A simple algorithm for edge-coloring bipartite multigraphs

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It is well known that the chromatic index of any bipartite multigraph G with n vertices and m edges is equal to its maximum degree k. The best algorithm currently known for finding a proper k-edge-coloring of such a multigraph runs in time $O(m \log k)$, see [2], or the forthcoming book [6], and applies rather elaborate data structures together with the basic approach of [5]. Another algorithm, of Cole and Hopcroft [1], finds such a coloring in time $O(m \log m)$. See [2] for the rather extensive history and references, and [4] for a more recent, related algorithm. Here we describe a new, simpler algorithm, that runs in time $O(m \log m)$ as well. Our description here is self-contained.

Start with the known fact that one may assume that the graph G is regular. Indeed, if not, identify repeatedly pairs of vertices that lie in the same side whose sum of degrees is at most k (as long as there is such a pair). When this process terminates add, if needed, vertices to the smaller side until both sides have the same number of vertices, and add edges to make sure the obtained multigraph is k-regular. It is easy to see that the number of edges in the new graph is at most $2m$ and a proper k-coloring of it supplies a proper coloring of the original graph. We thus assume from now on that G is k-regular, and that it has $n/2$ vertices in each color class, and $m = nk/2$ edges.

Fact: If $F = (V, E)$ is a bipartite graph, and H is a $2r$-regular multigraph obtained from F by replacing each edge $e \in E$ by $m(e)$ parallel edges, then one can split H into two r-regular spanning subgraphs H_R and H_B in time $O(|E|)$. (Note that the multigraph is given by its multiplicity function, and that the running time is linear in the number of edges of F, and not in that of H.)

Proof: For each edge $e \in E$ with $m(e) \geq 2$, take $\lfloor m(e)/2 \rfloor$ copies of e to H_R and $\lfloor m(e)/2 \rfloor$ copies to H_B, and omit these $2\lfloor m(e)/2 \rfloor$ copies of e from H. Next, find an Euler cycle in each connected component of the remaining subgraph of H and assign the edges along it alternately to H_R and H_B. $\square$

Corollary: One can find a perfect matching in G in time $O(m \log m)$.

Proof: Let 2^t denote the smallest power of 2 satisfying $2^t \geq m$ $(= kn/2)$. Define $\alpha = \lfloor 2^t/k \rfloor$. Let M be an arbitrary perfect matching between the two sides of G (which does not necessarily consist of edges of G), and define $\beta = 2^t - k\alpha$ $(< k)$. Let H be the graph obtained from G by replacing

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each edge of G by α parallel edges and by adding β copies of each edge of M, and let us call the copies of the edges of M the *bad* edges. Note that H is 2^t-regular. Now apply the fact above to $H = H^1$ to split it into two edge disjoint 2^{t-1} regular spanning subgraphs, and let H^2 be the one containing at most half of the bad edges of $H = H^1$. Thus, H^2 contains at most $n\beta/4 < nk/4$ bad edges. Applying the fact again to H^2 we get a 2^{t-2}-regular spanning subgraph H^3 of H with fewer than $nk/8$ bad edges. After t such steps we find a perfect matching of H containing no bad edges, which is a perfect matching in G. The total running time is $O(mt) = O(m \log m)$. □

Theorem: One can find a proper k-edge-coloring of G in time $O(m \log m)$.

Proof: We apply the method of Gabow [3]. Let $T(n, k)$ be the running time of the algorithm for a k regular bipartite graph on n vertices. If k is even, split G, in time $O(nk)$, to two edge-disjoint $k/2$-regular spanning bipartite subgraphs H_1 and H_2. Find, in time $T(n, k/2)$, a $k/2$ proper edge-coloring of H_1, add some of the color classes obtained by this coloring to H_2, to create a 2^r-regular graph H_3, with $k/2 \leq 2^r < k$. Now find a 2^r-coloring of H_3 in time $O(nk \log k)$ by repeatedly applying the fact mentioned above, and obtain, together with the remaining color classes of H_1, a proper coloring of H. Therefore, for even k, $T(n, k) \leq O(nk \log k) + T(n, k/2)$.

For odd k find, in time $O(nk \log(nk))$, a perfect matching in G, using the corollary above, omit it, and continue as before. This gives that for odd k, $T(n, k) \leq O(nk \log(nk)) + T(n, (k - 1)/2))$.

As $T(n, 1) = O(1)$ we conclude that the total running time is $O(m \log m)$, as needed. □

Remarks:

(i) A close look at the proof of the corollary shows that it actually implies that if G is a regular bipartite multigraph with m edges and only m' distinct ones, then one can find a perfect matching in G in time $O(m' \log m)$. In [1], (see also [6]), it is shown how to generate, in time $O(m)$, from any given k-regular bipartite multigraph G with n vertices and $m = nk/2$ edges, a k-regular multigraph G' with only $O(n \log k)$ distinct edges, each of which is an edge of G (but the multiplicities in G' may be bigger). Therefore, by the corollary, we can find a perfect matching in G in time $O(nk + n \log k \log(nk))$. This is linear in the number of edges m provided $m \geq \Omega(n \log n \log \log n)$. It also provides an $O(m \log k)$ algorithm for finding a proper edge-coloring if $m \geq \Omega(n \log n \log \log n)$.

(ii) The Proof of the corollary provides a proof of the marriage theorem for regular bipartite graphs (that is, the fact that a regular bipartite multigraph contains a perfect matching), using Euler's Theorem.

Acknowledgment: I would like to thank Lex Schrijver for helpful discussions and comments.

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