Weak ϵ-nets and interval chains

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Abstract

We construct weak ϵ-nets of almost linear size for certain types of point sets. Specifically, for planar point sets in convex position we construct weak $\frac{1}{r}$-nets of size $O(r\alpha(r))$, where $\alpha(r)$ denotes the inverse Ackermann function. For point sets along the moment curve in $\mathbb{R}^d$ we construct weak $\frac{1}{r}$-nets of size $r \cdot 2^{\text{poly}(\alpha(r))}$, where the degree of the polynomial in the exponent depends (quadratically) on d.

Our constructions result from a reduction to a new problem, which we call *stabbing interval chains with j-tuples*. Given the range of integers $N = [1, n]$, an interval chain of length k is a sequence of k consecutive, disjoint, nonempty intervals contained in N. A j-tuple $\overline{p} = (p_1, \ldots, p_j)$ is said to *stab* an interval chain $C = I_1 \cdots I_k$ if each p_i falls on a different interval of C. The problem is to construct a small-size family $\mathcal{Z}$ of j-tuples that stabs all k-interval chains in N.

Let $z_k^{(j)}(n)$ denote the minimum size of such a family $\mathcal{Z}$. We derive almost-tight upper and lower bounds for $z_k^{(j)}(n)$ for every fixed j; our bounds involve functions $\alpha_m(n)$ of the inverse Ackermann hierarchy. Specifically, we show that for $j = 3$ we have $z_k^{(3)}(n) = \Theta\big(n\alpha_{\lfloor k/2 \rfloor}(n)\big)$ for all $k \geq 6$. For each $j \geq 4$ we derive a pair of functions $P'_j(m)$, $Q'_j(m)$, almost equal asymptotically, such that $z^{(j)}_{P'_j(m)}(n) = O(n\alpha_m(n))$ and $z^{(j)}_{Q'_j(m)}(n) = \Omega(n\alpha_m(n))$.

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1 Introduction

Let S be an n-point set in $\mathbb{R}^d$, and let ϵ be a real number, $0 < \epsilon < 1$. A *weak ϵ-net for S with respect to convex sets* is a set of points $N \subset \mathbb{R}^d$, such that every convex set in $\mathbb{R}^d$ that contains at least ϵn points of S contains a point of N.[3]

In this paper we only consider weak ϵ-nets with respect to convex sets, so we simply call them "weak ϵ-nets". Also, for convenience, we let $r = 1/\epsilon$, and we speak of weak $\frac{1}{r}$-nets, $r > 1$, so our bounds increase with r.

Alon et al. [2] showed that for every finite $S \subset \mathbb{R}^d$ and every $r > 1$ there exists a weak $\frac{1}{r}$-net of size at most $f_d(r)$, for some family of functions f_d, each depending only on r.

The best known upper bound for the planar case is $f_2(r) = O(r^2)$, by Alon et al. [2] (see also Chazelle et al. [7]). For general $d \geq 3$ we have $f_d(r) = O\big(r^d (\log r)^{c(d)}\big)$, for some constants $c(d)$. This was first shown by Chazelle et al. [7], and later on by Matoušek and Wagner [12] via an alternative, simpler technique (which significantly reduced the exponents $c(d)$, to $c(d) = O(d^3 \log d)$).

On the other hand, there are no known lower bounds for fixed d, besides the trivial $f_d(r) = \Omega(r)$. (Matoušek [10] showed, though, that $f_d(r)$ increases exponentially in d for fixed r; specifically, $f_d(50) = \Omega\Big(e^{\sqrt{d/2}}\Big)$.)

If the points of S lie in certain special configurations, better bounds exist on the size of the weak ϵ-net. For example, Chazelle et al. [7] showed that if $S \subset \mathbb{R}^2$ is in convex position, then S has a weak $\frac{1}{r}$-net of size $O\big(r(\log r)^{\log_2 3}\big) = O\big(r(\log r)^{1.59}\big)$. Furthermore, if S is the vertex set of a regular n-gon, then S admits a weak $\frac{1}{r}$-net of size $\Theta(r)$.

The techniques of Matoušek and Wagner [12] also yield improved bounds for some special cases. Thus, they showed that if the points of $S \subset \mathbb{R}^d$ lie along the *moment curve*

$$\mu_d = \{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}, \tag{1}$$

then S has a weak $\frac{1}{r}$-net of size $O\Big(r(\log r)^{c'(d)}\Big)$, for some constants $c'(d) \approx 2d^2 \ln d$. They also obtained improved bounds for point sets on algebraic varieties of bounded degree, among other cases.

Bradford and Capoyleas [5] showed that if S is, in some sense, uniformly distributed on the $(d-1)$-dimensional sphere, then S has a weak $\frac{1}{r}$-net of size $O(r \log^2 r)$ (with the constant of proportionality depending on d).

(Aronov et al. [1] have tackled the weak ϵ-net problem from another angle, for the planar case: They seek to determine, given an integer $k \geq 1$, the maximum value r_k for which every set $S \subset \mathbb{R}^2$ has a weak $\frac{1}{r_k}$-net of size k. They derive upper and lower

[3]The set N is called a *weak* ϵ-net because we do not necessarily have $N \subseteq S$; otherwise, N would be a regular (or "strong") ϵ-net. The need to consider *weak* ϵ-nets here stems from the fact that the system of all convex sets in $\mathbb{R}^d$ has *infinite VC-dimension.* In contrast, consider a set system with *finite* VC-dimension, such as the system of all ellipsoids or all axis-parallel boxes in $\mathbb{R}^d$. Then, every finite set $S \subset \mathbb{R}^d$ has a *strong* ϵ-net of size $O\big(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\big)$ with respect to such a set system. See Matoušek [11, Ch. 10] for details.

bounds for r_k, for small values of k. Babazadeh and Zarrabi-Zadeh [4] extended this work to the case $d = 3$.

Mustafa and Ray [13] have found a connection between weak ϵ-nets with respect to convex sets, and "strong" ϵ-nets with respect to other set systems with finite VC-dimension.)

Algorithmic aspects. The constructions of Matoušek and Wagner [12] yield an algorithm for building, for a given n-point set $S \subset \mathrm{R}^d$, $d \geq 2$, a weak $\frac{1}{r}$-net of size $O(r^d \mathrm{polylog}(r))$ in time $O(n \log r)$. For the case $d = 2$, a weak $\frac{1}{r}$-net of size $O(r^2)$ can be constructed in time $O(nr^2)$, as was shown earlier by Chazelle et al. [6].

Chazelle et al. [6] also show how to determine, in time $O(n^3)$, the largest r for which a given set N is a weak $\frac{1}{r}$-net of a given planar n-point set S. There is no known polynomial-time algorithm for this problem for dimensions 3 and larger.

Our results. In this paper we derive improved upper bounds for two of the above-mentioned cases: namely, for planar point sets in convex position, and for point sets along the moment curve μ_d (1). Our bounds involve the inverse Ackermann function $\alpha(r)$, a function that grows extremely slowly. Our bounds are as follows:

Theorem 1.1 *Let S be an n-point set in convex position in the plane. Then, S has a weak $\frac{1}{r}$-net of size $O(r\alpha(r))$.*

Theorem 1.2 *Let S be a set of n points along the d-dimensional moment curve μ_d, $d \geq 3$. Let*

$$j = \begin{cases} (d^2+d)/2, & d \text{ even;} \\ (d^2+1)/2, & d \text{ odd;} \end{cases}$$

and let $s = \lfloor (j-2)/2 \rfloor$. Then, S has a weak $\frac{1}{r}$-net of size

$$\begin{array}{ll} r \cdot 2^{O(\alpha(r)^s)}, & j \text{ even;} \\ r \cdot 2^{O(\alpha(r)^s \log \alpha(r))}, & j \text{ odd.} \end{array}$$

(*Note that j is even if and only if d is divisible by* 4.)

Furthermore, these weak $\frac{1}{r}$-nets can easily be constructed in time $O(n \log r)$, as we will show.

1.1 The inverse Ackermann function

Let us introduce (our version of) the inverse Ackermann functions $\alpha_k(x)$ and $\alpha(x)$.

The *inverse Ackermann hierarchy* is a sequence of functions $\alpha_k(x)$, for $k = 1, 2, 3, \ldots$ and real $x \geq 0$, defined as follows. We let $\alpha_1(x) = x/2$, and for each $k \geq 2$, we let $\alpha_k(x)$ be the number of times we have to apply α_{k-1}, starting from x, until we reach a value not larger than 1. Formally, for $k \geq 2$, we let

$$\alpha_k(x) = \begin{cases} 0, & \text{if } x \leq 1; \\ 1 + \alpha_k(\alpha_{k-1}(x)), & \text{otherwise.} \end{cases}$$

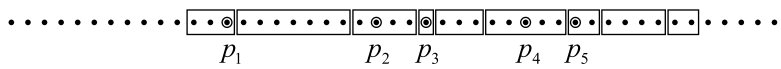

Figure 1: A 9-chain stabbed by a 5-tuple.

Then, we have $\alpha_2(x) = \lceil \log_2 x \rceil$ for $x \geq 1$, and $\alpha_3(x) = \log^* x$. (Note that $\alpha_k(x)$ is always an integer for $k \geq 2$.)

Each function in this hierarchy grows much more slowly than the previous one. In particular, for all fixed k and j, we have $\alpha_{k+1}(x) = o\left(\alpha_k^{(j)}(x)\right)$. (Here $f^{(j)}$ denotes the j-fold composition of f.)

Now, for every fixed $x \geq 6$, the sequence $\alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots$ decreases strictly until it settles at 3. The *inverse Ackermann function*[4] $\alpha(x)$ assigns to each real number x the smallest integer k for which $\alpha_k(x) \leq 3$:

$$\alpha(x) = \min\{k \mid \alpha_k(x) \leq 3\}.$$

The inverse Ackermann function satisfies $\alpha(x) = o(\alpha_k(x))$ for every fixed k.

In our constructions, we will sometimes work with variants $\widehat{\alpha}_k(x)$ of the inverse Ackermann function, which better suit our specific purposes (Lemmas 3.5 and 3.8). This makes no asymptotic difference, for in each case there exists an absolute constant c such that

$$|\widehat{\alpha}_k(x) - \alpha_k(x)| \leq c$$

for all large enough k and all x. We address this issue in Appendix B.

1.2 Interval chains

Our constructions of weak ϵ-nets follow by a reduction to a new problem, which we call *stabbing interval chains*.

Let $[i, j]$ denote the interval of integers $\{i, i+1, \ldots, j\}$; the case $i = j$ is also denoted as $[i]$. An *interval chain*[5] of size k (also called a k-*chain*) is a sequence of k consecutive, disjoint, nonempty intervals

$$\begin{aligned} C &= I_1 I_2 \cdots I_k \\ &= [a_1, a_2][a_2 + 1, a_3] \cdots [a_k + 1, a_{k+1}], \end{aligned}$$

where $a_1 \leq a_2 < a_3 < \cdots < a_{k+1}$. We say that a j-tuple of integers $(p_1, \ldots, p_j)$ *stabs* an interval chain C if each p_i lies on a different interval of C (see Figure 1).

[4] We follow Seidel [14, slide 85]. The inverse Ackermann function is usually defined as follows (see, for example, [11, p. 173], though there are other definitions). Define $A_k(n)$ for integers $k, n \geq 1$ by $A_1(n) = 2n$, and $A_k(n) = A_{k-1}^{(n)}(1)$ for $k \geq 2$. Then, let $\alpha'(x) = \min\{m \mid A_m(m) \geq x\}$. Now, we have $\alpha_k(x) = \min\{m \mid A_k(m) \geq x\}$ for $k \geq 2$, and $\alpha(x) = \min\{m \mid A_m(3) \geq x\}$. Thus, since $A_{m-2}(m-2) \leq A_m(3) \leq A_m(m)$ for $m \geq 3$, it follows that $0 \leq \alpha(x) - \alpha'(x) \leq 2$ for $x > 8$. We note that we make no explicit use of the functions $A_k(n)$ in this paper.

[5] An identical definition of interval chains has already been given by Condon and Saks [9, sec. 2.2], for an unrelated application.

Our problem is to stab, with as few j-tuples as possible, all interval chains of size k that lie within a given range $[1, n]$.

Definition 1.3: Let $z_k^{(j)}(n)$ denote the minimum size of a collection $\mathcal{Z}$ of j-tuples that stab all k-chains that lie in $[1, n]$.

Note that $z_k^{(j)}(n)$ is increasing in n, decreasing in k, and increasing in j.

In this paper we derive almost-tight upper and lower bounds for $z_k^{(j)}(n)$, involving functions in the inverse Ackermann hierarchy. Our upper bounds for $z_k^{(j)}(n)$ are used in the proofs of Theorems 1.1 and 1.2 above. The case $j = 3$ (which is the one needed for Theorem 1.1) is simpler (and tighter) than the general case $j \geq 4$, and we treat this case separately, both in the upper and the lower bounds.

Our bounds for stabbing interval chains are as follows:

Theorem 1.4 *$z_k^{(3)}(n)$ satisfies the following bounds:*

$$z_3^{(3)}(n) = \binom{n-1}{2}; \quad z_4^{(3)}(n) = \Theta(n \log n); \quad z_5^{(3)}(n) = \Theta(n \log \log n);$$

and, for every $k \geq 6$, we have

$$\begin{aligned} z_k^{(3)}(n) &\leq cn\alpha_{\lfloor k/2 \rfloor}(n) \quad \text{for all } n; \\ z_k^{(3)}(n) &\geq c'n\alpha_{\lfloor k/2 \rfloor}(n) \quad \text{for all } n \geq n_k; \end{aligned}$$

for some absolute constants c and c', and some constants n_k depending on k.

Theorem 1.5 *Let $j \geq 4$ be fixed, and let $s = \lfloor (j-2)/2 \rfloor$. Then there exist functions $P'_j(m)$, $Q'_j(m)$, both of the form*

$$P'_j(m), Q'_j(m) = \begin{cases} 2^{(1/s!)m^s + O(m^{s-1})}, & j \text{ even;} \\ 2^{(1/s!)m^s \log_2 m + O(m^s)}, & j \text{ odd;} \end{cases} \tag{2}$$

such that, for every $m \geq 2$, we have

$$\begin{aligned} z_{P'_j(m)}^{(j)}(n) &\leq cn\alpha_m(n) \quad \text{for all } n; \\ z_{Q'_j(m)}^{(j)}(n) &\geq n\alpha_m(n) \quad \text{for all } n \geq n_m. \end{aligned}$$

Here $c = c(j)$ is a constant that depends only on j, and $n_m = n_m(j)$ are constants that depend on j and m.

Thus, for every fixed j, once k is sufficiently large, $z_k^{(j)}(n)$ becomes barely superlinear in n. Moreover, if we let k grow as an appropriate function of $\alpha(n)$, then the upper bounds become *linear*. Namely, we have $z_k^{(3)}(n) = O(n)$ for $k \geq 2\alpha(n)$; and for $j \geq 4$, we have $z_k^{(j)}(n) = O(n)$ for $k \geq P'_j(\alpha(n))$.

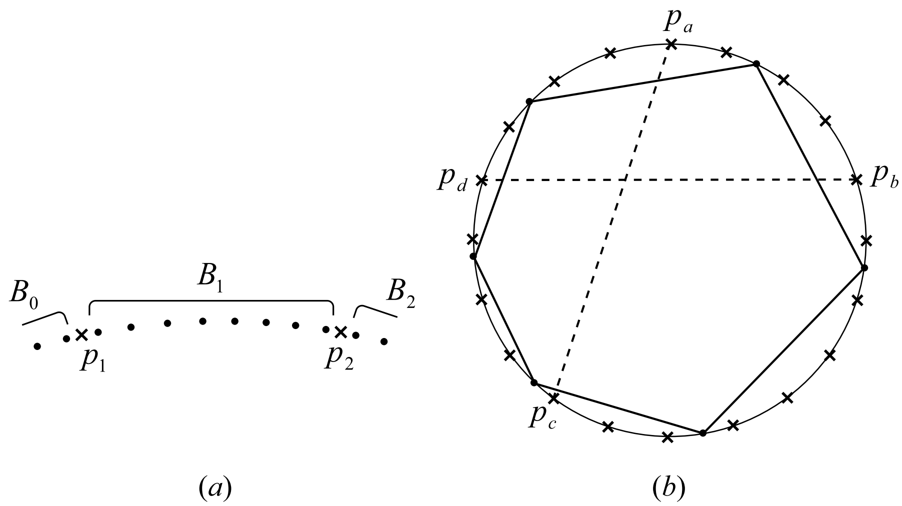

Figure 2: The case of planar point sets in convex position: (a) "Separator" points p_j between consecutive blocks. (b) The intersection between two chords joining pairs of points from four different intervals falls inside $\mathcal{CH}(S')$.

The rest of this paper is organized as follows. In Section 2 we reduce the problem of building weak ϵ-nets for our special point sets to problems of stabbing interval chains with j-tuples. In Section 3 we derive our upper bounds for stabbing interval chains, as asserted in Theorems 1.4 and 1.5, thus completing the proofs of Theorems 1.1 and 1.2 on the size of the weak ϵ-nets. At the end of Section 3 we address the issue of constructing our weak ϵ-nets efficiently.

In Section 4 we derive our almost-matching lower bounds for stabbing interval chains, as provided in Theorems 1.4 and 1.5. We end with a discussion of some open and related problems in Section 5.

Appendix A addresses the case $j = 2$ of the interval-chain stabbing problem (stabbing with *pairs*). Finally, Appendix B contains a technical lemma, used in bounding the difference between variants of the inverse Ackermann functions.

2 From weak ϵ-nets to interval chains

In this section we present constructions of weak ϵ-nets that reduce to problems of stabbing interval chains with j-tuples. We first address the case when S is planar and in convex position, and then we tackle the case where S lies on the moment curve in $\mathbb{R}^d$ (as well as some related cases).

Lemma 2.1 *Let S be a set of n points in convex position in the plane, and let $r > 1$. Then S has a weak $\frac{1}{r}$-net of size $z^{(3)}_{\ell/r-1}(\ell)$, where $\ell < n$ is a free parameter.*

Proof: Partition the points of S into ℓ "blocks" $B_0, B_1, \ldots, B_{\ell-1}$ of n/ℓ consecutive points, clockwise along the boundary of $\mathcal{CH}(S)$ (we ignore the rounding to integers). Construct a set of points $P = \{p_0, p_1, \ldots, p_{\ell-1}\}$, where each p_j lies on the boundary of $\mathcal{CH}(S)$ between the last point of B_{j-1} and the first point of B_j. (Indices are modulo ℓ. See Figure $2(a)$.)

Consider a subset $S' \subset S$ of size at least n/r. S' must contain $m = \ell/r$ points $q_0, q_1, \ldots, q_{m-1}$ lying on m distinct blocks. Let B_{j_k} be the block containing q_k; assume without loss of generality that $0 \le j_0 < j_1 < \cdots < j_{m-1} < \ell$. The blocks B_{j_k} partition P cyclically into m nonempty intervals

$$I_k = \{p_{j_k+1}, p_{j_k+2}, \ldots, p_{j_{k+1}}\}, \quad \text{for } 0 \le k < m.$$

(Indices are modulo ℓ or modulo m as appropriate.) Let $p_a, p_b, p_c, p_d \in P$ be four points belonging to four different intervals I_k, listed in cyclic order. Then the intersection between the segments $p_a p_c$ and $p_b p_d$ must lie inside $\mathcal{CH}(q_0, \ldots, q_{m-1}) \subseteq \mathcal{CH}(S')$. See Figure $2(b)$.[6]

Thus, it is enough to construct a set of quadruples of points of P, such that, no matter how P is cyclically partitioned into m intervals $I_0 I_1 \cdots I_{m-1}$, some quadruple will "stab" four different intervals. The set of chord-intersection points corresponding to these quadruples is our desired weak $\frac{1}{r}$-net.

We take point p_0 as the first point for all the quadruples; by construction, p_0 lies in the last interval I_{m-1}. Thus, it only remains to build a family $\mathcal{Z}$ of *triples* of the form (p_a, p_b, p_c), with $1 \le a < b < c < \ell$, such that some triple is guaranteed to fall on three distinct intervals among $I_0, \ldots, I_{m-2}$, in any given cyclic chain $I_0, \ldots, I_{m-1}$.

But this is isomorphic to the problem of stabbing all $(m-1)$-chains in $[1, \ell-1]$ with triples. Thus, there exists a family $\mathcal{Z}$ of size at most $z^{(3)}_{m-1}(\ell) = z^{(3)}_{\ell/r-1}(\ell)$. ∎

Remark: Including point p_0 in all the quadruples entails a penalty of at most a factor of 2 in the number of quadruples. Indeed, given an optimal family $\mathcal{Z}$ of quadruples that stab all cyclic partitions into m intervals, we can replace each quadruple $\overline{q} = (p_a, p_b, p_c, p_d) \in \mathcal{Z}$, with $0 < a < b < c < d < \ell$, by the two quadruples $\overline{q}_1 = (p_0, p_b, p_c, p_d)$, $\overline{q}_2 = (p_0, p_a, p_b, p_c)$. If $\overline{q}$ stabs four different intervals in such a partition, then one of $\overline{q}_1$, $\overline{q}_2$ must also do so.

Proof of Theorem 1.1: By Theorem 1.4 we have

$$z^{(3)}_{\ell/r-1}(\ell) = O(\ell \alpha_{\ell/(2r)-1}(\ell)). \tag{3}$$

Thus, we want to choose ℓ as a function of r so as to minimize this expression.

Let $\ell = 2r(1 + \alpha(r))$, so $\ell/(2r) - 1 = \alpha(r)$. We claim that $\alpha_{\alpha(r)}(\ell) \le 4$ for all large enough r. Indeed, for all $k \ge 3$ and $r \ge 0$ we have $\alpha_k(r^2) \le 1 + \alpha_k(r)$. Thus, once r is large enough, we have

$$\begin{aligned} \alpha_{\alpha(r)}(\ell) = \alpha_{\alpha(r)}(2r(1+\alpha(r))) &\le \alpha_{\alpha(r)}(r^2) \\ &\le 1 + \alpha_{\alpha(r)}(r) \\ &= 1 + 3 = 4, \end{aligned}$$

[6]This basic idea, initially observed by Emo Welzl, already appears in [7].

since $\alpha_{\alpha(r)}(r) \le 3$ by definition. Hence, the expression (3) becomes $O(r\alpha(r))$. ∎

2.1 Point sets along the moment curve

A similar reduction applies to the case when S is a set of n points along the moment curve μ_d (1). This curve has the property that every hyperplane intersects it in at most d points (see, e.g., Matoušek [11, p. 97]). In fact, our analysis applies to any curve that satisfies this property.

We can consider points along the moment curve to be ordered by increasing parameter t. If A and B are two finite sets of points along μ_d, we say that A and B are *interleaving* if between every two points of A there is a point of B and vice versa. In such a case, we must have $\Big||A| - |B|\Big| \le 1$.

Lemma 2.2 *Let $s = \lceil (d+1)/2 \rceil$, and let $j = (s-1)(d+1)+1$. (Thus, $j = (d^2+d+2)/2$ for d even, and $j = (d^2+1)/2$ for d odd.)*

Let A be a set of j points along the moment curve $\mu_d \subset \mathbb{R}^d$. Then there exists a point $x \in \mathcal{CH}(A)$ with the following property: For every point set $B \subset \mu_d$ interleaving with A, with

$$|B| = \begin{cases} j, & d \text{ even,} \\ j+1, & d \text{ odd,} \end{cases}$$

we have $x \in \mathcal{CH}(B)$.

Proof: By Tverberg's Theorem (see, e.g., [11, p. 200]), A can be partitioned into s pairwise disjoint subsets $A_1, \ldots, A_s$, whose convex hulls all contain some common point x. This point x satisfies the assertion of the lemma, for if $x \notin \mathcal{CH}(B)$, then there would exist a hyperplane h that separates x from B. But there must be at least s points of A in the same side of h as x (one for each part A_i). By continuity, and since A and B are interleaving, it follows that the curve μ_d must intersect h at least $2s-1$ times if d is even, or $2s$ times if d is odd. In either case, this quantity equals $d+1$.

This is a contradiction, since no hyperplane can intersect the moment curve more than d times.[7] ∎

Remark: We can derive a slightly weaker version of Lemma 2.2 more simply, by applying the Centerpoint Theorem [11, p. 14], instead of Tverberg's Theorem. Let $j = (d^2+d+2)/2$, let A be a j-point set along μ_d, and let $x \in \mathbb{R}^d$ be a centerpoint of A. If $x \in h^+$ for some hyperplane h, then there must be at least $\lceil j/(d+1) \rceil = \lceil (d+1)/2 \rceil$ points of A in h^+. Proceed as above. The resulting bound is slightly weaker than the one given above when d is odd.

Using Lemma 2.2, the reduction from weak ϵ-nets to stabbing interval chains with j-tuples is straightforward:

[7] The above argument is very similar to the one used by Matoušek and Wagner [12], applied to a different construction.

Lemma 2.3 *Let S be a set of n points along the moment curve μ_d, and let $r > 1$. Let*

$$j' = \begin{cases} (d^2+d)/2, & d \text{ even;} \\ (d^2+1)/2, & d \text{ odd.} \end{cases}$$

Then S has a weak $\frac{1}{r}$-net of size at most $z^{(j')}_{\ell/r-1}(\ell)$, where $\ell < n$ is a free parameter.

Proof: Partition S into ℓ blocks $B_0, B_1, \ldots, B_{\ell-1}$ of n/ℓ consecutive points. Construct a set of points $P = \{p_1, \ldots, p_{\ell-1}\} \subset \mu_d$, where each p_i lies between the last point of B_{i-1} and the first point of B_i. Take also a point $p_\ell \in \mu_d$ lying after $B_{\ell-1}$.

Consider a set $S' \subset S$ of size at least n/r. S' must contain $m = \ell/r$ points $q_1, \ldots, q_m$, lying on m different blocks $B_{i_1}, \ldots, B_{i_m}$. These points define on P an $(m-1)$-chain $C = I_1 \cdots I_{m-1}$, where

$$I_k = \{p_{i_k+1}, p_{i_k+2}, \ldots, p_{i_{k+1}}\}, \quad \text{for } 1 \le k \le m-1.$$

Construct an optimal family $\mathcal{Z}'$ of j'-tuples of points in P that stab all $(m-1)$-chains in P. Append the point p_ℓ to every j'-tuple in $\mathcal{Z}'$, obtaining a family $\mathcal{Z}$ of $(j'+1)$-tuples (actually, this is necessary only for d even). We have $|\mathcal{Z}| = z^{(j')}_{m-1}(\ell - 1)$.

There must exist some $\overline{p} \in \mathcal{Z}$ whose first j' points stab the chain C. Thus, the $j'+1$ points of $\overline{p}$ are interleaving with some $(j'+1)$-point subset of $\{q_1, \ldots, q_m\}$. By the choice of j', Lemma 2.2 applies, so the point $x = x(\overline{p})$ guaranteed by the lemma lies in $\mathcal{CH}(S')$. Therefore, the set of all points $x(\overline{p})$, $\overline{p} \in \mathcal{Z}$, is our desired weak $\frac{1}{r}$-net. ∎

Proof of Theorem 1.2: We want to choose ℓ as a function of r so as to minimize $z^{(j')}_{\ell/r-1}(\ell)$. Take $\ell = r(1 + P'_{j'}(\alpha(r)))$, with $P'_{j'}(m)$ as given in Theorem 1.5. Then, arguing as in the proof of Theorem 1.1,

$$z^{(j')}_{\ell/r-1}(\ell) = z^{(j')}_{P'_{j'}(\alpha(r))}(\ell) \le c\ell\alpha_{\alpha(r)}(\ell) \le 4c\ell.$$

The claim follows. ∎

Remark: The results in this section can be generalized to curves $\gamma \subset \mathbb{R}^d$ with the property that every hyperplane intersects γ at most q times, for some integer q. (We must have $q \ge d$, since we can always pass a hyperplane through d given points.) In Lemma 2.2, we take instead $s = \lceil (q+1)/2 \rceil$, and we let $|B| = j$ for q even and $|B| = j+1$ for q odd. Lemma 2.3 is also modified accordingly. We obtain weak $\frac{1}{r}$-nets of size $r \cdot 2^{\mathrm{poly}(\alpha(r))}$ for point sets along these curves γ. (Note that the methods of [12] yield weak $\frac{1}{r}$-nets of size $O(r \operatorname{polylog}(r))$ for these point sets.)

3 Upper bounds for stabbing interval chains

In this section we derive upper bounds on $z^{(j)}_k(n)$, the minimum number of j-tuples needed to stab all k-interval chains contained in the range $[1, n]$. We will always take j to be a constant, noting that the constants implicit in the asymptotic notations do depend on j (though neither on k nor on n).

We start with the easy case $k = j$, for which we have an exact bound.

Lemma 3.1 *We have*

$$z_j^{(j)}(n) = \binom{n - \lfloor j/2 \rfloor}{\lceil j/2 \rceil} = \Theta\left(n^{\lceil j/2 \rceil}\right)$$

for all $j \geq 2$.

Proof: Suppose first that j is odd. Consider all j-chains of the form

$$[a_1][a_1+1, a_2-1][a_2][a_2+1, a_3-1][a_3] \cdots [a_{(j+1)/2}],$$

where $1 \leq a_i \leq n$ and $a_i + 2 \leq a_{i+1}$ for all i. There are $\binom{n-(j-1)/2}{(j+1)/2}$ such chains, each of which must be stabbed by a different j-tuple. On the other hand, we can stab all j-chains by taking all j-tuples of the form

$$(b_1, b_1+1, b_2, b_2+1, b_3, \ldots, b_{(j+1)/2}),$$

where $1 \leq b_i \leq n$ and $b_i + 2 \leq b_{i+1}$ for all i. There are $\binom{n-(j-1)/2}{(j+1)/2}$ such j-tuples. Therefore, for j odd we have $z_j^{(j)}(n) = \binom{n-(j-1)/2}{(j+1)/2} = \binom{n-\lfloor j/2 \rfloor}{\lceil j/2 \rceil}$.

The case where j is even is similar. For the lower bound, we consider all j-chains of the form

$$[a_1][a_1+1, a_2-1][a_2] \cdots [a_{j/2}][a_{j/2+1}, n],$$

and, for the upper bound, we take all j-tuples of the form

$$(b_1, b_1+1, \ldots, b_{j/2}, b_{j/2}+1).$$

We get $z_j^{(j)}(n) = \binom{n-j/2}{j/2}$. ∎

Once k is large enough with respect to j, the number of j-tuples required to stab all k-chains becomes $O(n \operatorname{polylog}(n))$:

Lemma 3.2 *For every fixed* $j \geq 2$ *we have*[8]

$$z_{2^{j-1}}^{(j)}(n) = O\left(n \log^{j-2} n\right).$$

Proof: By induction on j. The base case $j = 2$ is given by Lemma 3.1, so let $j \geq 3$, and put $k = 2^{j-1}$.

Divide the range $[1, n]$ into two blocks B_1, B_2, each of size at most $n/2$, leaving between them the element $y = \lceil n/2 \rceil$.

For each block B_i we build an optimal family of j-tuples that stab all k-chains entirely contained in B_i. This requires at most $2z_k^{(j)}(n/2)$ j-tuples in total.

It remains to stab those k-chains that contain the element y. Every such chain C must have $k/2 = 2^{j-2}$ intervals entirely contained in either B_1 or B_2. Thus, it suffices

[8] A more careful analysis shows that the constant of proportionality actually decreases exponentially with j.

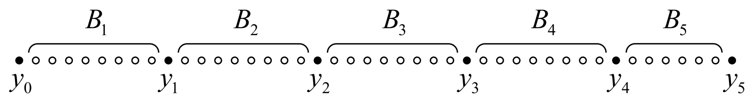

Figure 3: Range $[1, n]$ partitioned into blocks and separators.

to build on each B_i an optimal family of $(j-1)$-tuples that stab all $k/2$-chains in B_i, and append the element y to each $(j-1)$-tuple. The number of resulting j-tuples is at most $2z_{k/2}^{(j-1)}(n/2)$, which is $O\big(n \log^{j-3} n\big)$ by the induction hypothesis.

We obtain the recurrence relation

$$z_k^{(j)}(n) \le 2z_k^{(j)}\left(\frac{n}{2}\right) + O\big(n \log^{j-3} n\big),$$

which implies $z_k^{(j)}(n) = O\big(n \log^{j-2} n\big)$. $\blacksquare$

We now derive upper bounds for $z_k^{(j)}(n)$ for all k. We first tackle the case $j = 3$ (which is the one used in the proof of Theorem 1.1), and then we address the general case $j \ge 4$. For completeness, we address the case $j = 2$ in Appendix A.

Our derivations below (and of the lower bounds in Section 4) follow a recurring pattern: We first derive a recurrence relation for $z_k^{(j)}(n)$, and then we apply it with appropriately chosen parameters. For added clarity, we identify the lemmas stating the recurrence relations by the name *Recurrence*.

3.1 Upper bounds for triples

We have already established that $z_3^{(3)}(n) = \binom{n-1}{2}$ (Lemma 3.1) and $z_4^{(3)}(n) = O(n \log n)$ (Lemma 3.2). Our bounds for stabbing k-chains with triples, $k \ge 5$, are based on the following recurrence relation.

Recurrence 3.3 *Let t be an integer parameter, with $1 \le t \le \sqrt{n/2} - 1$. Then,*

$$z_k^{(3)}(n) \le \frac{n}{t} z_k^{(3)}(t) + z_{k-2}^{(3)}\left(\frac{n}{t}\right) + 2n.$$

Proof: Partition the range $[1, n]$ into blocks $B_1, B_2, \ldots, B_b$ of size t (except for the last block, which might be smaller), leaving between each pair of adjacent blocks, as well as before the first block and after the last one, a single "separator" element. Let the set of separators be $Y = \{y_0, \ldots, y_b\}$, such that block B_i lies between separators y_{i-1} and y_i (see Figure 3).

The number of blocks is $b = \left\lceil \frac{n-1}{t+1} \right\rceil$. We have $b \le n/t - 1$, since $n \ge 2(t+1)^2 \ge 2t^2 + t$.

Consider an arbitrary k-chain $C = I_1 \cdots I_k$. C must satisfy exactly one of the following properties (see Figure 4):

1. C is entirely contained within a block B_i.

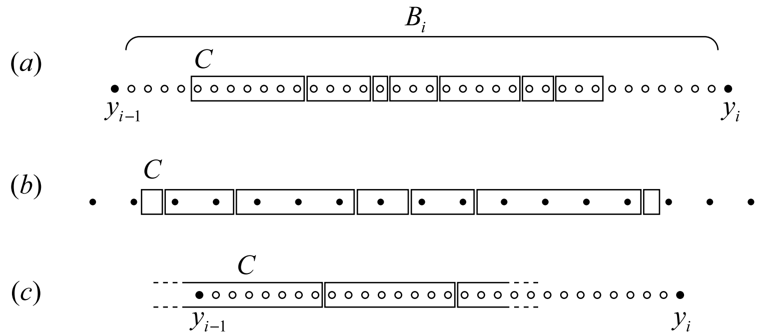

Figure 4: A k-chain C must satisfy exactly one of these properties: Either C is contained within a block (a); or every interval of C, except possibly the first and last, contains a separator (b); or some interval of C, besides the first and last, falls entirely within a block, and another interval contains an adjacent separator (c).

2. Every interval of C, except possibly the first and the last, contains a separator.
3. Some interval I_j of C, $2 \le j \le k-1$, falls entirely within a block B_i, but not all of C is contained in the block. Thus, some other interval of C contains either y_{i-1} or y_i.

We can take care of the first case by constructing within each block B_i an optimal family of triples that stab all k-chains. This requires at most $bz_k^{(3)}(t) \le (n/t)z_k^{(3)}(t)$ triples.

The second case is handled by constructing on the separators Y an optimal family of triples that stab all $(k-2)$-chains. This requires at most $z_{k-2}^{(3)}(b+1) \le z_{k-2}^{(3)}(n/t)$ triples.

Finally, the third case is handled by taking all triples of the forms

$$\begin{aligned} (a, a+1, y_i), &\quad \text{for} \quad y_{i-1} \le a \le y_i - 2, \\ (y_{i-1}, a, a+1), &\quad \text{for} \quad y_{i-1} < a \le y_i - 1, \end{aligned}$$

for all y_i. There are at most $2n$ such triples. ∎

Lemma 3.4 *We have $z_5^{(3)}(n) = O(n \log\log n)$.*

Proof: Apply Recurrence 3.3 with $k = 5$ and $t = \sqrt{n/3}$, and use Lemma 3.1. ∎

Lemma 3.5 *There exists an absolute constant c such that, for every $k \ge 6$, we have*

$$z_k^{(3)}(n) \le cn\alpha_{\lfloor k/2 \rfloor}(n) \quad \textit{for all } n.$$

Proof: Here it is convenient to work with a slight variant of the inverse Ackermann function. Let $n_0 = 2000$. For this proof, let $\widehat{\alpha}_m(x)$, $m \ge 2$, be given by $\widehat{\alpha}_2(x) = \alpha_2(x) =$

$\lceil \log_2 x \rceil$, and, for $m \geq 3$, by the recurrence

$$\widehat{\alpha}_m(x) = \begin{cases} 1, & \text{if } x \leq n_0; \\ 1 + \widehat{\alpha}_m(2\widehat{\alpha}_{m-1}(x)), & \text{otherwise.} \end{cases}$$

There exists a constant c_0 such that $|\widehat{\alpha}_m(x) - \alpha_m(x)| \leq c_0$ for all m and x (see Appendix B).

Let $k \geq 4$, and let $m = \lfloor k/2 \rfloor$. We prove, by induction on k, that

$$z_k^{(3)}(n) \leq c_1 n \widehat{\alpha}_m(n) \quad \text{for all } n,$$

for some absolute constant c_1. The base cases of the induction are $z_4^{(3)}(n), z_5^{(3)}(n) = O(n \log n)$, by Lemmas 3.2 and 3.4, respectively. Without loss of generality, assume that $c_1 \geq 4$ and that $c_1 \geq z_4^{(3)}(n)/n$ for all $n \leq n_0$.

Let now $k \geq 6$, and assume that the bound holds for $k-2$. To establish the bound for k, assume first that $n \leq n_0$. Then, we have

$$z_k^{(3)}(n) \leq z_4^{(3)}(n) \leq c_1 n = c_1 n \widehat{\alpha}_m(n).$$

Thus, let $n > n_0$. We apply Recurrence 3.3 with $t = 2\widehat{\alpha}_{m-1}(n)$. (Note that $t \leq \sqrt{n/2} - 1$ for $n > n_0$.) Letting $z_k^{(3)}(n) = n g(n)$, and using the fact that $c_1 \geq 4$, we obtain

$$\begin{aligned} g(n) \leq g(t) + \frac{c_1}{t}\widehat{\alpha}_{m-1}\left(\frac{n}{t}\right) + 2 \quad &\leq \quad g(t) + \frac{c_1}{t}\widehat{\alpha}_{m-1}(n) + 2 \\ &= \quad g(t) + \frac{c_1}{2} + 2 \\ &\leq \quad g(t) + c_1. \end{aligned}$$

Since $\widehat{\alpha}_m(t) = \widehat{\alpha}_m(n) - 1$, it follows by induction on n (with base case $n \leq n_0$) that

$$g(n) \leq c_1 \widehat{\alpha}_m(n) \quad \text{for all } n.$$

Therefore,

$$z_k^{(3)}(n) \leq c_1 n \widehat{\alpha}_m(n) \quad \text{for all } n. \quad \blacksquare$$

This proves the upper bounds of Theorem 1.4.

Remark: Had we not been careful to add the factor 2 in the definition of $\widehat{\alpha}_m(x)$ and in the choice of t, we would have got a weaker bound of $z_k^{(3)}(n) = O(nk\alpha_{\lfloor k/2 \rfloor}(n))$. Then, the bound of Theorem 1.1 would have deteriorated to $O(r\alpha^2(r))$.

3.2 From triples to j-tuples

We now extend our techniques of the previous section and derive upper bounds for $z_k^{(j)}(n)$, the minimum number of j-tuples needed to stab all k-chains in $[1, n]$, for $j \geq 4$.

Our bounds are based on the following recurrence relation.

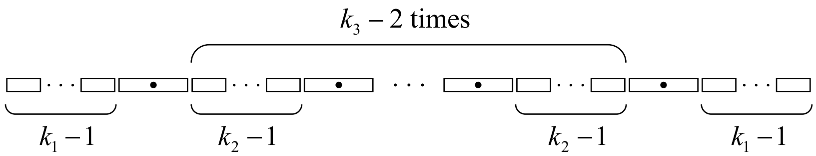

Figure 5: A chain which violates all three properties, like the one shown, can have at most $k-1$ intervals.

Recurrence 3.6 *Let $j \geq 4$ be fixed. Let t be a parameter, $1 \leq t \leq \sqrt{n/2}-1$, and let k_1, k_2, k_3 be integers. Put $k = 2k_1 + k_2(k_3-2)$. Then,*

$$z_k^{(j)}(n) \leq \frac{n}{t}\left(z_k^{(j)}(t) + 2z_{k_1}^{(j-1)}(t) + z_{k_2}^{(j-2)}(t)\right) + z_{k_3}^{(j)}\left(\frac{n}{t}\right).$$

Proof: As before, partition the range $[1,n]$ into blocks $B_1, \ldots, B_b$ of size t (except for the last block, which might be smaller), such that each block B_i is surrounded by separator elements y_{i-1}, y_i. Denote the set of separators by $Y = \{y_0, \ldots, y_b\}$. Again, since $t \leq \sqrt{n/2}-1$, we have $b \leq n/t - 1$.

Let k_1, k_2, k_3 be given, and put $k = 2k_1 + k_2(k_3-2)$. Then, every k-chain $C = I_1 \cdots I_k$ satisfies at least one of the following properties:

1. C is entirely contained within a block B_i.

2. The first k_1 intervals of C, or the last k_1 intervals of C, fall entirely within a block B_i, and some other interval of C contains the separator y_i or y_{i-1}, respectively.

3. Some k_2 consecutive intervals of C fall within a block B_i, and two other intervals contain the separators y_{i-1} and y_i.

4. At least k_3 distinct intervals of C contain separators.

Indeed, the largest number of intervals for which a chain might possibly violate *all* the above properties is

$$(k_3-1) + (k_3-2)(k_2-1) + 2(k_1-1) = k-1.$$

(See Figure 5.) Hence, by our choice of k, one of the above properties must hold.

Thus, we can stab all k-chains by building the following family of j-tuples. Within each block B_i we build

- an optimal family of j-tuples that stab all k-chains;

- an optimal family of $(j-1)$-tuples that stab all k_1-chains, where each of these tuples is extended into a j-tuple in two ways, by appending either of the surrounding separators y_{i-1}, y_i;

- an optimal family of $(j-2)$-tuples that stab all k_2-chains, where each of these tuples is extended into a j-tuple by appending both separators y_{i-1}, y_i.

$m =$	2	3	4	5	6	7
$P_2(m) =$	2	2	2	2	2	2
$P_3(m) =$	4	6	8	10	12	14
$P_4(m) =$	8	24	60	136	292	608
$P_5(m) =$	16	132	1160	11852	142784	2000164
$P_6(m) =$	32	984	61240	8352072	$\cdots$	

Table 1: Values of $P_j(m)$ for small j and m.

So far we have, in total, at most

$$\frac{n}{t}\left(z_k^{(j)}(t) + 2z_{k_1}^{(j-1)}(t) + z_{k_2}^{(j-2)}(t)\right)$$

j-tuples. In addition, we construct on the set of separators Y an optimal family of j-tuples that stab all k_3-chains. The number of such j-tuples is at most $z_{k_3}^{(j)}(n/t)$. Every k-chain C must be stabbed by some j-tuple in this family. ∎

Define integer-valued functions $P_j(m)$, $j, m \geq 2$, by

$$P_2(m) = 2; \quad P_3(m) = 2m;$$

$$P_j(m) = \begin{cases} 2^{j-1}, & m = 2; \\ 2P_{j-1}(m) + P_{j-2}(m)\left(P_j(m-1) - 2\right), & m \geq 3; \end{cases} \quad \text{for } j \geq 4.$$

See Table 1. We can give an explicit formula for $P_4(m)$:

$$P_4(m) = 5 \cdot 2^m - 4m - 4.$$

Lemma 3.7 *Let $j \geq 3$ be fixed, and let $s = \lfloor (j-2)/2 \rfloor$. Then,*

$$P_j(m) = \begin{cases} 2^{(1/s!)m^s + O(m^{s-1})}, & \textit{for } j \textit{ even;} \\ 2^{(1/s!)m^s \log_2 m + O(m^s)}, & \textit{for } j \textit{ odd.} \end{cases}$$

Proof: By induction on j. The base cases $j = 3, 4$ are clear, so let $j \geq 5$. Let $p_j(m) = \log_2 P_j(m)$. Using the bounds

$$\log_2 x \leq \log_2(x+y) \leq \frac{1}{\ln 2} \cdot \frac{y}{x} + \log_2 x, \quad \text{for } y \geq 0,$$

we obtain

$$p_{j-2}(m) + p_j(m-1) \leq p_j(m) \leq R_j(m) + p_{j-2}(m) + p_j(m-1), \tag{4}$$

where

$$R_j(m) = \frac{2P_{j-1}(m)}{\ln 2 \cdot P_{j-2}(m) P_j(m-1)}.$$

Thus, by the left-hand side of (4), we have

$$p_j(m) \geq \sum_{i=3}^{m} p_{j-2}(i).$$

The lower bound for $P_j(m)$ follows by bounding this sum by an integral, since

$$\begin{aligned}\int \left(\frac{1}{(s-1)!}x^{s-1}\log_2 x + cx^{s-1}\right)dx &= \frac{1}{s!}x^s \log_2 x + O(x^s), \quad \text{for } s \geq 1;\\ \int \left(\frac{1}{(s-1)!}x^{s-1} + cx^{s-2}\right)dx &= \frac{1}{s!}x^s + O\big(x^{s-1}\big), \qquad \text{for } s \geq 2.\end{aligned}$$

Thus, applying the lower bound for $P_j(m)$, and assuming by induction the upper bound for $P_{j-1}(m)$, it follows that $\lim_{m\to\infty} P_{j-1}(m)/P_j(m-1) = 0$, so $R_j(m)$ tends to zero with m. Therefore, by the right-hand side of (4),

$$p_j(m) = o(m) + \sum_{i=3}^{m} p_{j-2}(i),$$

and the upper bound for $P_j(m)$ follows similarly. ∎

Lemma 3.8 *Let $j \geq 2$ be fixed. Then, there exists a constant $c = c(j)$ such that, for every $m \geq 2$, we have*

$$z^{(j)}_{P_j(m)}(n) \leq cn\alpha_m(n)^{j-2} \quad \textit{for all } n. \tag{5}$$

Proof: We proceed along the lines of the proof of Lemma 3.5, except that now we also use induction on j. The case $j = 3$ was proven already (Lemmas 3.2 and 3.5), so let $j \geq 4$ be fixed.

We again work with a slight variant of the inverse Ackermann function. Let $n_0 = j^{4j}$. For this proof, let $\widehat{\alpha}_m(x)$, $m \geq 2$, be given by $\widehat{\alpha}_2(x) = \alpha_2(x) = \lceil \log_2 x \rceil$, and for $m \geq 3$ by the recurrence

$$\widehat{\alpha}_m(x) = \begin{cases} 1, & \text{if } x \leq n_0;\\ 1 + \widehat{\alpha}_m\big(4\widehat{\alpha}_{m-1}(x)^{j-2}\big), & \text{otherwise.}\end{cases}$$

Again, there exists a constant c_0 (depending only on j) such that $|\widehat{\alpha}_k(x) - \alpha_k(x)| \leq c_0$ for all k and x (see Appendix B).

We will show, by induction on m, that there exists a constant c_1 (depending only on j) such that

$$z^{(j)}_{P_j(m)}(n) \leq c_1 n \left(2\widehat{\alpha}_m(n)^{j-2} + \widehat{\alpha}_m(n)^{j-3} + \widehat{\alpha}_m(n)\right) \tag{6}$$

for all $m \geq 2$ and all n. This is easily seen to imply the claim.

The base case $m = 2$ is given by Lemma 3.2, so assume c_1 is large enough that (6) holds for $m = 2$. Assume further that

$$c_1 \geq z^{(j)}_{P_j(3)}(n)/(4n), \quad \text{for } n \leq n_0. \tag{7}$$

By induction on j, we know there exist constants c_2, c_3 (depending on j), such that

$$\begin{aligned} z^{(j-1)}_{P_{j-1}(m)}(n) &\leq c_2 n \widehat{\alpha}_m(n)^{j-3}, \\ z^{(j-2)}_{P_{j-2}(m)}(n) &\leq c_3 n \widehat{\alpha}_m(n)^{j-4}, \end{aligned}$$

for all $m \geq 3$ and all n. Without loss of generality, assume $c_1 \geq c_2, c_3$.

Now, let $m \geq 3$, and suppose (6) holds for $m-1$. To establish (6) for m, assume first that $n \leq n_0$. Then, by (7), we have

$$\begin{aligned} z^{(j)}_{P_j(m)}(n) \leq z^{(j)}_{P_j(3)}(n) &\leq 4c_1 n \\ &= c_1 n \left(2\widehat{\alpha}_m(n)^{j-2} + \widehat{\alpha}_m(n)^{j-3} + \widehat{\alpha}_m(n)\right). \end{aligned}$$

Thus, let $n > n_0$. Apply Recurrence 3.6 with the following parameters:

$$k_1 = P_{j-1}(m), \quad k_2 = P_{j-2}(m), \quad k_3 = P_j(m-1),$$
$$k = P_j(m), \quad t = 4\widehat{\alpha}_{m-1}(n)^{j-2}.$$

(By our choice of n_0, we have $t \leq \sqrt{n/2} - 1$ for $n > n_0$.) Using $t \leq n$ and $n/t \leq n$, we have

$$\begin{aligned} 2z^{(j-1)}_{k_1}(t) &\leq 2c_1 t \widehat{\alpha}_m(n)^{j-3}; \\ z^{(j-2)}_{k_2}(t) &\leq c_1 t \widehat{\alpha}_m(n)^{j-4}; \\ z^{(j)}_{k_3}\left(\frac{n}{t}\right) &\leq \frac{c_1 n}{t} \left(2\widehat{\alpha}_{m-1}(n)^{j-2} + \widehat{\alpha}_{m-1}(n)^{j-3} + \widehat{\alpha}_{m-1}(n)\right) \\ &= \frac{c_1 n}{4} \left(2 + \widehat{\alpha}_{m-1}(n)^{-1} + \widehat{\alpha}_{m-1}(n)^{-j+3}\right) \leq c_1 n. \end{aligned}$$

Plugging these expressions into Recurrence 3.6 and letting $z^{(j)}_k(n) = ng(n)$, we get

$$g(n) \leq g(t) + 2c_1\widehat{\alpha}_m(n)^{j-3} + c_1\widehat{\alpha}_m(n)^{j-4} + c_1.$$

Since $\widehat{\alpha}_m(t) = \widehat{\alpha}_m(n) - 1$, it follows by induction on n that

$$g(n) \leq c_1 \left(2\widehat{\alpha}_m(n)^{j-2} + \widehat{\alpha}_m(n)^{j-3} + \widehat{\alpha}_m(n)\right).$$

(The base case $n \leq n_0$ follows from (7), and for the induction on n we apply

$$(\widehat{\alpha}_m(n) - 1)^{j-x} \leq (\widehat{\alpha}_m(n) - 1)\widehat{\alpha}_m(n)^{j-x-1}$$

for $x = 2, 3$.) Thus,

$$z^{(j)}_{P_j(m)}(n) \leq c_1 n \left(2\widehat{\alpha}_m(n)^{j-2} + \widehat{\alpha}_m(n)^{j-3} + \widehat{\alpha}_m(n)\right),$$

as claimed. ∎

Let $P'_j(m) = P_j(m+1)$ for $j \geq 4$, $m \geq 2$. Clearly, $P'_j(m)$ satisfies (2). There exists a constant c', depending only on j, such that $\alpha_{m+1}(n)^{j-2} \leq c'\alpha_m(n)$ for all m and n. Therefore,

$$z^{(j)}_{P'_j(m)}(n) \leq c'' n \alpha_m(n) \quad \text{for all } n,$$

for some constant $c'' = c''(j)$. This proves the upper bounds of Theorem 1.5.

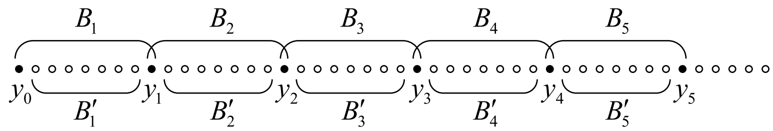

Figure 6: Blocks and contracted blocks defined on the range $[1, n]$.

Computational aspects. The upper bound constructions given in this Section yield algorithms for building stabbing families of j-tuples in linear time in the size of the output.

Thus, the weak $\frac{1}{r}$-nets of Theorems 1.1 and 1.2 can be easily built in time $O(n \log r)$, for a given n-point set S with the appropriate properties. Consider first the planar case (of Theorem 1.1):

Let $S = (q_0, \ldots, q_{n-1})$ be a given list of n points in the plane in convex position (listed in no particular order). We arbitrarily fix q_0 as the first point of S around the boundary of $\mathcal{CH}(S)$. Then, we can determine the relative order of any two other points q_a, q_b, $a, b \geq 1$, around this boundary, by testing whether $q_0 q_a q_b$ makes a right or a left turn. With this comparison predicate, we can build the ℓ-point list $P = (p_0, \ldots, p_{\ell-1})$, as given in the proof of Lemma 2.1, in time $O(n \log \ell)$; we do this by divide and conquer, applying linear-time selection in each step.

From the list P, we can obtain our desired weak $\frac{1}{r}$-net, of size $O(\ell) = O(r\alpha(r))$, in time $O(\ell)$. Thus, the total running time is $O(\ell + n \log \ell) = O(n \log r)$. (We may assume that $\ell \leq n$, for otherwise we can just return S itself as the desired weak $\frac{1}{r}$-net.)

The case of the moment curve is analogous. (Finding the point x of Lemma 2.2 involves examining a finite number of partitions—a constant-time operation, since d is constant.)

4 Lower bounds for stabbing interval chains

We now derive asymptotic lower bounds for $z_k^{(j)}(n)$. As before, we take j to be fixed, recalling that the implicit constants do depend on j.

As a warm-up, we first derive lower bounds of the form $z_k^{(j)}(n) = \Omega(n \log n)$ for appropriate k, for each $j \geq 3$. (We do not use these bounds in our later arguments, but we are interested in the case $j = 3$, since it yields $z_4^{(3)}(n) = \Theta(n \log n)$.)

Lemma 4.1 *For every fixed $j \geq 3$ we have*

$$z_{(j-1)^2}^{(j)}(n) = \Omega(n \log n),$$

where the constant of proportionality depends on j.

Proof: Let $t = \lceil n/j \rceil$. We define on the range $[1, n]$ a sequence of j blocks of size t, in which every two consecutive blocks overlap on exactly one element. For this, let

$y_i = 1 + i(t-1)$ for $0 \le i \le j$. Note that $y_0 = 1$ and $y_j \le n$. Then let

$$B_i = [y_{i-1}, y_i], \quad \text{for } 1 \le i \le j.$$

We also define "contracted blocks" that do not contain the elements y_i:

$$B'_i = [y_{i-1}+1, y_i - 1], \quad \text{for } 1 \le i \le j.$$

(See Figure 6.) We have $|B'_i| = t-2$ for all i.

Let $k = (j-1)^2$, and let $\mathcal{Z}$ be a family of j-tuples that stab all k-chains in $[1, n]$. $\mathcal{Z}$ must contain families $\mathcal{Z}_1, \ldots, \mathcal{Z}_j$ of "local" j-tuples that stab all k-chains in $B_1, \ldots, B_j$, respectively. Further, these local families must be disjoint, since every two blocks overlap on at most one element. Thus,

$$|\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_j| \ge j z_k^{(j)}(t) \ge j z_k^{(j)}\left(\frac{n}{j}\right).$$

Now, consider the "global" j-tuples of $\mathcal{Z}$—those that are not contained in any block B_i. Consider the elements of the contracted blocks B'_i that are not contained in any global j-tuple. Call these elements "unused".

Suppose each of the blocks B'_1, B'_j contains a run of $j-2$ consecutive unused elements, and each of the intermediate blocks $B'_2, \ldots, B'_{j-1}$ contains a run of $j-3$ consecutive unused elements. Construct an interval chain C that has these $j^2 - 3j + 2$ unused elements as singleton intervals, plus $j-1$ "long" intervals between the runs of singletons. (If $j = 3$ then the two long intervals meet at an arbitrary place in B'_2.) Note that each long interval is nonempty, since it contains an element y_i.

The chain C has $j^2 - 2j + 1 = k$ intervals, but it cannot be stabbed by any j-tuple in $\mathcal{Z}$: It cannot be stabbed by a local j-tuple, since each block B_i contains at most $j-1$ intervals or parts thereof; and it cannot be stabbed by a global j-tuple, since the global j-tuples can only stab the long intervals, which number only $j-1$.

Therefore, there cannot exist such runs of unused elements. This implies that there are $\Omega(n)$ global j-tuples: At the very least, there must exist some B'_i in which every $(j-2)$-nd element is "used" by some global j-tuple.

We obtain the following recurrence relation:

$$z_k^{(j)}(n) \ge j z_k^{(j)}\left(\frac{n}{j}\right) + \Omega(n).$$

Thus, $z_k^{(j)}(n) = \Omega(n \log n)$. ∎

We now derive lower bounds for $z_k^{(j)}(n)$ for all k. As in the case of the upper bounds, we first deal with $j = 3$, and then with $j \ge 4$.

4.1 Lower bounds for triples

Our asymptotically tight lower bounds for triples are based on the following recurrence relation.

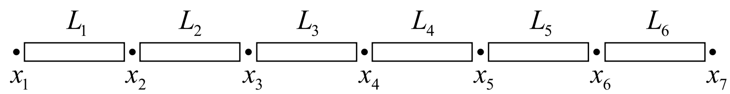

Figure 7: The m unused elements $x_1, \ldots, x_m$, from m distinct blocks, define $m-1$ nonempty "links" $L_1, \ldots, L_{m-1}$.

Recurrence 4.2 *Let t be an integer parameter, with $3 \le t \le \sqrt{n}$. Then,*

$$z^{(3)}_{k+2}(n) \ge \frac{n}{t} z^{(3)}_{k+2}(t) + \min\left(\frac{n}{18},\ z^{(3)}_k\left(\frac{n}{3t}\right)\right)$$

for all $n \ge 36$.

Proof: Let $b = \lceil n/t \rceil$. We define on the range $[1, n]$ a sequence of b blocks of size t, in which every two consecutive blocks overlap on exactly one element: Let $y_i = 1 + i(t-1)$ for $0 \le i \le b$. Note that $y_0 = 1$; and it can be checked that $y_b \le n$, since $n \ge t^2$. Then let

$$B_i = [y_{i-1}, y_i],$$

for $1 \le i \le b$. As before, we also let

$$B'_i = [y_{i-1} + 1, y_i - 1],$$

for $1 \le i \le b$ (refer again to Figure 6). Then, $|B_i| = t$ and $|B'_i| = t - 2$ for all i.

Let $\mathcal{Z}$ be a family of triples that stab all $(k+2)$-chains in $[1, n]$. As before, $\mathcal{Z}$ must contain b disjoint families of "local" triples that stab all chains in each block B_i. The total size of these families is at least $bz^{(3)}_{k+2}(t) \ge (n/t) z^{(3)}_{k+2}(t)$.

Now consider the "global" triples of $\mathcal{Z}$—those that are not contained in any block B_i. As before, consider the elements of the contracted blocks B'_i that are not contained in any global triple, and call them "unused".

Suppose that at most half the blocks B'_i contain unused elements. Then there must be $\Omega(n)$ global triples. More precisely, the number of global triples must be at least

$$\frac{1}{3} \cdot \frac{b}{2}(t-2) \ge \frac{n}{6}\left(1 - \frac{2}{t}\right) \ge \frac{n}{18},$$

since $t \ge 3$. In this case we are done.

Thus, suppose that at least half the blocks B'_i contain unused elements. Let $x_1, \ldots, x_m$ be m unused elements from m distinct blocks, with $m \ge b/2$. These elements define a sequence of $m-1$ intervals $L_i = [x_i + 1, x_{i+1} - 1]$ for $1 \le i \le m-1$, which we call "links" (see Figure 7). Each link L_i contains at least one element $y_{i'}$, so the links are nonempty.

Consider a k-chain $C' = I'_1 \cdots I'_k$ on the links, where $I'_i = [L_{a_i}, L_{a_{i+1}-1}]$ for some integers a_i, $1 \le i \le k+1$. We can translate C' into a $(k+2)$-chain $C = I_0 I_1 \cdots I_{k+1}$ on $[1, n]$, as follows: We make the unused elements right before I'_1 and after I'_k into

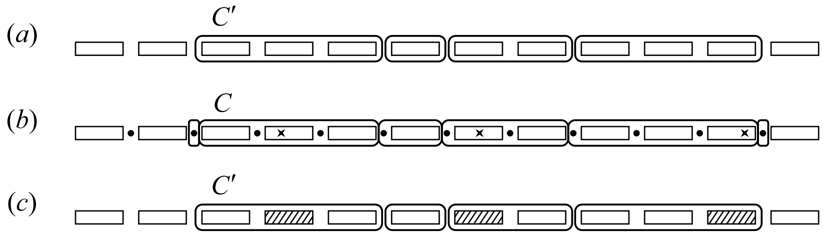

Figure 8: Every k-chain C' on the links (a) can be translated into a $(k+2)$-chain C on $[1, n]$ (b). A global triple (marked by x's) must stab C on three links. We can translate this triple back into a triple of links that stabs C' (c).

singleton intervals, and we append each intermediate unused element to the link at its right. Then we fuse the links in each I'_i into one interval. See Figure 8(a,b).

This chain C cannot be stabbed by any local triple, since each block B_i contains parts of at most two intervals of C. Thus, C must be stabbed by a global triple τ. Since τ does not contain any unused elements, it cannot stab the singleton intervals I_0 or I_{k+1}. Therefore, τ must stab three links on three different intervals among $I_1, \ldots, I_k$. Thus, we can translate τ back into a triple of links τ' that stabs C'. See Figure 8(c).

Hence, we have enough triples of links τ' to stab all k-chains on the $m-1$ links. The number of original global triples τ must be at least as large. Thus, there are at least $z_k^{(3)}(m-1)$ global triples. Finally, note that $m - 1 \geq n/(3t)$, since $n \geq 6\sqrt{n}$ for $n \geq 36$. ■

Lemma 4.3 *We have*

$$z_5^{(3)}(n) = \Omega(n \log \log n).$$

Proof: Apply Recurrence 4.2 with $k = 3$ and $t = \sqrt{n}$, and use Lemma 3.1. ■

Lemma 4.4 *There exists an absolute constant c_1 such that, for all $k \geq 6$, we have*

$$z_k^{(3)}(n) \geq c_1 n \alpha_{\lfloor k/2 \rfloor}(n) \quad \text{for all } n \geq n_k, \tag{8}$$

for some integers n_k that depend on k.

Proof: By induction from k to $k+2$. The base cases are $k = 6, 7$, which we derive from Recurrence 4.2 with $k = 4$ and $t = \log n$, and with $k = 5$ and $t = \log \log n$, respectively. We use the lower bounds for $z_4^{(3)}(n)$ and $z_5^{(3)}(n)$ of Lemmas 4.1 and 4.3, respectively, and we obtain

$$z_6^{(3)}(n), z_7^{(3)}(n) = \Omega(n \log^* n) = \Omega(n\alpha_3(n)).$$

(The recursion depth is $\log^* n$ for $z_6^{(3)}(n)$ and $\frac{1}{2}\log^* n$ for $z_7^{(3)}(n)$.)

Now, let $k \geq 6$, and let $m = \lfloor k/2 \rfloor$. Assume by induction that

$$z_k^{(3)}(n) \geq c_1 n \alpha_m(n), \quad \text{for all } n \geq n_k, \tag{9}$$

for some constants c_1 and n_k. Assume without loss of generality that $2c_1 \leq 1/18$. We apply Recurrence 4.2 with

$$t = \frac{1}{6}(\alpha_m(n) - 1).$$

Note that $\alpha_m(n)$ grows slowly enough that $\alpha_m(n/(3t)) \geq \alpha_m(n) - 1$ for all large enough n. Thus, let n' be a large enough constant (depending on k) such that this holds for all $n \geq n'$. Assume further that n' is large enough so that $3 \leq t \leq \sqrt{n}$ and $n/(3t) \geq n_k$ for all $n \geq n'$.

Then, by (9), for all $n \geq n'$ we have

$$z_k^{(3)}\left(\frac{n}{3t}\right) \geq c_1 \frac{n}{3t} \alpha_m\left(\frac{n}{3t}\right) \geq c_1 \frac{n}{3t}(\alpha_m(n) - 1) = 2c_1 n.$$

Plugging this into Recurrence 4.2 and letting $z_{k+2}^{(3)}(n) = ng(n)$, we obtain

$$g(n) \geq g(t) + 2c_1, \qquad \text{for all } n \geq n'.$$

It follows by Lemma B.1, given in Appendix B, that

$$g(n) \geq 2c_1\alpha_{m+1}(n) - O(1).$$

Thus, there exists an integer $n_{k+2} \geq n'$, such that $g(n) \geq c_1\alpha_{m+1}(n)$ for all $n \geq n_{k+2}$. We conclude that

$$z_{k+2}^{(3)}(n) \geq c_1 n\alpha_{m+1}(n), \quad \text{for all } n \geq n_{k+2},$$

completing our induction on k. ∎

Remark: We cannot expect (8) to hold for *all* n, since $z_k^{(3)}(k) = 1$. The integers n_k implied by the proof above actually grow very fast with k; the condition $t \geq 3$ for $n \geq n'$, together with $n_{k+2} \geq n'$, implies that $\alpha_{\lfloor k/2 \rfloor}(n_{k+2}) \geq 19$.

This proves the lower bounds of Theorem 1.4.

4.2 Lower bounds for j-tuples, $j \geq 4$

We now derive general lower bounds for $z_k^{(j)}(n)$, $j \geq 4$. We will construct a sequence of integer-valued functions $Q_j(m)$, $m \geq 2$, such that

$$z_{Q_j(2)}^{(j)}(n) \;=\; \Omega\Big(n \log^{(j-1)} n\Big); \qquad (10)$$

$$z_{Q_j(m)}^{(j)}(n) \;=\; \Omega\Big(n\alpha_m^{(j-2)}(n)\Big) = \omega(n\alpha_{m+1}(n)), \quad m \geq 3; \qquad (11)$$

for all $j \geq 4$. (Recall that $f^{(j)}$ denotes the j-fold composition of f.) Our arguments become more involved, because we now divide each block into *sub-blocks*. Let us start with the case $m = 2$ given by (10).

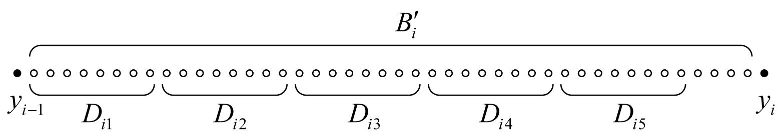

Figure 9: Sub-blocks defined within a contracted block B_i'.

Recurrence 4.5 *Let $j \geq 3$ be fixed. Let q be a parameter, with $q \leq n/(3j) - 2$. Let k_1, k_2 be integers, and put $k = 2k_1 + (j-2)k_2 + j - 1$. Then,*

$$z_k^{(j)}(n) \geq \min\left(\frac{n}{3jq} z_{k_1}^{(j-1)}(q),\ \frac{n}{3jq} z_{k_2}^{(j-2)}(q),\ j z_k^{(j)}\left(\frac{n}{j}\right) + \frac{n}{3j^2 q}\right)$$

for all $n \geq 6j$.

Proof: Let $t = \lceil n/j \rceil$. Define the elements $y_0, \ldots, y_j$, the blocks $B_1, \ldots, B_j$, and the contracted blocks $B_1', \ldots, B_j'$, as in the proof of Lemma 4.1. We have $|B_i| = t$ and $|B_i'| = t - 2$ for all i.

Define on each contracted block B_i' a sequence $D_{i1}, \ldots, D_{id}$ of $d = \lfloor (t-2)/q \rfloor$ disjoint *sub-blocks* of size q (these sub-blocks do not necessarily cover B_i' completely; see Figure 9). Note that $d \geq 2n/(3jq)$, since $q \leq n/(3j) - 2$.

Let $\mathcal{Z}$ be a family of j-tuples that stab all k-chains in $[1, n]$. For each i, let $\mathcal{Z}_i$ contain those j-tuples of $\mathcal{Z}$ that lie entirely inside B_i. Note that the families $\mathcal{Z}_i$ are pairwise disjoint.

Let $\mathcal{Z}_1'$ (resp., $\mathcal{Z}_j'$) be the family of $(j-1)$-tuples obtained by deleting the last (resp., first) element of each j-tuple in $\mathcal{Z}_1$ (resp., $\mathcal{Z}_j$). For each $2 \leq i \leq j-1$, let $\mathcal{Z}_i'$ be the family of $(j-2)$-tuples obtained by deleting the first *and* last elements of each j-tuple in $\mathcal{Z}_i$.

We say that a sub-block $D_{i\ell}$, $i \in \{1, j\}$, is *cleared* if the $(j-1)$-tuples in $\mathcal{Z}_i'$ stab all the k_1-chains in $D_{i\ell}$. And a sub-block $D_{i\ell}$, $2 \leq i \leq j-1$ is *cleared* if the $(j-2)$-tuples in $\mathcal{Z}_i'$ stab all the k_2-chains in $D_{i\ell}$.

A block B_i is *cleared* if at least *half* of its sub-blocks are cleared.

Now consider the "global" j-tuples of $\mathcal{Z}$—those that are not contained in any $\mathcal{Z}_i$. Let B_i' be an uncleared block. We say that B_i' is *safe* if every uncleared sub-block $D_{i\ell}$ within B_i' (of which there are at least $d/2$) contains some point of a global j-tuple.

Suppose all the blocks are uncleared and unsafe. Then we can build a k-chain C that cannot be stabbed by any j-tuple in $\mathcal{Z}$: For each $1 \leq i \leq j$, we take an uncleared sub-block $D_{i\ell_i}$ of block B_i' that is not "touched" by any global j-tuple. We take a "hardy" k_1-chain from each of the sub-blocks $D_{1\ell_1}$, $D_{j\ell_j}$, and a "hardy" k_2-chain from each intermediate block $D_{i\ell_i}$, $2 \leq i \leq j-1$. These "hardy" chains are chains that are not stabbed by any tuple in the respective families $\mathcal{Z}_i'$, and are also not touched any global j-tuple.

We connect the hardy chains together with $j - 1$ "long intervals" (see Figure 10). As before, the long intervals are nonempty, since each one contains an element y_i. The

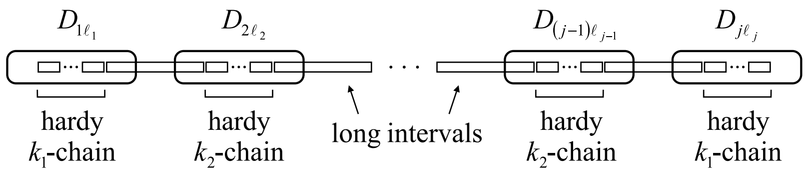

Figure 10: A k-chain which cannot be stabbed by any j-tuple, local or global.

total length of C is

$$2k_1 + (j-2)k_2 + j - 1 = k.$$

Now, C cannot be stabbed by a local j-tuple, because then the corresponding $(j-1)$- or $(j-2)$-tuple in $\mathcal{Z}'_i$ would stab a hardy chain. And C cannot be stabbed by a global j-tuple, since the global j-tuples can only stab the long intervals, which number only $j-1$.

Therefore, there are two possibilities. The first one is that all the blocks are uncleared, but at least one of them is safe. This implies that there are at least

$$\frac{1}{j} \cdot \frac{d}{2} \geq \frac{n}{3j^2 q}$$

global j-tuples. There must also be at least $j z_k^{(j)}(t)$ local j-tuples.[9]

The second possibility is that some block B'_i is cleared. If $i \in \{1, j\}$, this implies that

$$|\mathcal{Z}_i| \geq |\mathcal{Z}'_i| \geq \frac{d}{2} z_{k_1}^{(j-1)}(q) \geq \frac{n}{3jq} z_{k_1}^{(j-1)}(q).$$

And if $2 \leq i \leq j-1$, this implies that

$$|\mathcal{Z}_i| \geq |\mathcal{Z}'_i| \geq \frac{n}{3jq} z_{k_2}^{(j-2)}(q). \qquad \blacksquare$$

Now, let

$$Q_2(2) = 1; \quad Q_3(2) = 5;$$
$$Q_j(2) = 2Q_{j-1}(2) + (j-2)Q_{j-2}(2) + j - 1, \quad j \geq 4.$$

For $j \geq 4$ we have $Q_j(2) = 15, 49, 163, 577, 2139, \ldots$.

Lemma 4.6 *For every fixed $j \geq 2$ we have*

$$z_{Q_j(2)}^{(j)}(n) = \Omega\Big(n \log^{(j-1)} n\Big),$$

where the constant of proportionality depends on j.

[9] This of course holds in any case.

Proof: By induction on j. The case $j = 2$ is trivial, since it is impossible to stab a 1-chain with a pair, so $z_1^{(2)}(n) = \infty$. And the case $j = 3$ is given by Lemma 4.3. So let $j \geq 4$. Apply Recurrence 4.5 with

$$k_1 = Q_{j-1}(2), \quad k_2 = Q_{j-2}(2), \quad k = Q_j(2), \quad q = \log n.$$

By induction, we have

$$\begin{aligned} \frac{n}{3jq} z_{k_1}^{(j-1)}(q) &= \Omega\Big(n \log^{(j-1)} n\Big); \\ \frac{n}{3jq} z_{k_2}^{(j-2)}(q) &= \Omega\Big(n \log^{(j-2)} n\Big) = \omega\Big(n \log^{(j-1)} n\Big). \end{aligned}$$

Now, consider the recurrence relation[10]

$$f(n) \geq j f\left(\frac{n}{j}\right) + \frac{n}{\log n}.$$

This recurrence has solution $f(n) = \Omega(n \log \log n) = \omega\Big(n \log^{(j-1)} n\Big)$. Therefore, substituting into Recurrence 4.5, we get $z_{Q_j(2)}^{(j)}(n) = \Omega\Big(n \log^{(j-1)} n\Big)$, as desired. ∎

We now derive the bounds (11). We use the following recurrence.

Recurrence 4.7 *Let j be fixed. Let t and q be parameters, with $t \leq \sqrt{n}$ and $q \leq t/9 - 2$. Let k_1, k_2, k_3 be integers, and put $k = 2k_1 + (k_2 + 1)(k_3 - 1) + 1$. Then,*

$$z_k^{(j)}(n) \geq \min\left(\frac{n}{9q} z_{k_1}^{(j-1)}(q),\ \frac{n}{9q} z_{k_2}^{(j-2)}(q),\ \frac{n}{t} z_k^{(j)}(t) + \min\left(\frac{n}{9jq},\ z_{k_3}^{(j)}\left(\frac{n}{3t}\right)\right)\right)$$

for all $n \geq 36$.

Proof: Let $b = \lceil n/t \rceil$, and define the elements $y_0, \ldots, y_b$, the blocks $B_1, \ldots, B_b$, and the contracted blocks $B'_1, \ldots, B'_b$ as in the proof of Recurrence 4.2. We have $|B_i| = t$ and $|B'_i| = t - 2$ for all i.

As in the proof of Recurrence 4.5, define on each contracted block B'_i a sequence $D_{i1}, \ldots, D_{id}$ of $d = \lfloor (t-2)/q \rfloor$ disjoint "sub-blocks" of size q.

Let $\mathcal{Z}$ be a family of j-tuples that stab all k-chains in $[1, n]$. For each i, let $\mathcal{Z}_i$ be the family of j-tuples of $\mathcal{Z}$ that are entirely contained in block B_i. The families $\mathcal{Z}_i$ are pairwise disjoint, and each one has size at least $z_k^{(j)}(t)$.

For each i, let $\mathcal{Z}_i^{(1)}$ be the family of $(j-1)$-tuples obtained by removing the *last* element of each j-tuple in $\mathcal{Z}_i$. Let $\mathcal{Z}_i^{(2)}$ be the family of $(j-2)$-tuples obtained by removing the *first and last* elements of each j-tuple in $\mathcal{Z}_i$. And let $\mathcal{Z}_i^{(3)}$ be the family of $(j-1)$-tuples obtained by removing the *first* element of each j-tuple in $\mathcal{Z}_i$.

[10] It is correct, in a recurrence like Recurrence 4.5, to consider each case separately and then take the minimum.

Let $D_{i\ell}$ be a sub-block within block B_i'. We say that $D_{i\ell}$ is *left-cleared* (resp., *right-cleared*) if the $(j-1)$-tuples of $\mathcal{Z}_i^{(1)}$ (resp., $\mathcal{Z}_i^{(3)}$) stab all the k_1-chains in $D_{i\ell}$. And we say that $D_{i\ell}$ is *middle-cleared* if the $(j-2)$-tuples of $\mathcal{Z}_i^{(2)}$ stab all the k_2-chains in $D_{i\ell}$.

Now consider the "global" j-tuples of $\mathcal{Z}$—those that are not contained in any block B_i. We say that a sub-block $D_{i\ell}$ is *visited* if it contains some point of a global j-tuple.

If a sub-block $D_{i\ell}$ is neither left-, middle-, nor right-cleared, nor is it visited, then $D_{i\ell}$ is *hot*; otherwise, it is cold. A hot sub-block contains three hardy chains $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ (not necessarily disjoint), of lengths k_1, k_2, and k_1, respectively, which are not stabbed by any tuple in $\mathcal{Z}_i^{(1)}$, $\mathcal{Z}_i^{(2)}$, $\mathcal{Z}_i^{(3)}$, respectively, and are not "touched" by any global j-tuple.

A block B_i' is *hot* if it contains some hot sub-block $D_{i\ell}$; otherwise, it is cold.

Now, suppose that at least half the blocks B_i' are cold. Then, there is a total of at least $bd/2$ cold sub-blocks. Therefore, there must be at least

$$\frac{1}{4} \cdot \frac{bd}{2} \geq \frac{n}{9q}$$

sub-blocks which are either all left-cleared, or all middle-cleared, or all right-cleared, or all visited. (Note that $d \geq 8t/(9q)$, since $q \leq t/9 - 2$.)

The first or third case implies

$$|\mathcal{Z}| \geq \frac{n}{9q} z_{k_1}^{(j-1)}(q);$$

while the second case implies

$$|\mathcal{Z}| \geq \frac{n}{9q} z_{k_2}^{(j-2)}(q).$$

Finally, the fourth case implies that $\mathcal{Z}$ contains at least $n/(9jq)$ global j-tuples, plus at least $(n/t)z_k^{(j)}(t)$ local j-tuples.

Suppose, then, that there are $m \geq n/(2t)$ hot blocks B_i'. Let $K_1, \ldots, K_m$ be m hot sub-blocks from m distinct such blocks. These sub-blocks define a sequence of $m-1$ nonempty "links" $L_1, \ldots, L_{m-1}$ between them, as in the proof of Recurrence 4.2. Each sub-block K_i contains the hardy chains $H_i^{(1)}$, $H_i^{(2)}$, $H_i^{(3)}$ mentioned above.

Consider a k_3-chain $C' = I_1' \ldots I_{k_3}'$ on the links. This chain is uniquely determined by a sequence of $k_3 + 1$ sub-blocks

$$K_{a_1}, K_{a_2}, \ldots, K_{a_{k_3+1}}, \tag{12}$$

where each interval I_i' contains those links that lie between K_{a_i} and $K_{a_{i+1}}$.

We can translate C' into the k-chain

$$C = H_{a_1}^{(1)} I_1 H_{a_2}^{(2)} I_2 \cdots I_{k_3-1} H_{a_{k_3}}^{(2)} I_{k_3} H_{a_{k_3+1}}^{(3)},$$

on $[1, n]$, where each interval I_i extends from the end of one hardy chain to the beginning of the next. The number of intervals in C is

$$2k_1 + (k_3 - 1)k_2 + k_3 = k.$$

$m =$	2	3	4	5	6	7
$Q_2(m) =$	1	1	1	1	1	1
$Q_3(m) =$	5	7	9	11	13	15
$Q_4(m) =$	15	43	103	227	479	987
$Q_5(m) =$	49	471	4907	59327	831523	13306327
$Q_6(m) =$	163	8071	849095	193712087	$\cdots$	

Table 2: Values of $Q_j(m)$ for small j and m.

Now, C cannot be stabbed by any local j-tuple from a block K_{a_i}, since then the corresponding hardy chain $H_{a_i}^{(x)}$ would be stabbed by a tuple from $\mathcal{Z}_{a_i}^{(x)}$ (for an appropriate $x \in \{1, 2, 3\}$). Therefore, C must be stabbed by a global j-tuple $\tau \in \mathcal{Z}$. Further, τ must stab j links from j different intervals I_i (since none of the chains $H_{a_i}^{(x)}$ is touched by τ). Thus, we can translate τ back into a j-tuple of links τ' that stabs C'.

Hence, there are at least $z_{k_3}^{(j)}(m-1) \geq z_{k_3}^{(j)}(n/(3t))$ global j-tuples, plus at least $(n/t)z_k^{(j)}(t)$ local j-tuples. ∎

Define integer-valued functions $Q_j(m)$, for $j, m \geq 2$, by

$$Q_2(m) = 1; \quad Q_3(m) = 2m + 1;$$

and for $j \geq 4$,

$$Q_j(m) = 2Q_{j-1}(m) + \Big(1 + Q_{j-2}(m)\Big)\Big(Q_j(m-1) - 1\Big) + 1, \quad m \geq 3;$$

with $Q_j(2)$ as defined above. See Table 2.

We have $Q_4(m) = 8 \cdot 2^m - 4m - 9$, and in general, letting $s = \lfloor (j-2)/2 \rfloor$,

$$Q_j(m) = \begin{cases} 2^{(1/s!)m^s + O(m^{s-1})}, & \text{for } j \geq 4 \text{ even;} \\ 2^{(1/s!)m^s \log_2 m + O(m^s)}, & \text{for } j \geq 3 \text{ odd;} \end{cases}$$

just as in the case of $P_j(m)$.

Lemma 4.8 *For every $j \geq 2$ and $m \geq 3$ we have*

$$z_{Q_j(m)}^{(j)}(n) = \Omega\Big(n\alpha_m^{(j-2)}(n)\Big)$$

(where the implicit constants might depend on both m and j).

Proof: The case $j = 2$ is trivial, and the case $j = 3$ is given by Lemma 4.4. So let $j \geq 4$.

We apply Recurrence 4.7 with the following parameters:

$$k_1 = Q_{j-1}(m), \quad k_2 = Q_{j-2}(m), \quad k_3 = Q_j(m-1), \quad k = Q_j(m).$$

We first handle the case $m = 3$, by induction on j. For this, let $t = \log^{(j-1)} n$ and $q = \alpha_3(n)$. Then, by induction we have

$$\begin{aligned} \frac{n}{9q} z_{k_1}^{(j-1)}(q) &= \Omega\Big(n\alpha_3^{(j-2)}(n)\Big), \\ \frac{n}{9q} z_{k_2}^{(j-2)}(q) &= \Omega\Big(n\alpha_3^{(j-3)}(n)\Big) = \omega\Big(n\alpha_3^{(j-2)}(n)\Big). \end{aligned}$$

Now, consider the recurrence relation

$$f(n) \geq \frac{n}{t} f(t) + \frac{n}{q}. \tag{13}$$

We have $\alpha_3\Big(\log^{(i)} n\Big) = \alpha_3(n) - i$ for every integer $i \geq 0$. Hence, (13) expands into an harmonic-like series, which yields $f(n) = \Omega(n \log \alpha_3(n)) = \omega\Big(n\alpha_3^{(j-2)}(n)\Big)$. Finally, by Lemma 4.6 we have

$$z_{k_3}^{(j)}\Big(\frac{n}{3t}\Big) = \Omega\Big(\frac{n}{t} \log^{(j-1)} \frac{n}{3t}\Big) = \Omega(n).$$

The solution of the recurrence $f(n) \geq (n/t)f(t) + \Omega(n)$ is $f(n) = \Omega(n\alpha_3(n))$, which is also $\omega\Big(n\alpha_3^{(j-2)}(n)\Big)$. Plugging into Recurrence 4.7, we get $z_{Q_j(3)}^{(j)}(n) = \Omega\Big(n\alpha_3^{(j-2)}(n)\Big)$, as desired.

Now we handle the general case $m \geq 4$ by induction. Let $t = \alpha_{m-1}^{(j-2)}(n)$ and $q = \alpha_m(n)$. Then, by induction on j we have

$$\begin{aligned} \frac{n}{9q} z_{k_1}^{(j-1)}(q) &= \Omega\Big(n\alpha_m^{(j-2)}(n)\Big), \\ \frac{n}{9q} z_{k_2}^{(j-2)}(q) &= \Omega\Big(n\alpha_m^{(j-3)}(n)\Big) = \omega\Big(n\alpha_m^{(j-2)}(n)\Big). \end{aligned}$$

Again, consider the recurrence relation (13). This time, we get $f(n) = \Omega(n \log \alpha_m(n)) = \omega\Big(n\alpha_m^{(j-2)}(n)\Big)$. And by induction on m we have

$$z_{k_3}^{(j)}\Big(\frac{n}{3t}\Big) = \Omega\Big(\frac{n}{t} \alpha_{m-1}^{(j-2)}\Big(\frac{n}{3t}\Big)\Big) = \Omega(n).$$

The solution of the recurrence $f(n) \geq (n/t)f(t) + \Omega(n)$, for our choice of t, is $f(n) = \Omega(n\alpha_m(n))$, which is $\omega\Big(n\alpha_m^{(j-2)}(n)\Big)$. Plugging into Recurrence 4.7, we get $z_{Q_j(m)}^{(j)}(n) = \Omega\Big(n\alpha_m^{(j-2)}(n)\Big)$, as desired. ∎

Define $Q'_j(m)$ for $j \geq 4$, $m \geq 2$, by

$$\begin{aligned} Q'_j(2) &= j; \\ Q'_j(m) &= Q_j(m-1), \quad m \geq 3. \end{aligned}$$

Then, using the fact that $\alpha_{m-1}^{(j-1)}(n) = \omega(\alpha_m(n))$ for $m \geq 2$, we conclude by Lemmas 3.1, 4.6, and 4.8 that

$$z_{Q'_j(m)}^{(j)}(n) = \omega(n\alpha_m(n)), \quad \text{for all } j \geq 4, m \geq 2.$$

This proves the lower bounds in Theorem 1.5.

Remark: We could have derived the asymptotic lower bounds of Theorem 1.5 somewhat more simply, as follows: We omit Recurrence 4.5 and the resulting Lemma 4.6, and instead we start our induction on m with the $\Omega(n \log n)$ bound of Lemma 4.1. Further, in Recurrence 4.7 we can omit the role of the k_1-chains and the families of $(j-1)$-tuples $\mathcal{Z}_i^{(1)}$ and $\mathcal{Z}_i^{(3)}$. This would not have affected the asymptotic growth of the sequences $Q_j(m)$, $Q'_j(m)$.

However, we chose to present the largest values of $Q'_j(m)$ we were able to obtain with our techniques, especially since the extra effort involved is not significant.

5 Discussion

Open problems. The most pressing issue is to close the gap between the bounds $\Omega(r)$ and $O(r\alpha(r))$ for the size of weak $\frac{1}{r}$-nets for planar sets in convex position. A worst-case bound of $\Theta(r\alpha(r))$ would be a major achievement, since there are no known superlinear lower bounds for weak ϵ-nets for any fixed dimension d, even for arbitrary point sets.

Another open issue is to determine how tight the bounds are for the case of point sets along the moment curve μ_d. For example, does j really have to be quadratic in d in Lemma 2.2?

It would also be nice to find the exact asymptotic form of $z_k^{(j)}(n)$ for every fixed j and k.

Partial sums in semigroups. Our divide-and-conquer approach to the problem of stabbing interval chains with triples ($j = 3$) is very similar to the approach of Alon and Schieber [3], for a problem related to offline computation of partial sums in semigroups (see also [8, 16]). The problem there can be abstractly formulated as follows.

We are given the range $[1, n]$ and an integer k. We want to construct a family $\mathcal{Y}$ of subsets of $[1, n]$, with $|\mathcal{Y}|$ as small as possible, such that every interval $[a, b]$, $1 \le a \le b \le n$, can be expressed as the union of at most k sets from $\mathcal{Y}$. Let $y_k(n)$ denote the minimum size of such a family $\mathcal{Y}$. Then,

$$y_1(n) = \binom{n+1}{2}; \quad y_2(n) = \Theta(n \log n); \quad y_3(n) = \Theta(n \log \log n);$$

$$y_k(n) = \Theta\big(n\alpha_{\lfloor k/2 \rfloor + 1}(n)\big), \quad k \ge 4.$$

In fact, these upper bounds can be achieved even if we require the sets in $\mathcal{Y}$ to be intervals, and we require every $[a, b]$ to be expressed as a *disjoint* union of such intervals. (We note that, even though the proof techniques are very similar, we are not aware of any explicit reduction between the two problems.)

Davenport–Schinzel sequences. The similarities between our bounds for interval chains and the bounds for Davenport–Schinzel sequences are nothing short of remarkable.

The current bounds for $\lambda_s(n)$, the maximum length of a Davenport–Schinzel sequence of order s on n symbols, are as follows (Sharir and Agarwal [15]). Let $t = \lfloor (s-2)/2 \rfloor$. Then,

$$\lambda_3(n) = \Theta(n\alpha(n)); \quad \lambda_4(n) = \Theta\Big(n \cdot 2^{\alpha(n)}\Big).$$

For $s \geq 5$ there are upper bounds of

$$\begin{aligned} \lambda_s(n) &\leq n \cdot 2^{\alpha(n)^t + C_s(n)}, && \text{for } s \text{ even;} \\ \lambda_s(n) &\leq n \cdot 2^{\alpha(n)^t \log_2 \alpha(n) + C_s(n)}, && \text{for } s \text{ odd;} \end{aligned}$$

where $C_s(n)$ are functions of $\alpha(n)$ of lower order than the first term in the exponent. And for $s \geq 6$ there is a lower bound of

$$\lambda_s(n) \geq n \cdot 2^{(1/t!)\alpha(n)^t + O(\alpha(n)^{t-1})}.$$

Note that, for s even, there are gaps in the coefficients of $\alpha(n)^t$ between the upper and lower bounds, and for s odd there are no lower bounds with the $\log \alpha(n)$ factor in the exponent.

Compare these bounds to our bounds for $P'_j(m)$, $Q'_j(m)$ in (2), and to the resulting bounds for weak ϵ-nets in Theorems 1.1 and 1.2. The similarity is striking.

There is a significant difference, however. The bounds for $\lambda_s(n)$ involve the inverse Ackermann function $\alpha(n)$, while the bounds for interval chains involve functions $\alpha_m(n)$ of the inverse Ackermann hierarchy. However, once we go from interval chains to weak $\frac{1}{r}$-nets, we obtain upper bounds involving $\alpha(r)$.

In any case, in light of these similarities, the following conjecture suggests itself (and perhaps also a line of attack for proving it):

Conjecture 5.1 *The true bounds for $\lambda_s(n)$, $s \geq 5$, are*

$$\begin{aligned} \lambda_s(n) &= n \cdot 2^{(1/t!)\alpha(n)^t + O(\alpha(n)^{t-1})}, && \text{for } s \text{ even;} \\ \lambda_s(n) &= n \cdot 2^{(1/t!)\alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)}, && \text{for } s \text{ odd;} \end{aligned}$$

where $t = \lfloor (s-2)/2 \rfloor$.

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A Bounds for stabbing with pairs

We give almost-tight bounds on the number of *pairs* needed to stab all k-chains in $[1, n]$.

Lemma A.1 *We have*

$$\frac{n}{\lfloor k/2 \rfloor} - 3 \leq z_k^{(2)}(n) \leq \frac{n}{\lfloor k/2 \rfloor} - 1.$$

Proof: For the upper bound, let k be even, and let $q = k/2$. Take the family of pairs

$$\mathcal{Z} = \{(iq, (i+1)q) \mid 1 \leq i \leq n/q - 1\}.$$

It is easily verified that in any k-chain C, there must be at least two different intervals that contain elements of the form iq. Therefore, there must be two adjacent elements iq, $(i+1)q$ that fall on two different intervals, so C is stabbed. We have

$$|\mathcal{Z}| = \left\lfloor \frac{n}{q} - 1 \right\rfloor \leq \frac{n}{\lfloor k/2 \rfloor} - 1,$$

and we are done.

For the lower bound, let k be odd, and let $q = (k-1)/2$. Let

$$\mathcal{Z} = \{(x_i, y_i) \mid 1 \leq i \leq m\}$$

be a family of pairs that stabs all k-chains in $[1, n]$, with $x_i < y_i$ for all i. Let $X = \{x_i \mid 1 \leq i \leq m\}$.

We may assume that there exists an integer $a_1 \in [1, n - q + 1]$ such that

$$X \cap [a_1, a_1 + q - 1] = \emptyset,$$

for otherwise we have $|\mathcal{Z}| = |X| \geq \lfloor n/q \rfloor \geq n/q - 1$ and we are done. Let a_1 be the smallest integer with the above property. Partition X into

$$\begin{aligned} X_1 &= X \cap [1, a_1 - 1], \\ X_2 &= X \cap [a_1, n]. \end{aligned}$$

By the minimality of a_1, we have

$$|X_1| \geq \left\lfloor \frac{a_1 - 1}{q} \right\rfloor \geq \frac{a_1}{q} - 1.$$

Let $Y = \{y_i \mid x_i \in X_2\}$. Suppose there exists an integer $a_2 \in [a_1 + q + 1, n - q + 1]$ such that

$$Y \cap [a_2, a_2 + q - 1] = \emptyset.$$

Then the k-chain consisting of the q singletons $[a_1] \cdots [a_1 + q - 1]$, followed by the interval $[a_1 + q, a_2 - 1]$, followed by the q singletons $[a_2] \cdots [a_2 + q - 1]$, cannot be stabbed by any pair in $\mathcal{Z}$, as is easily checked.

Thus, such an integer a_2 cannot exist, so we have

$$|X_2| = |Y| \geq \left\lfloor \frac{n - a_1 - q}{q} \right\rfloor \geq \frac{n - a_1}{q} - 2,$$

so $|X| = |X_1| + |X_2| \geq n/q - 3$. ∎

B Comparing functions defined by recurrence relations

Let $f(x)$ and $g(x)$ be functions satisfying $f(x), g(x) < x$ for all large enough x, and let $f^*(x)$, $g^*(x)$ be given by the recurrence relations

$$\begin{aligned} f^*(x) &= 1 + f^*(f(x)), \\ g^*(x) &= 1 + g^*(g(x)), \end{aligned}$$

with appropriate initial conditions for small enough x. In this appendix we show a sufficient condition for establishing that

$$|f^*(x) - g^*(x)| = O(1). \tag{14}$$

In this paper we make frequent use of bounds of this type.

Let us assume for simplicity that $g(x) \geq f(x)$ for all large enough x. Then, it is enough to establish an upper bound on $g^*(x) - f^*(x)$.

Lemma B.1 *Let $f(x), g(x), f^*(x), g^*(x)$ be functions as given above. Suppose there exists a function $\delta(x)$ and a real number x_1 such that*

$$x \leq \delta(x), \tag{15}$$

$$g(\delta(x)) \leq \delta(f(x)), \tag{16}$$

for all $x \geq x_1$. Then,

$$g^*(x) - f^*(x) \leq 1 + g^*(\delta(x_1)) - f^*(x_1) \tag{17}$$

for all $x \geq x_1$.

Proof: Given $x \geq x_1$, let $j = j(x)$ be the smallest integer such that $f^{(j)}(x) < x_1$. (Here $f^{(j)}$ denotes the j-fold composition of f.) Thus, $f^{(j-1)}(x) \geq x_1$, so

$$f^*(x) = (j-1) + f^*\left(f^{(j-1)}(x)\right) \geq (j-1) + f^*(x_1). \tag{18}$$

Then, by (15) and repeated application of (16),

$$\begin{aligned} g^{(j)}(x) \leq g^{(j)}(\delta(x)) &= g^{(j-1)}(g(\delta(x))) \\ &\leq g^{(j-1)}(\delta(f(x))) \\ &\vdots \\ &\leq g\left(\delta\left(f^{(j-1)}(x)\right)\right) \leq \delta\left(f^{(j)}(x)\right) \leq \delta(x_1). \end{aligned}$$

Thus,

$$g^*(x) = j + g^*\left(g^{(j)}(x)\right) \leq j + g^*(\delta(x_1)).$$

This, together with (18), yields (17), as desired. ∎

Thus, the problem of establishing a bound of the form (14) reduces to finding an appropriate function δ. We illustrate the utility of Lemma B.1 with a few examples.

Example 1: Let $f(x) = cx$ and $g(x) = cx + d$, for some constants $0 < c < 1$ and $d > 0$. If we let $\delta(x) = x + d/(1-c)$, then we have $g(\delta(x)) = \delta(f(x))$. Thus, by Lemma B.1, we have $|f^*(x) - g^*(x)| = O(1)$. Since we have $f^*(x) = \log_{1/c} x + O(1)$ (where the additive constant depends on the initial condition for small x), we conclude that also $g^*(x) = \log_{1/c} x + O(1)$.

Example 2: Let $f(x) = x^c$ and $g(x) = dx^c$, for some constants $0 < c < 1$ and $d > 1$. If we let $\delta(x) = d^{1/(1-c)}x^{1/c}$, then again we have $g(\delta(x)) = \delta(f(x))$. Thus, $f^*(x)$ and $g^*(x)$ are both of the form $\log_{1/c} \log x + O(1)$.

Example 3: Let $f(x) = \alpha_k(x)$ and $g(x) = \alpha_k(x)^c$, for some integer $k \geq 2$ and some $c > 1$. Suppose first that $k = 2$ (and recall that $\alpha_2(x) = \lceil \log_2 x \rceil$). Let

$$\delta(x) = (cx + c\log_2 c + c + 1)^c.$$

Using the fact that $\log_2(cx + k) \leq 1 + \log_2 cx$ for $x \geq k/c$, we have for all large enough x,

$$\begin{aligned}
g(\delta(x)) &\leq (1 + \log_2 \delta(x))^c \\
&= (1 + c\log_2(cx + c\log_2 c + c + 1))^c \\
&\leq (1 + c(\log_2 cx + 1))^c \\
&= \delta(\log_2 x) \;\leq\; \delta(f(x)),
\end{aligned}$$

so $\delta(x)$ satisfies (16). We conclude that $f^*(x)$ and $g^*(x)$ are both of the form $\log^* x + O(1)$.

If $k \geq 3$, then we simply take $\delta(x) = (x + 1)^c$. Once x is large enough, we have $\alpha_k(\delta(x)) \leq 1 + \alpha_k(x)$, so

$$g(\delta(x)) \leq (1 + \alpha_k(x))^c = \delta(f(x)),$$

again satisfying (16). Thus, $f^*(x)$ and $g^*(x)$ are both of the form $\alpha_{k+1}(x) + O(1)$.

It can be shown that the functions $\widehat{\alpha}_k(x)$ used in the proofs of Lemmas 3.5 and 3.8 satisfy

$$|\widehat{\alpha}_k(x) - \alpha_k(x)| \leq c$$

for all large enough k and all x, for some absolute constant c. This is done by an argument similar to that of Example 3 above, though slightly more involved, using induction on k. We omit the details.