Privileged users in zero-error transmission over a noisy channel

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Abstract

The k-th power of a graph G is the graph whose vertex set is $V(G)^k$, where two distinct k-tuples are adjacent iff they are equal or adjacent in G in each coordinate. The Shannon capacity of G, $c(G)$, is $\lim_{k\to\infty} \alpha(G^k)^{\frac{1}{k}}$, where $\alpha(G)$ denotes the independence number of G. When G is the characteristic graph of a channel $\mathcal{C}$, $c(G)$ measures the effective alphabet size of $\mathcal{C}$ in a zero-error protocol. A sum of channels, $\mathcal{C} = \sum_i \mathcal{C}_i$, describes a setting when there are $t \geq 2$ senders, each with his own channel $\mathcal{C}_i$, and each letter in a word can be selected from any of the channels. This corresponds to a disjoint union of the characteristic graphs, $G = \sum_i G_i$. It is well known that $c(G) \geq \sum_i c(G_i)$, and in [1] it is shown that in fact $c(G)$ can be larger than any fixed power of the above sum.

We extend the ideas of [1] and show that for every $\mathcal{F}$, a family of subsets of $[t]$, it is possible to assign a channel $\mathcal{C}_i$ to each sender $i \in [t]$, such that the capacity of a group of senders $X \subset [t]$ is high iff X contains some $F \in \mathcal{F}$. This corresponds to a case where only privileged subsets of senders are allowed to transmit in a high rate. For instance, as an analogue to secret sharing, it is possible to ensure that whenever at least k senders combine their channels, they obtain a high capacity, however every group of $k-1$ senders has a low capacity (and yet is not totally denied of service). In the process, we obtain an explicit Ramsey construction of an edge-coloring of the complete graph on n vertices by t colors, where every induced subgraph on $\exp\left(\Omega(\sqrt{\log n \log\log n})\right)$ vertices contains all t colors.

1 Introduction

A channel $\mathcal{C}$ on an input alphabet V and an output alphabet U maps each $x \in V$ to some $S(x) \subset U$, such that transmitting x results in one of the letters of $S(x)$. The characteristic graph of the channel $\mathcal{C}$, $G = G(\mathcal{C})$, has a vertex set V, and two vertices $x \neq y \in V$ are adjacent iff $S(x) \cap S(y) \neq \emptyset$,

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i.e., the corresponding input letters are confusable in the channel. Clearly, a maximum set of predefined letters which can be transmitted in $\mathcal{C}$ without possibility of error corresponds to a maximum independent set in the graph G, whose size is $\alpha(G)$ (the independence number of G).

The strong product of two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph, $G_1 \cdot G_2$, on the vertex set $V_1 \times V_2$, where two vertices $(u_1, u_2) \neq (v_1, v_2)$ are adjacent iff for all $i = 1, 2$, either $u_i = v_i$ or $u_i v_i \in E_i$. In other words, the pairs of vertices in both coordinates are either equal or adjacent. This product is associative and commutative, hence we can define G^k to be the k-th power of G, where two vertices $(u_1, \ldots, u_k) \neq (v_1, \ldots, v_k)$ are adjacent iff for all $i = 1, \ldots, k$, either $u_i = v_i$ or $u_i v_i \in E(G)$.

Note that if I, J are independent sets of two graphs, G, H, then $I \times J$ is an independent set of $G \cdot H$. Therefore, $\alpha(G^{n+m}) \geq \alpha(G^n)\alpha(G^m)$ for every $m, n \geq 1$, and by Fekete's lemma (cf., e.g., [4], p. 85), the limit $\lim_{n\to\infty} \alpha(G^n)^{\frac{1}{n}}$ exists and equals $\sup_n \alpha(G^n)^{\frac{1}{n}}$. This parameter, introduced by Shannon in [5], is the Shannon capacity of G, denoted by $c(G)$.

When sending k-letter words in the channel $\mathcal{C}$, two words are confusable iff the pairs of letters in each of their k-coordinates are confusable. Thus, the maximal number of k-letter words which can be sent in $\mathcal{C}$ without possibility of error is precisely $\alpha(G^k)$, where $G = G(\mathcal{C})$. It follows that for sufficiently large values of k, the maximal number of k-letter words which can be sent without possibility of error is roughly $c(G)^k$. Hence, $c(G)$ represents the effective alphabet size of the channel in zero-error transmission.

The sum of two channels, $\mathcal{C}_1 + \mathcal{C}_2$, describes the setting where each letter can be sent from either of the two channels, and letters from $\mathcal{C}_1$ cannot be confused with letters from $\mathcal{C}_2$. The characteristic graph in this case is the disjoint union $G_1 + G_2$, where G_i is the characteristic graph of $\mathcal{C}_i$. Shannon showed in [5] that $c(G_1 + G_2) \geq c(G_1) + c(G_2)$ for every two graphs G_1 and G_2, and conjectured that in fact $c(G_1 + G_2) = c(G_1) + c(G_2)$ for all G_1 and G_2. This was disproved in [1], where the first author gives an explicit construction of two graphs G_1, G_2 with a capacity $c(G_i) \leq k$, satisfying $c(G_1 + G_2) \geq k^{\Omega(\frac{\log k}{\log\log k})}$.

We extend the ideas of [1] and show that it is possible to construct t graphs, G_i ($i \in [t] = \{1, 2, \ldots, t\}$), such that for every subset $X \subseteq [t]$, the Shannon capacity of $\sum_{i\in X} G_i$ is high iff X contains some subset of a predefined family $\mathcal{F}$ of subsets of $[t]$. This corresponds to assigning t channels to t senders, such that designated groups of senders $F \in \mathcal{F}$ can obtain a high capacity by combining their channels ($\sum_{i\in F} \mathcal{C}_i$), and yet every group of senders $X \subseteq [t]$ not containing any $F \in \mathcal{F}$ has a low capacity. In particular, a choice of $\mathcal{F} = \{F \subset [t] \ : \ |F| = k\}$ implies that every set X of senders has a high Shannon capacity of $\sum_{i\in X} \mathcal{C}_i$ if $|X| \geq k$, and a low capacity otherwise. The following theorem, proved in Section 2, formalizes the claims above:

Theorem 1.1. *Let $T = \{1, \ldots, t\}$ for some fixed $t \geq 2$, and let $\mathcal{F}$ be a family of subsets of T. For every (large) n it is possible to construct graphs G_i, $i \in T$, each on n vertices, such that the following two statements hold for all $X \subseteq T$:*

1. *If X contains some $F \in \mathcal{F}$, then $c(\sum_{i \in X} G_i) \geq n^{1/|F|} \geq n^{1/t}$.*

2. *If X does not contain any $F \in \mathcal{F}$, then*

$$c(\sum_{i \in X} G_i) \leq \mathrm{e}^{(1+o(1))\sqrt{2 \log n \log\log n}} ,$$

where the $o(1)$-term tends to 0 as $n \to \infty$.

As a by-product, we obtain the following Ramsey construction, where instead of forbidding monochromatic subgraphs, we require "rainbow" subgraphs (containing all the colors used for the edge-coloring). This is stated by the next proposition, which is proved in Section 3:

Proposition 1.2. *For every (large) n and $t \leq \sqrt{\frac{2 \log n}{(\log\log n)^3}}$ there is an explicit t-edge-coloring of the complete graph on n vertices, such that every induced subgraph on*

$$\mathrm{e}^{(1+o(1))\sqrt{8 \log n \log\log n}}$$

vertices contains all t colors.

This extends the construction of Frankl and Wilson [2] that deals with the case $t = 2$ (using a slightly different construction).

2 Graphs with high capacities for unions of predefined subsets

The upper bound on the capacities of subsets not containing any $F \in \mathcal{F}$ relies on the algebraic bound for the Shannon capacity using representations by polynomials, proved in [1]. See also Haemers [3] for a related approach.

Definition. *Let $\mathbb{K}$ be a field, and let $\mathcal{H}$ be a linear subspace of polynomials in r variables over $\mathbb{K}$. A* ***representation*** *of a graph $G = (V, E)$ over $\mathcal{H}$ is an assignment of a polynomial $f_v \in \mathcal{H}$ and a value $c_v \in \mathbb{K}^r$ to every $v \in V$, such that the following holds: for every $v \in V$, $f_v(c_v) \neq 0$, and for every $u \neq v \in V$ such that $uv \notin E$, $f_u(c_v) = 0$.*

Theorem 2.1 ([1])**.** *Let $G = (V, E)$ be a graph and let $\mathcal{H}$ be a space of polynomials in r variables over a field $\mathbb{K}$. If G has a representation over $\mathcal{H}$ then $c(G) \leq \dim(\mathcal{H})$.*

We need the following simple lemma:

Lemma 2.2. *Let $T = [t]$ for $t \geq 1$, and let $\mathcal{F}$ be a family of subsets of T. There exist sets $A_1, A_2, \ldots, A_t$ such that for every $X \subseteq T$:*

$$X \text{ does not contain any } F \in \mathcal{F} \iff \bigcap_{i \in X} A_i \neq \emptyset .$$

Furthermore, $|\bigcup_{i=1}^{t} A_i| \leq \binom{t}{\lfloor t/2 \rfloor}$.

Proof of lemma. Let $\mathcal{Y}$ denote the family of all maximal sets Y such that Y does not contain any $F \in \mathcal{F}$. Assign a unique element p_Y to every $Y \in \mathcal{Y}$, and define:

$$A_i = \{p_Y \ : \ i \in Y \ , \ Y \in \mathcal{Y}\} \ . \tag{1}$$

Let $X \subseteq T$, and note that (1) implies that $\bigcap_{i \in X} A_i = \{p_Y \ : \ X \subseteq Y\}$. Thus, if X does not contain any $F \in \mathcal{F}$, then $X \subseteq Y$ for some $Y \in \mathcal{Y}$, and hence $p_Y \in \bigcap_{i \in X} A_i$. Otherwise, X contains some $F \in \mathcal{F}$ and hence is not a subset of any $Y \in \mathcal{Y}$, implying that $\bigcap_{i \in X} A_i = \emptyset$.

Finally, observe that $\mathcal{Y}$ is an anti-chain and that $|\bigcup_{i=1}^t A_i| \leq |\mathcal{Y}|$, hence the bound on $|\bigcup_{i=1}^t A_i|$ follows from Sperner's Theorem [6]. ■

Proof of Theorem 1.1. Let p be a large prime, and let $\{p_Y \ : \ Y \in \mathcal{Y}\}$ be the first $|\mathcal{Y}|$ primes succeeding p. Define $s = p^2$ and $r = p^3$, and note that, as t and hence $|\mathcal{Y}|$ are fixed, by well-known results about the distribution of prime numbers, $p_Y = (1+o(1))p < s$ for all Y, where the $o(1)$-term tends to 0 as $p \to \infty$.

The graph $G_i = (V_i, E_i)$ is defined as follows: its vertex set V_i consists of all $\binom{r}{s}$ possible s-element subsets of $[r]$, and for every $A \neq B \in V_i$:

$$(A, B) \in E_i \iff |A \cap B| \equiv s \pmod{p_Y} \text{ for some } p_Y \in A_i \ . \tag{2}$$

Let $X \subseteq T$. If X does not contain any $F \in \mathcal{F}$, then, by Lemma 2.2, $\bigcap_{i \in X} A_i \neq \emptyset$, hence there exists some q such that $q \in A_i$ for every $i \in X$. Therefore, for every $i \in X$, if A, B are disconnected in G_i, then $|A \cap B| \not\equiv s \pmod{q}$. It follows that the graph $\sum_{i \in X} G_i$ has a representation over a subspace of the multi-linear polynomials in $|X|r$ variables over $\mathbb{Z}_q$ with a degree smaller than q. To see this, take the variables $x_j^{(i)}$, $i = 1, \ldots, |X|$, $j = 1, \ldots, r$, and assign the following polynomial to each vertex $A \in V_i$:

$$f_A(\overline{x}) = \prod_{u \not\equiv s} (u - \sum_{j \in A} x_j^{(i)}) \ .$$

The assignment c_A is defined as follows: $x_j^{(i')} = 1$ if $i' = i$ and $j \in A$, otherwise $x_j^{(i')} = 0$. As every assignment $c_{A'}$ gives values in $\{0, 1\}$ to all $x_j^{(i)}$, it is possible to reduce every f_A modulo the polynomials $(x_j^{(i)})^2 - x_j^{(i)}$ for all i and j, and obtain multi-linear polynomials, equivalent on all the assignments $c_{A'}$.

The following holds for all $A \in V_i$:

$$f_A(c_A) = \prod_{u \not\equiv s} (u - s) \not\equiv 0 \pmod{q} \ ,$$

and for every $B \neq A$:

$$B \in V_i \ , \ (A, B) \notin E_i \implies f_A(c_B) = \prod_{u \not\equiv s} (u - |A \cap B|) \equiv 0 \pmod{q} \ ,$$

$$B \notin V_i \implies f_A(c_B) = \prod_{u \not\equiv s} u \equiv 0 \pmod{q} \ ,$$

where the last equality is by the fact that $s \not\equiv 0 \pmod q$, as $s = p^2$ and $p < q$. As the polynomials f_A lie in the direct sum of $|X|$ copies of the space of multi-linear polynomials in r variables of degree less than q, it follows from Theorem 2.1 that the Shannon capacity of $\sum_{i \in X} G_i$ is at most:

$$|X| \sum_{i=0}^{q-1} \binom{r}{i} \leq t \sum_{i=0}^{q-1} \binom{r}{i} < t \binom{r}{q} .$$

Recalling that $q = (1 + o(1))p$ and writing $t\binom{r}{q}$ in terms of $n = \binom{r}{s}$ gives the required upper bound on $c(\sum_{i \in X} G_i)$.

Assume now that X contains some $F \in \mathcal{F}$, $F = \{i_1, \ldots, i_{|F|}\}$. We claim that the following set is an independent set in $\left(\sum_{i \in X} G_i\right)^{|F|}$:

$$\{(A^{(i_1)}, A^{(i_2)}, \ldots, A^{(i_{|F|})}) \ : \ A \subseteq [r] \ , \ |A| = s\} \ ,$$

where $A^{(i_j)}$ is the vertex corresponding to A in V_{i_j}. Indeed, if $(A, A, \ldots, A)$ and $(B, B, \ldots, B)$ are adjacent, then for every $i \in F$, $|A \cap B| \equiv s \pmod{p_Y}$ for some $p_Y \in A_i$. However, $\bigcap_{i \in F} A_i = \emptyset$, hence there exist $p_Y \neq p'_Y$ such that $|A \cap B|$ is equivalent both to $s \pmod{p_Y}$ and to $s \pmod{p'_Y}$. By the Chinese Remainder Lemma, it follows that $|A \cap B| = s$ (as $|A \cap B| < p_Y p'_Y$), thus $A = B$. Therefore, the Shannon capacity of $\sum_{i \in X} G_i$ is at least $\binom{r}{s}^{1/|F|} = n^{1/|F|}$. ■

3 Explicit construction for rainbow Ramsey graphs

Proof of Proposition 1.2. Let p be a large prime, and let $p_1 < \ldots < p_t$ denote the first t primes succeeding p. We define r, s as in the proof of Theorem 1.1: $s = p^2$, $r = p^3$, and consider the complete graph on n vertices, K_n, where $n = \binom{r}{s}$, and each vertex corresponds to an s-element subset of $[r]$. The fact that $t \leq \sqrt{\frac{2 \log n}{(\log \log n)^3}}$ implies that $t \leq (\frac{1}{2} + o(1)) \frac{p}{\log p}$, and hence, by the distribution of prime numbers, $p_t < 2p$ (with room to spare) for a sufficiently large value of p.

We define an edge-coloring γ of K_n by t colors in the following manner: for every $A, B \in V$, $\gamma(A, B) = i$ if $|A \cap B| \equiv s \pmod{p_i}$ for some $i \in [t]$, and is arbitrary otherwise. Note that for every $i \neq j \in \{1, \ldots, t\}$, $s < p_i p_j$. Hence, if $|A \cap B| \equiv s \pmod{p_i}$ and $|A \cap B| \equiv s \pmod{p_j}$ for such i and j, then by the Chinese Remainder Lemma, $|A \cap B| = s$, and in particular, $A = B$. Therefore, the coloring γ is well-defined.

It remains to show that every large induced subgraph of K_n has all t colors according to γ. Indeed, this follows from the same consideration used in the proof of Theorem 1.1. To see this, let G_i denote the spanning subgraph of K_n whose edge set consists of all (A, B) such that $\gamma(A, B) = i$. Each pair $A \neq B$, which is disconnected in G_i, satisfies $|A \cap B| \not\equiv s \pmod{p_i}$. Therefore, G_i has a representation over the multi-linear polynomials in r variables over $\mathbb{Z}_{p_i}$ with a degree smaller than p_i (define $f_A(x_1, \ldots, x_r)$ as is in the proof of Theorem 1.1, and take c_A to be the characteristic vector of A). Thus, $c(G_i) < \binom{r}{p_i}$, and in particular, $\alpha(G_i) < \binom{r}{p_i}$. This ensures that every induced subgraph on at least $\binom{r}{p_i} \leq \binom{r}{2p}$ vertices contains an i-colored edge, and the result follows. ■

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