc of radius ℓ (excluding $\mathcal{S}$) is colored green. Any valid segment with non-regular target that intersects the boundary of $\mathcal{Q}$ must have its source within the L-shaped (dark) region.

LEMMA 4.2. *For any $\ell > 0$, $0 < \epsilon < \min(\ell/4, 1/(8\ell))$, $n \leq \ell/\epsilon^3$, and for any regular point q at distance ℓ from $\mathcal{M} = \mathcal{M}(\ell, \epsilon)$,*

$$\text{we have} \qquad \Pr\Big[q \text{ is well bounded in } \mathcal{A}(\mathcal{C})\Big] \leq 1/2,$$

over $\mathcal{C} \in \mathbf{C}^n_\ell$.

We proceed by constructing a special threshold curve that encloses $\mathcal{M}$ and satisfies the following properties:

1. the expected number of segments that intersect this curve is $O(n\epsilon^{3/2}(-\log\epsilon))$ and $\Omega(n\epsilon^{3/2})$, where the segments are drawn within our familiar random model;
2. Lemma 4.1 applies uniformly to any sufficiently small disc centered anywhere along this curve; and
3. the length of the curve is at most $4 + 2\pi\ell$.

LEMMA 4.3. *For any $\ell > 0$ and $0 < \epsilon < \min(\ell/4, 1/(8\ell))$ let $\mathcal{Q} = \mathcal{Q}(\ell, \epsilon)$ denote the Minkowski sum of $\mathcal{M} = \mathcal{M}(\ell, \epsilon)$ with the disk of radius ℓ centered at the origin. Then the expected number of segments in $\mathcal{C}$, with $\mathcal{C}$ drawn uniformly at random from $\mathbf{C}^n_\ell$ (n, ℓ fixed), that intersect the boundary of $\mathcal{Q}$ is $O(n\epsilon^{3/2}(-\log\epsilon))$ and $\Omega(n\epsilon^{3/2})$ as $\epsilon \downarrow 0$.*

By Lemmas 4.1 and 4.3, the boundary of $\mathcal{Q} = \mathcal{Q}(\ell, \epsilon)$ satisfies the desired threshold properties. See Figure 7 for an illustration of $\mathcal{Q}$.

We also use the following variant of Lemma 3.1.

LEMMA 4.4. *For each $0 < p < 1$ there exists $\mu = \mu(p)$, such that the following statement holds. Suppose $\ell > 0$, $k = \max(\lfloor 1/\ell\rfloor, 1)$ and let ζ be a $1/k \times 1/k$ square within $\mathcal{S}$. Consider a random collection of μk^2 segments, each of length ℓ, where each segment is chosen by picking its source*

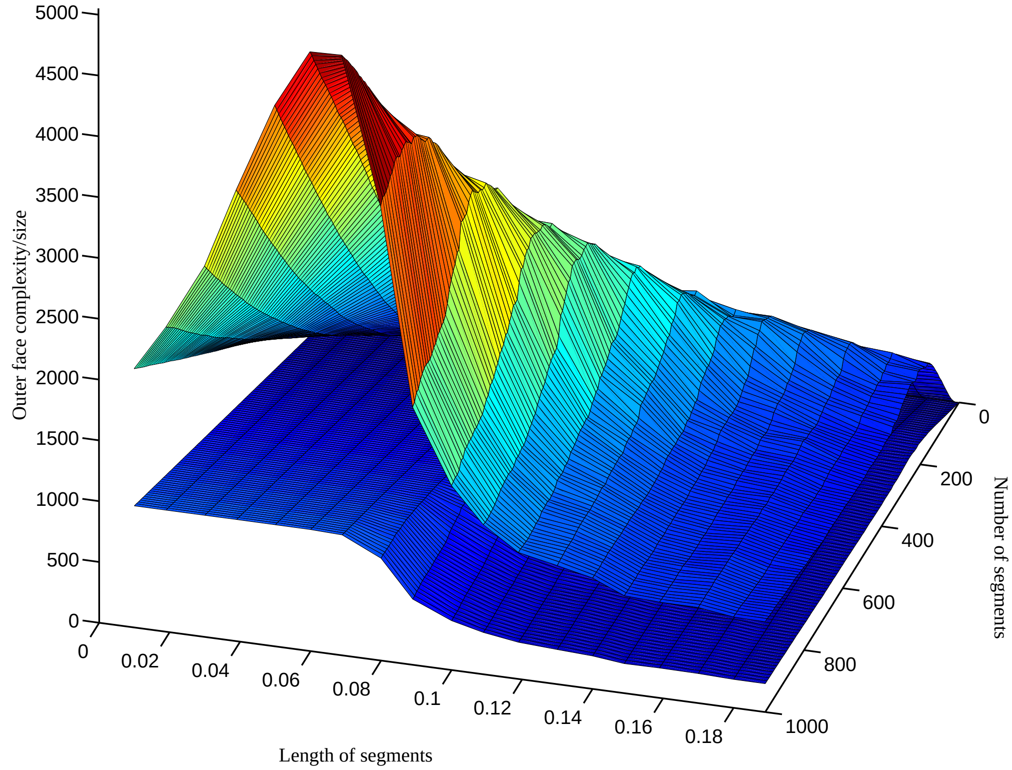

Figure 8: The two depicted surfaces are outer face complexity (higher) and number of segments on the face (lower), drawn as a function of the total number of segments and the segment length.

uniformly at random in $\mathcal{S}$, and by choosing its orientation uniformly at random on the unit circle. Then, the probability that ζ is well bounded in the arrangement formed by these segments is at least p.

Proof **[Theorem 1.3]**. Suppose $n_0 = \lceil c_1 \max \ell^2, \ell^{-4} \rceil$, where c_1 is a parameter to be fixed later, $n > 3n_0 \lceil \log_2 n \rceil$, and suppose that $\mathcal{C}$ is drawn uniformly at random from $\mathbf{C}_\ell^n$. Let $\epsilon = 15000 \ell n^{-1/3}$, $\mathcal{M} = \mathcal{M}(\ell, \epsilon)$ and $\delta = \epsilon^{3/2}/(60\sqrt{\ell})$. Fix some sequence of points $(p_1, p_2, \ldots)$, of smallest possible size, which goes around M and satisfies $dist_{\mathcal{M}}(p_i) = \ell$ and $dist(p_i, p_{i+1}) < \delta$ cyclically. Let $(d_1, d_2, \ldots)$ be the corresponding sequence of discs of radii δ centered at the respective points p_i. Let $k = \max(\lfloor \frac{1}{\ell} \rfloor, 1)$, and split $\mathcal{S}$ into the grid $G(k)$ of k by k squares. Using Lemma 4.4, there exists some $m > 0$ such that each fixed square in the grid is well bounded in $\mathbf{C}_\ell^m$, with probability greater than or equal to $1/2$. By setting $c_1 = 4km$ and drawing the $n > 3n_0 \lceil \log_2 n \rceil \geq 3c_1 \lceil \log_2 n \rceil > 3m \lceil \log_2 n \rceil$ segments in $\mathcal{S}$, in $3\lceil \log_2 n \rceil$ phases, revealing m or more new segments in each phase, we get that the probability a fixed grid square is not well bounded, is at most $(1/2)^{3 \log_2 n} = 1/n^3$. In particular, the probability that $\mathcal{S}$ is not well bounded in $\mathbf{C}_\ell^n$, is at most $4k/n^3 \leq 1/n^2$, by considering only the boundary grid squares. Note that from the lower bound on n $\epsilon \leq \min((15000\ell^3)^{-1/3}, (15000/\ell^3)^{-1/3}) \leq \min(1/(8\ell), \ell/4)$. If $\mathcal{S}$ is well bounded in $\mathcal{A}(\mathcal{C})$, then, using Lemma 4.1, we deduce that each fixed disc d_i is not well bounded in $\mathcal{A}(\mathcal{C})$ with probability less than or equal to $(1/2)^{3 \log_2 n} = 1/n^3$, and that any of the discs $\{d_i\}$ is not well bounded in $\mathcal{A}(\mathcal{C})$ with probability less than or equal to $((4 + 2\pi\ell)/\delta)/n^3 \leq O(k)\sqrt{n}/n^3 < 1/n^2$. Using Lemma 4.3, the probability that a single segment will protrude out of the ring of discs $\cup d_i$ is $O(\epsilon^{3/2}(-\log \epsilon))$. It follows that the expected number of segments on the outer face is at most

$$n(1/n^2 + 1/n^2 + O(\epsilon^{3/2}(-\log \epsilon))) = O(n^{1/2+o(1)}).$$

For a lower bound, set $\epsilon = \sqrt[3]{\ell/n}$. By Lemma 4.3 we have the right number of segments outside $\mathcal{Q}$. Using Lemma 4.2 at least half of these segments contribute to the outer face and the theorem follows. $\square$

We can also show that, if we let $f_{disc}(n, \ell)$ denote the analogous function of $f(n, \ell)$ for a disc, namely the segments source points are drawn uniformly at random within a unit disc, then

Theorem 4.5. *For any $\ell > 0$, $f_{disc}(n, \ell) = \Theta(n^{1/2})$, as $n \to \infty$.*

The proof follows on the same lines as in the previous proof.

5. EXPERIMENTS

We now present actual measurements obtained experimentally for (i) the number of segments on the outer face, and (ii) the face complexity, averaged over ten runs, where we let the number n of segments go up till 1000 and the segment length ℓ range between 0.01 and 0.2. The experimental results were generated using the Cgal `Arrangements` package [22] with the Gmpq[2] number type. The results are

[2] `http://gmplib.org`.

summarized in Figure 8. We defer a full description of the implementation and the measures we took to speed up the experiments to the full version of the paper.

6. REFERENCES

[1] P. K. Agarwal and M. Sharir. Arrangements and their applications. In J.-R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pages 49–119. Elsevier Science Publishers B.V. North-Holland, Amsterdam, 2000.

[2] P. Bose and L. Devroye. Intersections with random geometric objects. *Comput. Geom. Theory Appl.*, 10(3):139–154, 1998.

[3] P. Bose and L. Devroye. On the stabbing number of a random Delaunay triangulation. *Comput. Geom. Theory Appl.*, 36(2):89–105, 2007.

[4] R. Cole. Searching and storing similar lists. *J. Algorithms*, 7(2):202–220, 1986.

[5] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. *Computational Geometry: Algorithms and Applications*. Springer-Verlag, Berlin, 2nd edition, 2000.

[6] O. Devillers. The Delaunay hierarchy. *Internat. J. Found. Comput. Sci.*, 13:163–180, 2002.

[7] O. Devillers, S. Pion, and M. Teillaud. Walking in a triangulation. *Internat. J. Found. Comput. Sci.*, 13:181–199, 2002.

[8] L. Devroye, C. Lemaire, and J.-M. Moreau. Expected time analysis for Delaunay point location. *Comput. Geom. Theory Appl.*, 29(2):61–89, 2004.

[9] L. Devroye, E. P. Mucke, and B. Zhu. A note on point location in Delaunay triangulations of random points. *Algorithmica*, 22:477–482, 1998.

[10] L. Devroye and G. Toussaint. Convex hulls for random lines. *J. Algorithms*, 14(3):381–394, 1993.

[11] H. Edelsbrunner, L. J. Guibas, and J. Stolfi. Optimal point location in a monotone subdivision. *SIAM J. Comput.*, 15(2):317–340, 1986.

[12] E. Flato, D. Halperin, I. Hanniel, O. Nechushtan, and E. Ezra. The design and implementation of planar maps in CGAL. *ACM J. Exp. Algorithmics*, 5:13, 2000.

[13] E. Fogel, D. Halperin, L. Kettner, M. Teillaud, R. Wein, and N. Wolpert. Arrangements. In J.-D. Boissonat and M. Teillaud, editors, *Effective Computational Geometry for Curves and Surfaces*, chapter 1, pages 1–66. Springer-Verlag, Berlin, 2007.

[14] M. Golin, S. Langerman, and W. Steiger. The convex hull for random lines in the plane. In *Discrete and Computational Geometry*, volume 2866 of *Lecture Notes in Comput. Sci.*, pages 172–175. Springer-Verlag, Berlin, 2003. Japan Conference on Discrete and Computational Geometry (JCDCG) 2002, Tokyo, Japan.

[15] D. Halperin. Arrangements. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 24, pages 529–562. Chapman & Hall/CRC, 2nd edition, 2004.

[16] I. Haran and D. Halperin. An experimental study of point location in general planar arrangements. In *Proc. 8th Workshop Alg. Eng. Exper.(Alenex'06)*, 2006.

[17] D. Kirkpatrick. Optimal search in planar subdivisions. *SIAM J. Comput.*, 12(1):28–35, 1983.

[18] K. Mulmuley. A fast planar partition algorithm. *J. Symbolic Comput.*, 10(3-4):253–280, 1990.

[19] N. Sarnak and R. E. Tarjan. Planar point location using persistent search trees. *Comm. ACM*, 29(7):669–679, 1986.

[20] R. Seidel. A simple and fast incremental randomized algorithm for computing trapezoidal decompositions and for triangulating polygons. *Comput. Geom. Theory Appl.*, 1(1):51–64, 1991.

[21] M. Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications*, pages 12–42, 86–114. Cambridge University Press, New York, 1995.

[22] R. Wein, E. Fogel, B. Zukerman, and D. Halperin. Advanced programming techniques applied to CGAL's arrangement package. *Comput. Geom. Theory Appl.*, 38(1-2):37–63, 2007.