Problems and results in Extremal Combinatorics, Part I

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Abstract

Extremal Combinatorics is an area in Discrete Mathematics that has developed spectacularly during the last decades. This paper contains a collection of problems and results in the area, including solutions or partial solutions to open problems suggested by various researchers in Extremal Graph Theory, Extremal Finite Set Theory and Combinatorial Geometry. This is not meant to be a comprehensive survey of the area, it is merely a collection of various extremal problems, which are hopefully interesting. The choice of the problems is inevitably somewhat biased, and as the title of the paper suggests I hope to write a related paper in the future. Each section of this paper is essentially self contained, and can be read separately.

1 Introduction

Extremal Combinatorics deals with the problem of determining or estimating the maximum or minimum possible cardinality of a collection of finite objects that satisfies certain requirements. Such problems are often related to other areas including Computer Science, Information Theory, Number Theory and Geometry. This branch of Combinatorics has developed spectacularly over the last few decades, see, e.g., [10], [31], and their many references.

This paper contains a collection of problems and results in the area, including solutions or partial solutions to open problems suggested by various researchers in Extremal Graph Theory, Extremal Finite Set Theory and Combinatorial Geometry. This is not meant to be a comprehensive survey of the area, but rather a collection of several extremal problems, which are hopefully interesting. The techniques used include combinatorial, probabilistic and algebraic tools. Each section of this paper is essentially self contained, and can be read separately.

2 Multiple intersection of set systems

Let $\mathcal{F}$ be a finite family of sets. For each $k \geq 1$, define $\mathcal{F}_k = \{F_1 \cap \ldots \cap F_k : \; F_i \in \mathcal{F}\}$. Thus $\mathcal{F}_k$ is the set of all intersections of k (not necessarily distinct) members of $\mathcal{F}$. Motivated by a question studied in [26], Gyárfás and Ruszinkó [27] raised the problem of estimating the maximum possible

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cardinality of $\mathcal{F}_3$, given the cardinality of $\mathcal{F}_2$. They noticed that there are examples of families $\mathcal{F}$ for which $|\mathcal{F}_3| \geq \Omega(|\mathcal{F}_2|^{3/2})$, and observed that $|\mathcal{F}_3| \leq O(|\mathcal{F}_2|^2)$. Indeed, if $\mathcal{F}$ is the family of all subsets of cardinality $n-1$ of an n element set, then $|\mathcal{F}_2| = n + \binom{n}{2}$ and $|\mathcal{F}_3| = n + \binom{n}{2} + \binom{n}{3} = \Omega(|\mathcal{F}_2|^{3/2})$. The upper bound $|\mathcal{F}_3| \leq O(|\mathcal{F}_2|^2)$ follows from the fact that every member of $\mathcal{F}_3$ is the intersection of a member of $\mathcal{F}_2$ with one of $\mathcal{F}_1$ ($\subset \mathcal{F}_2$).

It turns out that a tight upper bound for $|\mathcal{F}_3|$ in terms of $|\mathcal{F}_2|$ can be derived from the Kruskal-Katona Theorem, using a simple shifting technique. Moreover, the same technique applies for bounding $|\mathcal{F}_r|$ as a function of $|\mathcal{F}_k|$ for all $r > k$.

Theorem 2.1 *Let $\mathcal{F}$ be a finite collection of sets, and suppose $r > k \geq 1$ are integers. Then, for every real $x \geq r$, if*

$$|\mathcal{F}_k| < \sum_{i=1}^{k} \binom{x}{i}. \tag{1}$$

then

$$|\mathcal{F}_r| < \sum_{i=1}^{r} \binom{x}{i}. \tag{2}$$

This is tight for every integral value of x in the sense that for every integral x there are examples in which equality holds in both inequalities above.

As mentioned above, the proof relies on the Kruskal-Katona Theorem [34], [32]. For convenience, we use here the version of Lovász ([35], Problem 13.31), which simplifies the formulae. It is, however, worth noting that by using the precise statement of the theorem one can slightly improve the assertion above for non-integral values of x.

Theorem 2.2 *[The Kruskal-Katona Theorem, [34], [32], [35]] Let r be an integer, and let H be an r-uniform hypergraph with no multiple edges which contains at least $\binom{x}{r}$ edges, where $x \geq r$ is real. Then for every j, $1 \leq j \leq r$, there are at least $\binom{x}{j}$ j-tuples contained in edges of H.*

Proof of Theorem 2.1: Let $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ be an arbitrary enumeration of the sets in $\mathcal{F}$. Order all nonempty subsets of $\mathcal{F}$ lexicographically as follows. For two subsets $\mathcal{G} = \{F_{i_1}, F_{i_2}, \ldots, F_{i_s}\}$ and $\mathcal{T} = \{F_{j_1}, F_{j_2}, \ldots, F_{j_t}\}$ of $\mathcal{F}$, where $i_1 < i_2 \ldots < i_s$ and $j_1 < j_2 < \ldots < j_t$, $\mathcal{G}$ precedes $\mathcal{T}$ in the order if $s < t$, or if $s = t$ and $i_m < j_m$ where m is the smallest index such that i_m and j_m differ.

As $F \cap F = F$ for all $F \in \mathcal{F}$, we can express each member of $\mathcal{F}_j$ as an intersection of at most j distinct members of $\mathcal{F}$. For each positive integer $j \leq r$, let $H(j)$ be the hypergraph whose vertices are the members of $\mathcal{F}$ and whose edges are all sets $\mathcal{G}$ defined as follows: for each $A \in \mathcal{F}_j$, express A as an intersection of the sets in $\mathcal{G}$, where $\mathcal{G}$ is the lexicographically first subset of $\mathcal{F}$ the intersection of whose members is A. Note that the cardinality of each such $\mathcal{G}$ is at most j, and that all singletons $\{F_i\}$ are edges of $H(j)$. Obviously, the number of edges of $H(j)$ is precisely $|\mathcal{F}_j|$. The crucial point is that if $\mathcal{G}$ is an edge of $H(r)$, $\mathcal{T}$ is a subset of $\mathcal{G}$, and $|\mathcal{T}| = j$, then $\mathcal{T}$ is an edge of $H(j)$, as well as an edge of each $H(i)$ for $i \geq j$. Indeed, if $\mathcal{T}$ is not an edge of $H(i)$ for some $i \geq j$, then there is a subset $\mathcal{T}' \subset \mathcal{F}$ which is lexicographically smaller than $\mathcal{T}$, with the same intersection. Thus $|\mathcal{T}'| \leq |\mathcal{T}| = j$ and hence $\mathcal{G}' = (\mathcal{G} - \mathcal{T}) \cup \mathcal{T}'$ is lexicographically smaller than $\mathcal{G}$, and its members have the same intersection, contradicting the fact that $\mathcal{G}$ is an edge of $H(r)$.

Suppose, now, that (1) holds. We claim that in this case for every i, $k \le i \le r$ the number of edges of cardinality i in $H(r)$ is smaller than $\binom{x}{i}$. Indeed, otherwise, by the last paragraph and by Theorem 2.2 it would follow that the number of edges of $H(k)$ of cardinality j, for every $1 \le j \le k$, is at least $\binom{x}{j}$, contradicting (1). As, by the last paragraph, every edge of cardinality at most k of $H(r)$ is also an edge of $H(k)$, this implies (2).

The set $\mathcal{F}$ of all subsets of cardinality $n-1$ of a set of cardinality n shows that equality can hold in (1) and (2) for all integral values of x ($= n$). This completes the proof of the theorem. □

Remark: The proof above works not only for subset intersection, but for any commutative, associative semi-group of idempotents. Let G be a (finite or infinite) semi-group, with a commutative, associative binary operation satisfying $g \cdot g = g$ for all $g \in G$. For a finite subset A of G and for each $k \ge 1$, define $A_k = \{g_1 \cdot g_2 \ldots \cdot g_k : \ g_i \in A\}$. In this notation, the proof above, with essentially no difference, gives the following.

Theorem 2.3 *Let G and A be as above and suppose $r > k \ge 1$ are integers. Then, for every real $x \ge r$, if*

$$|A_k| < \sum_{i=1}^{k} \binom{x}{i}$$

then

$$|A_r| < \sum_{i=1}^{r} \binom{x}{i}.$$

When G is the set of all subsets of some ground set, and the semi-group operation is set intersection, the above reduces to Theorem 2.1. Another (equivalent) example is obtained by letting the operation be union rather than intersection. The requirement that all elements are idempotents is not essential, and a similar result can be proved without this assumption.

3 Nearly spanning regular subgraphs of regular graphs

All graphs considered in this section are finite and simple. It is known (see [5]) that for every integer k there is some $r_0 = r_0(k)$ so that every r-regular graph with $r > r_0$ contains a nonempty k-regular subgraph. By the Petersen Theorem (see, e.g., [36], p. 218), if $r > k$, and r, k are both even, then every r-regular graph contains a spanning k-regular subgraph, but in any other case, the existence of a spanning k-regular subgraph is not ensured. Still, it seems plausible that if r is sufficiently large, there is always a nearly spanning k-regular subgraph.

The following conjecture was raised in discussions with Dhruv Mubayi [37].

Conjecture 3.1 *For every integer $k \ge 1$ and every real $\epsilon > 0$, there is an $r_0 = r_0(k, \epsilon)$ such that for $r > r_0$, every r-regular graph on n vertices contains a k-regular subgraph on at least $(1-\epsilon)n$ vertices.*

The above is certainly true (and easy) for $k = 1$. Indeed, by Vizing's Theorem [41], the edges of any r-regular graph on n vertices can be partitioned into at most $r+1$ matchings, and hence the largest of those is a 1-regular subgraph that covers at least $\frac{r}{r+1}n$ vertices. Here we show that the assertion of the conjecture holds for $k = 2$ as well.

Theorem 3.2 *For every real $\epsilon > 0$ there is an $r_0 = r_0(\epsilon)$ such that for $r > r_0$, every r-regular graph on n vertices contains a 2-regular subgraph on at least $(1-\epsilon)n$ vertices.*

The proof is short, and relies on the validity of two well known conjectures for permanents; the Minc conjecture, proved by Brégman, and the van der Waerden conjecture, proved by Falikman and Egorichev.

The Minc conjecture states that the permanent of any n by n matrix A with $(0,1)$ entries satisfies

$$Per(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i} \ ,$$

where r_i is the sum of the entries of the i-th row of A. This was proved by Brégman [16] (c.f. also [9] for another proof, based on a method of Schrijver).

The van der Waerden conjecture, proved by Falikman [22] and Egorichev [21], asserts that the permanent of every nonnegative n by n matrix in which each row and column sum is r is at least $(r/n)^n n!$ ($> (r/e)^n$). Schrijver [38] proved that if, in addition, each entry is an integer, then the permanent is at least $((r-1)^{r-1}/r^{r-2})^n$.

Armed with these powerful tools, we can now prove Theorem 3.2.

Proof of Theorem 3.2: Let $G = (V, E)$ be an r-regular graph on n vertices. We claim that if

$$\binom{n}{s} (\frac{r}{2})^s (r!)^{(n-2s)/r} < [\frac{(r-1)^{r-1}}{r^{r-2}}]^n, \tag{3}$$

then there is a 2-regular subgraph of G on more than $n - 2s$ vertices.

To prove this claim, consider the adjacency matrix A of the graph G. This is the $(0,1)$-matrix $A = (A(u,v))_{u,v \in V}$ in which $A(u,v) = 1$ iff $uv \in E$. As the sum of each row and each column of A is r, it follows, by Schrijver's result stated above, that the permanent of A is at least the right hand side of (3). This permanent can be interpreted combinatorially. Let $\mathcal{F}$ denote the set of all spanning subgraphs of G in which every connected component is either a single edge, or a cycle. For each $F \in \mathcal{F}$, define a weight $w(F) = 2^c$, where c is the number of connected components of F which are not single edges. It is not difficult to see that $Per(A) = \sum_{F \in \mathcal{F}} w(F)$. Call a member of $\mathcal{F}$ *small* if at least s of its components are single edges. We next bound the contribution of all small members of $\mathcal{F}$ to $Per(A)$. There are at most $\binom{n}{s}(r/2)^s$ ways to choose s single edges in G (as we can first choose a vertex of each edge, then choose a neighbor of each such vertex, and finally observe that in this way each collection of s edges has been counted 2^s times). The total contribution of all the members of $\mathcal{F}$ that contain a fixed set of s edges as connected components is the permanent of an $(n-2s)$ by $(n-2s)$ submatrix of A (which is the adjacency matrix of the induced subgraph of G obtained by deleting the $2s$ vertices covered by the single edges). By the Minc Conjecture stated above each such permanent does not exceed $(r!)^{(n-2s)/r}$. Therefore, the total contribution of all small members of $\mathcal{F}$ to $Per(A)$ is at most the left hand side of (3). As the permanent itself is at least the right hand side, it follows that if (3) holds then there is at least one member of $\mathcal{F}$ which is not small, and hence contains a 2-regular subgraph on more than $n - 2s$ vertices.

The assertion of Theorem 3.2 follows from the above claim by a simple computation. The right hand side of (3) exceeds $(r/e)^n$ (which lower bounds the permanent by the van der Waerden conjecture). By Stirling's formula and some standard estimates, if $s = \epsilon n$ and r is large, then the left hand side is at most

$$2^{H(\epsilon)n}(\frac{r}{2})^{\epsilon n}(\frac{r}{e})^{n-2\epsilon n}2(2\pi r)^{\frac{n}{2r}},$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. As $H(\epsilon) \le \epsilon\log_2(1/\epsilon) + O(\epsilon)$ it follows that if

$$O(\epsilon) + \frac{\log_2(2\pi r)}{2r} < \epsilon\log_2(\epsilon r)$$

then (3) holds. Therefore, if we choose, say, r such that $\epsilon = \frac{\log r}{r\log\log r}$, then (3) holds, ensuring a 2-regular subgraph of G on at least $(1-2\epsilon)n$ vertices. This completes the proof. □

The method above can be used to prove some stronger variants of Theorem 3.2. In particular, it can be used to prove the following.

Theorem 3.3 *For every integer $g \ge 3$ and every real $\epsilon > 0$ there is an $r_0 = r_0(g,\epsilon)$ such that for $r > r_0$, every r-regular graph on n vertices contains a 2-regular subgraph H on at least $(1-\epsilon)n$ vertices such that the number of vertices in each connected component of H exceeds g.*

The proof is similar to the one above; one simply has to bound the maximum possible contribution of members of $\mathcal{F}$ that contain more than s vertices on cycles of length at most g to $Per(A)$. We omit the (simple) details. Jacques Verstraete [40] raised the following conjecture.

Conjecture 3.4 *For every fixed graph H and every real $\epsilon > 0$, there is an $r_0 = r_0(H,\epsilon)$ such that for $r > r_0$, every r-regular graph on n vertices contains a subgraph on at least $(1-\epsilon)n$ vertices which is a collection of pairwise vertex disjoint graphs each of which is a topological copy of H.*

This holds for acyclic graphs H (where one can even omit the word "topological"), as shown in [33], and by Theorem 3.3 above it holds for any fixed cycle H. The result can also be extended to deal with any unicyclic graph H, but the case of more complicated graphs H remains open.

4 Monotone paths in bounded degree graphs

Let $G = (V,E)$ be a simple, finite graph with $|E| = m$ edges, and let $f : E \mapsto \{1,2\ldots,m\}$ be a bijective numbering of its edges. A monotone path of length k in (G,f) is a simple path with k edges and increasing f-values. Let $\alpha(G)$ denote the minimum, over all bijective functions f, of the maximum length of a monotone path. The problem of estimating $\alpha(K_n)$ for the complete graph K_n on n vertices was raised by Chvátal and Komlós [18], and the best known bounds for it are

$$\frac{1}{2}(\sqrt{4n-3}-1) \le \alpha(K_n) \le (\frac{1}{2}+o(1))n.$$

The lower bound is due to Graham and Kleitman [24], and the upper bound was proved by Calderbank, Chung and Sturtevant [17].

Yuster [44] has studied the problem of the maximum possible value of $\alpha(G)$, where G ranges over all graphs with maximum degree d. Denote this maximum by α_d. It is observed in [44] that $\alpha_d \le d+1$. This follows from Vizing's Theorem [41]. Given a graph G with maximum degree d, partition its edges into $d+1$ matchings $M_1,\ldots,M_{d+1}$, and let f map all edges of M_1 to the smallest numbers $1, 2, \ldots, |M_1|$, then all edges of M_2 to the next block of M_2 numbers, etc. Obviously, no monotone path in G can contain more than one edge from a single M_i, showing that $\alpha(G) \leq d+1$. Trivially, $\alpha_1 = 1$ and $\alpha_2 = 3$, as shown by any odd cycle of length at least 5. Yuster proved in [44] that for any $\epsilon > 0$ there is some $d_0 = d_0(\epsilon)$ such that $\alpha_d \geq (1-\epsilon)d$ for all $d > d_0$. It seems plausible to conjecture that $\alpha_d = d+1$ for every $d > 1$. Here is a very short proof that $\alpha_d \geq d$ (and is thus either d or $d+1$ for all $d > 1$.)

Proposition 4.1 *For every graph G on n vertices with maximum degree d, with more than $n(d-1)/2$ edges, and with girth bigger than d, $\alpha(G) \geq d$. Therefore, $\alpha_d \geq d$.*

Proof: We prove, by induction on m, that for every graph $G = (V, E)$ with n vertices $v_1, v_2, \ldots v_n$ and m edges, and for every bijection $f : E \mapsto \{1, 2 \ldots, m\}$, there is a collection $\mathcal{F} = \{P_1, \ldots, P_n\}$ of n monotone paths (each of which may have repeated vertices, but no repeated edges) in G, where P_r ends in the vertex v_r for all r, and the total length of all paths P_i is $2m$. This is trivially true for $m = 0$ (as we simply take the trivial paths of length 0 at each vertex). Assuming it holds for $m-1$, we prove it for m. Let $G = (V, E)$ have m edges and let $f : E \mapsto \{1, 2, \ldots, m\}$ be a bijection. Suppose $f(v_i, v_j) = m$, and let G' be the graph obtained from g by omitting the edge $v_i v_j$. By applying the induction hypothesis to G' and the restriction of f to the set of its edges, we conclude that there is a collection $\mathcal{F} = \{P_1, \ldots P_n\}$ of monotone paths in G', where P_r ends in v_r, and the sum of lengths of all paths is $2m-2$. We can now extend P_i and P_j by appending the edge $v_i v_j$ at the end of each of them, thus obtaining the required collection of monotone paths in G.

In particular if G has more than $n(d-1)/2$ edges, then at least one of the paths in the collection $\mathcal{F}$ is of length at least d. As the girth exceeds d, the first d edges of this path form a simple path of length d, showing that $\alpha(G) \geq d$. Note that if G is of class 1 (that is, its chromatic index is d), then in fact $\alpha(G) = d$. Note also that as there are d-regular graphs of girth that exceeds d, this implies that $\alpha_d \geq d$. □

5 Independent transversals

For any integer $r \geq 2$, let $d(r)$ denote the supremum of all reals d so that if all degrees in an r-partite graph with vertex classes V_i of size n each are smaller than dn, then there is an *independent transversal*, that is, an independent set containing one point from each V_i. These numbers have been studied in [12], [13], [29], [43], [28], and their precise determination appears to be a difficult problem even for small fixed values of r. In this section we prove that the precise value of $d(r)$ for every r which is a power of 2 is $\frac{r}{2(r-1)}$. This follows almost immediately from a recent result of Haxell [28]. We also obtain bounds for $d(r)$ for some other values of r, and in particular, disprove a conjecture of Jin [29]. Yet, we are unable to determine the precise value of $d(r)$ even for a single value of $r > 5$ which is not a power of 2.

Note, first, that trivially, $d(2) = 1$. Graver (c.f., [12]) proved that $d(3) = 1$, and Bollobás, Erdős and Szemerédi [13] proved that $d(4) \leq 8/9$ and that

$$\frac{2}{r} \leq d(r) \leq \frac{1}{2} + \frac{1}{r-2}$$

for all $r \geq 2$.

The fact that all numbers $d(r)$ are bounded away from 0 was first proved in [3] (see also [9], Chapter 5 for a proof that $d(r) \geq \frac{1}{2e}$ for every r), and an example in [4] shows that $d(4) \leq 2/3$. In [43] Yuster proved that in fact $d(4) = 2/3$ and that for every $r \geq 3$,

$$Max\{\frac{1}{2e}, \frac{1}{2^{\lceil \log(r/3) \rceil}}, \frac{1}{3 \cdot 2^{\lceil \log r \rceil - 3}}\} \leq d(r) \leq \frac{2^{\lfloor \log r \rfloor - 1}}{2^{\lfloor \log r \rfloor} - 1},$$

where here and throughout the section all logarithms are in base 2. The upper bound in the above inequality and the equality $d(4) = 2/3$ was, in fact, proved earlier by Jin [29], who also proved that $d(5) = 2/3$ and conjectured that for every r,

$$d(r) = \frac{2^{\lfloor \log r \rfloor - 1}}{2^{\lfloor \log r \rfloor} - 1}. \tag{4}$$

Haxell [28] proved that $d(r) \geq 1/2$ for all r. This proves a conjecture raised in [13], (see also [10], page 363.) The same method actually implies that $d(r) \geq \frac{r}{2(r-1)}$ for all $r \geq 2$. The short proof is based on the following result of Aharoni and Haxell [1].

Theorem 5.1 ([1]) *Let $G = (V, E)$ be a graph, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of its vertex set. Suppose that for every set $S \subset \{1, 2 \ldots, r\}$, the set $V_S = \cup_{i \in S} V_i$ is not dominated by any subset of V_S of cardinality smaller than $2|S| - 1$. Then G has an independent set containing a vertex from each V_i.*

This easily implies the following.

Proposition 5.2 *Let G be an r-partite graph with vertex classes V_i of size n each, in which all degrees are smaller than $\frac{r}{2(r-1)}n$. Then there is an* independent transversal *in G. Therefore, $d(r) \geq \frac{r}{2(r-1)}$.*

Proof If S is a subset of $\{1, 2 \ldots, r\}$, then every subset of cardinality at most $2|S| - 2$ of $V_S = \cup_{i \in S} V_i$ dominates less than $(2|S| - 2)\frac{r}{2(r-1)}n \leq |S|n = |V_S|$ vertices of V_S, and the result thus follows from Theorem 5.1. □

Note that this, together with the upper bound of [29] mentioned above, implies that

$$d(r) = \frac{r}{2(r-1)} \text{ for every } r \text{ which is a power of 2.}$$

This formula, however, does not hold for $r = 3, 5$, as $d(3) = 1$ and $d(5) = 2/3$. We next describe some constructions that yield improved upper bounds for $d(r)$ for many values of r.

To simplify the presentation, we assume from now on, whenever this is needed, that the number of vertices in each vertex class in the graphs constructed here are divisible by any required number. It is easy to check that this may be assumed without loss of generality.

Proposition 5.3 *For every two integers $r, s \geq 2$,*

$$d(rs) \leq \frac{rd(r)d(s)}{rd(s) + d(r)}.$$

Proof: Formally, we should start by replacing $d(r)$ by $d'(r)$, where $d'(r) > d(r)$ is close to $d(r)$, do the same for $d(s)$, and end the argument by letting $d'(r)$ approach $d(r)$ and $d'(s)$ approach $d(s)$. Since this does not cause any real difficulty, we omit this somewhat tedious, formal description in what follows.

Let x and y be two integers such that $d(r)x = d(s)y$. Let H_1 be an r-partite graph with x vertices in each vertex class and with maximum degree $d(r)x$, which does not contain an independent transversal. Similarly, let H_2 be an s-partite graph with y vertices in each vertex class, with maximum degree $d(s)y$, containing no independent transversal. Take s pairwise disjoint copies of H_1, and let $A_{i1}, \ldots, A_{ir}$ denote the vertex classes of copy number i, $(1 \le i \le s)$. Let $B_1, \ldots, B_s$ be the vertex classes of a copy of H_2, and suppose each B_i is a disjoint union of r sets of equal size $B_i = B_{i1} \cup B_{i2} \cup \ldots \cup B_{ir}$. Let G be the rs-partite graph whose vertex classes are $A_{ij} \cup B_{ij}$. It is not difficult to check that G contains no independent transversal. The maximum degree of a vertex of G is $d(r)x = d(s)y$, and the number of vertices in each of its vertex classes is $x + y/r$. Therefore,

$$d(rs) \le \frac{d(r)x}{x + y/r} = \frac{d(r)}{1 + d(r)/(rd(s))} = \frac{rd(r)d(s)}{rd(s) + d(r)},$$

as needed. □

Note that the above proposition implies that if $d(r) \le \frac{r}{2(r-1)}$ and $d(s) \le \frac{s}{2(s-1)}$, then $d(rs) \le \frac{rs}{2(rs-1)}$. In particular, as $d(2) = 1$, we conclude that (as we have already noted above) for every $r = 2^k$, $d(r) \le \frac{r}{2(r-1)}$ (and hence equality holds for each such r).

There are several additional constructions that provide upper bounds for the numbers $d(r)$. We illustrate these with one example, which is a special case of a more general construction. This disproves the conjecture of Jin mentioned in (4).

Proposition 5.4 *The number $d(14)$ satisfies*

$$d(14) \le \frac{12}{2 \cdot 11}.$$

Proof: Let $A_1, \ldots, A_4, B_1, \ldots, B_4, C_1, \ldots, C_4, D_1, \ldots, D_4, X$ and Y be pairwise disjoint sets, where each of the 16 sets A_i, B_i, C_i, D_i is of size x, and each of the two sets X and Y is of size $2x/3$. Let $A_4 = A_{41} \cup A_{42} \cup A_{43} \cup A_{44}$ be a partition of A_4 into four pairwise disjoint sets of equal cardinality, and let $D_4 = D_{41} \cup \ldots \cup D_{44}$ be a similar partition for D_4. Let $X = X_1 \cup X_2 \cup X_3$ be a partition of X into three disjoint sets of equal cardinality and let $Y = Y_1 \cup Y_2 \cup Y_3$ be a similar partition of Y.

Let G be the 14-partite graph on the classes of vertices $A_1 \cup X_1$, $A_2 \cup X_2$, $A_3 \cup X_3$, $B_1 \cup A_{41}, B_2 \cup A_{42}, \ldots, B_4 \cup A_{44}$, $C_1 \cup D_{41}, C_2 \cup D_{42}, \ldots, C_4 \cup D_{44}$, $D_1 \cup Y_1$, $D_2 \cup Y_2$, $D_3 \cup Y_3$.

The edges are defined as follows. Let H be a 4-partite graph with vertex classes of size x each, maximum degree $2x/3$ and no independent transversal. (Although $d(4)$ is defined as a supremum, it is easy to check that there is indeed such a graph for every x divisible by 3-see [4].) Then the induced subgraph of G on the sets A_i is a copy of H (where the sets A_i are the vertex classes), and each of the induced subgraphs of G on the sets B_i, the sets C_i and the sets D_i is also a copy of H. In addition, every vertex of X is connected to every vertex of Y and there are no additional edges.

Note that the maximum degree in G is at most $2x/3$. In addition, each of its vertex classes contains either $x + 2x/9 = 11x/9$ vertices or $x + x/4 > 11x/9$ vertices. Moreover, G contains no independent transversal. Indeed, assume there is such a transversal. Then it cannot contain a vertex in X and a vertex in Y. Assume, without loss of generality, it contains no vertex of X. Then it must contain a vertex in each of the three sets A_1, A_2, A_3 and hence it cannot contain a vertex of A_4. But then it must contain a vertex in each of the four sets B_i, which is impossible.

Therefore,

$$d(14) \leq \frac{2x/3}{11x/9} = \frac{12}{2 \cdot 11},$$

as claimed. □

The above proposition can be extended to obtain improved bounds for $d(r)$ for various values of r. Even without this extension, we can combine it with the first proposition to obtain improved bounds for $d(14 \cdot 2^k)$ for every k. Yet, as mentioned above, we are unable to determine the precise value of $d(r)$ even for a single value of $r > 3$ which is not a power of 2.

6 Binary prefix graphs

Let $F = (V(F), E(F))$ denote the (infinite) graph whose set of vertices is the set of all infinite sequences over $\{0,1\}$, where two sequences are joined by an edge if and only if their longest common initial segment has an even length. Answering a question of Geschke, Goldstern and Kojman [23], we prove the following.

Theorem 6.1 *Every induced finite subgraph of F on n vertices has either a clique or an independent set of size at least $\sqrt{n}$.*

Theorem 6.2 *For any finite graph H, there is a finite graph G so that for every injective function f from $V(G)$ to $V(F)$ there is an induced copy of H in G which is mapped by the function f to either a clique or an independent set of F.*

Geschke, Goldstern and Kojman can apply these properties to produce two nontrivial continuous graphs on the Cantor set, with different co-chromatic numbers. Here continuous means that the edge and non-edge relations are continuous, nontrivial means that no open set is a clique or an independent set, and the co-chromatic number is the least possible number of cliques and independent sets whose union covers all vertices of the graph.

Proof of Theorem 6.1: Let $T = (V(T), E(T))$ be a finite induced subgraph of F on n vertices. Then there is a finite k so that each vertex of T is a vector in $\{0,1\}^k$, and two vectors are adjacent iff their longest common initial segment has an even length. Let $w(T)$ denote the maximum size of a clique of T, and let $\alpha(T)$ denote the maximum size of an independent set of T. To complete the proof, we show that

$$w(T) \cdot \alpha(T) \geq |V(T)|. \tag{5}$$

We present two (related) proofs. The first requires some nontrivial facts about perfect graphs and the second is self contained. Recall that a graph is perfect iff the chromatic number $\chi(S)$ of every induced subgraph S of it is equal to its maximum clique $w(S)$. We prove, by induction on k, that T is perfect. This is trivial for $k = 1$. Assuming the result holds for $k - 1$, we prove it for k. Let $V(T)$ be a collection of vectors in $\{0,1\}^k$. Split these vectors into two sets, according to the first coordinate, that is, define $V_0 = \{v \in V(T) : \ v_1 = 0\}$ and $V_1 = \{v \in V(T) : \ v_1 = 1\}$. Let T_0 be the induced subgraph of T on V_0, and let T_1 be the induced subgraph on V_1. For each $v \in V(T)$, let v' denote the vector of length $k-1$ obtained by omitting the first coordinate of v. By the induction hypothesis, the graph on the vectors $\{v' : v \in V_0\}$ in which any two vectors are adjacent iff their longest common initial segment has an even length is perfect. But this is clearly the complement of T_0. By the perfect graph theorem of Lovász (c.f., e.g., [35], Exercise 13.57), a graph is perfect iff its complement is perfect. This implies that T_0 is perfect. By the same reasoning, T_1 is perfect as well. Note that T is obtained from the vertex disjoint union of T_0 and T_1 by connecting each vertex of T_0 to each vertex of T_1. It is known (see [35], Exercise 9.35) that if we substitute a perfect graph for each point of a perfect graph we get a perfect graph. As T is obtained from a single edge by substituting T_0 and T_1 for its vertices, we conclude that T is perfect.

Since T is perfect, $\chi(T) = w(T)$, implying that the vertices of T can be covered by $w(T)$ independent sets. This implies (5) and completes the (first) proof.

A more direct proof proceeds by showing directly that (5) holds, by induction on k. For $k = 1$ this is trivial. Assuming it holds for $k-1$, let $V(T)$ be a collection of vectors in $\{0,1\}^k$, and define V_0, V_1, T_0 and T_1 as before. By the induction hypothesis, and since the condition (5) is the same for a graph and its complement, it follows that $w(T_0)\alpha(T_0) \geq |V(T_0|$ and $w(T_1)\alpha(T_1) \geq |V(T_1|$.

In addition, $\alpha(T) = max\{\alpha(T_0), \alpha(T_1)\}$ and $w(T) = w(T_0) + w(T_1)$, implying that

$$w(T)\alpha(T) = (w(T_0) + w(T_1))max\{\alpha(T_0), \alpha(T_1)\}$$

$$\geq w(T_0)\alpha(T_0) + w(T_1)\alpha(T_1) \geq |V(T_0)| + |V(T_1)| = |V(T)|.$$

This completes the (second) proof. □

To prove Theorem 6.2 we prove the following.

Lemma 6.3 *For every integer m there is an integer $n = n(m)$ such that for every finite graph H on m vertices, there is a finite graph G on n vertices such that every induced subgraph of G on at least $\sqrt{n}$ vertices contains an induced copy of H.*

Proof: The proof is based on some simple properties of random graphs. We make no attempt to optimize the dependence of n on m, and prove the lemma with $n = 2^{2m^2}$, where we also assume (as we obviously may) that m is sufficiently large. A more complicated proof, based on the technique in [8], can be used to prove the lemma with much smaller $n = n(m)$. Throughout the proof we omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation.

Let $G = (V, E)$ be the random graph on the set $V = \{1, 2, \ldots, n\}$ of n labelled vertices obtained by picking each pair of distinct vertices i, j in V, randomly and independently, to form an edge with probability $1/2$. There is an extensive amount of literature on the properties of random graphs, see, e.g., [11] or [9]. The argument here is elementary, and uses mainly some simple computation.

Put $k = \sqrt{n}$ and let K be a fixed set of k vertices of G. We claim that the probability that the induced subgraph of G on K contains no induced copy of H is at most

$$(1 - 2^{-\binom{m}{2}})^{k^2/m^3} \leq \exp\{-\frac{k^2}{m^3 2^{\binom{m}{2}}}\}. \tag{6}$$

It is worth noting that it is not difficult to improve this estimate significantly, but for our purpose here the above estimate suffices.

To prove (6) note, first, that there is a collection $\mathcal{F}$ of at least k^2/m^3 subsets of K, each of cardinality m, so that the members of $\mathcal{F}$ are nearly disjoint, that is, no two of them share more than one common vertex. (Here, too, the estimate can be improved to $(1+o(1))\binom{k}{2}/\binom{m}{2}$, using, say, Wilson's Theorem [42], but the above estimate suffices for our purpose.) To get the required collection $\mathcal{F}$ note that it is equivalent to a collection of k^2/m^3 pairwise edge disjoint cliques of size m in a clique of size k. To get such a collection simply omit m-cliques one by one, as long as this is possible, where with each chosen clique we omit all its edges. By Turán's Theorem (c.f., e.g., [9], pp. 91-92), as long as there are at least $(1-\frac{1}{m})\binom{k}{2}$ edges, the remaining graph still contains an m-clique, and thus our greedy procedure will not terminate before producing at least

$$\frac{\binom{k}{2}}{m\binom{m}{2}} \geq \frac{k^2}{m^3}$$

pairwise edge-disjoint cliques, giving the required collection $\mathcal{F}$.

Once we have the collection $\mathcal{F}$, note that for each $F \in \mathcal{F}$, the probability that the induced subgraph of G on F is isomorphic to H is at least $2^{-\binom{m}{2}}$, and the events corresponding to distinct members F are mutually independent (as the sets F are nearly disjoint). Therefore, the probability that none of these induced subgraphs is a copy of H is at most the quantity in (6).

As there are $\binom{n}{k}$ subsets of cardinality k of V, it follows that if

$$\binom{n}{k} \exp\{-\frac{k^2}{m^3 2^{\binom{m}{2}}}\} < 1,$$

then with positive probability every subset of k vertices of G contains an induced copy of H. An easy computation shows that for $k = \sqrt{n}$ and $n = 2^{2m^2}$ the above inequality holds, implying the assertion of the lemma. □

Proof of Theorem 6.2: Given H, let G be as in Lemma 6.3. By Theorem 6.1, for every injective function f from $V(G)$ to $V(F)$ there is a set of at least $\sqrt{n}$ vertices of G mapped by f to a clique or to an independent set, and by Lemma 6.3 this set contains an induced copy of H, completing the proof. □

7 Induced acyclic subgraphs in sparse bipartite graphs

All graphs considered in this section are finite and simple. M. Albertson and R. Haas [2] raised the following conjecture.

Conjecture 7.1 *Every planar, bipartite graph on n vertices contains an induced acyclic subgraph on at least $5n/8$ vertices.*

Motivated by this conjecture, we consider induced acyclic subgraphs in sparse bipartite graphs. The main result here is that every bipartite graph on n vertices with average degree at most d (≥ 1) contains an induced acyclic subgraph on at least $(1/2+e^{-bd^2})n$ vertices, for some absolute constant $b > 0$. On the other hand, there exist bipartite graphs on n vertices and average degree at most d (≥ 1) that contain no induced acyclic subgraphs on at least $(1/2+e^{-b'\sqrt{d}})n$ vertices. In particular, as the average degree of any planar bipartite graph is at most 4, there is an absolute positive constant δ such that every planar, bipartite graph on n vertices contains an induced acyclic subgraph on at least $(1/2+\delta)n$ vertices. This provides some (weak but nontrivial) result on the question of Albertson and Haas mentioned above. Of course, there may well be some better ways (which we failed to find) to apply the planarity in order to try and prove Conjecture 7.1, but we believe that the results on sparse bipartite graphs are of independent interest.

For a graph $G = (V, E)$, let $a(G)$ denote the maximum number of vertices in an induced acyclic subgraph of it. In [6] the authors determine precisely, for every n and m, the minimum possible value of $a(G)$ where G ranges over all graphs on n vertices and m edges. In particular, it follows that if G has n vertices and its average degree is at most $d \geq 2$, then $a(G) \geq \frac{2n}{d+1}$. This is tight for all integers n, d where $d \geq 2$ and $d+1$ divides n, as shown by a disjoint union of cliques of size $d+1$.

If the graph $G = (V, E)$ with $|V| = n$ and $|E| \leq \frac{1}{2}dn$ is bipartite, then one can obtain a better lower bound for $a(G)$, since, trivially, $a(G) \geq n/2$. In this section we show that this trivial estimate can be slightly improved as a function of the average degree.

Theorem 7.2 *There exists an absolute positive constant b such that for every bipartite graph $G = (V, E)$ with n vertices and average degree at most d (≥ 1),*

$$a(G) \geq (\frac{1}{2} + e^{-bd^2})n.$$

The exponential dependence on d cannot be replaced by a polynomial one, as shown in the following result.

Theorem 7.3 *There exists an absolute constant $b' > 0$ such that for every $d \geq 1$ and all sufficiently large n there exists a bipartite graph with n vertices and average degree at most d such that*

$$a(G) \leq (\frac{1}{2} + \frac{1}{e^{b'\sqrt{d}}})n.$$

The proofs of both theorems are probabilistic and are presented in the following two subsections. The final short subsection contains some related remarks.

7.1 Large acyclic subgraphs

In this subsection we prove Theorem 7.2. We make no attempt to optimize the various constants in our estimates. Throughout the section, let $G = (V, E)$ be a bipartite graph with classes of vertices A and B, where $|A| = p,\ |B| = q$. Without loss of generality assume that $p \geq q$, let $n = p + q$ be the total number of vertices of G, and suppose that G contains at most dp edges (this certainly holds if the average degree of G is at most d.) Let $deg(v)$ denote the degree of the vertex v of G.

Lemma 7.4 *There exists an integer $x \leq e^{4d^2}$ such that*

$$|\{a \in A\ :\ deg(a) \geq x\}| < \frac{p}{4dx}.$$

Proof: Otherwise

$$dp \geq |E| = \sum_{x \geq 1} |\{a \in A \ : \ deg(a) \geq x\}| \geq \sum_{x=1}^{\lfloor e^{4d^2} \rfloor} \frac{p}{4dx} > dp,$$

contradiction. □

Lemma 7.5 *The size of the maximum induced acyclic subgraph of G satisfies*

$$a(G) \geq \frac{q}{dx} + p - \frac{p}{4dx} - \frac{p}{2dx} = p + \frac{1}{4dx}(4q - 3p). \tag{7}$$

Proof: Let S be a random subset of B obtained by picking each member of B, randomly and independently, to be a member of S with probability $\frac{1}{dx}$, where x satisfies the conclusion of Lemma 7.4. Let T be the set of all vertices in A whose degrees are smaller than x, which have at most one neighbor in S. Then $S \cup T$ spans an acyclic subgraph of G. To complete the proof, it suffices to show that the expected size of $S \cup T$ is at least the right hand side of (7).
To do so note, first, that the expected size of S is $\frac{q}{dx}$. The number of vertices in A with degrees at least x is, by Lemma 7.4 and the choice of x, at most $\frac{p}{4dx}$. It thus remains to prove the following.
Claim: The expected number of vertices $a \in A$ that satisfy $deg(a) < x$ and have at least two neighbors in S is at most $\frac{p}{2dx}$.
Proof (of claim): This expectation does not exceed

$$\sum_{a \in A, deg(a) < x} \frac{\binom{deg(a)}{2}}{d^2x^2} \leq \frac{1}{2d^2x^2} \sum_{a \in A, deg(a) < x} deg(a)^2 \leq \frac{x}{2d^2x^2} \sum_{a \in A, deg(a) < x} deg(a) \leq \frac{1}{2d^2x} dp = \frac{p}{2dx}.$$

This completes the proof of the claim. □
The assertion of the claim, the paragraph preceding it, and linearity of expectation, complete the proof of the lemma. □
We can now prove Theorem 7.2 in the following more precise form (in which the constants 16 and 4 can be improved).

Proposition 7.6 *For every bipartite graph $G = (V, E)$ with n vertices and average degree at most d (≥ 1),*

$$a(G) \geq \left(\frac{1}{2} + \frac{1}{16de^{4d^2}}\right)n.$$

Proof: If, say, $4q \leq 7p/2$, then, trivially,

$$a(G) \geq |A| = p \geq \frac{16}{30}p + \frac{14}{30} \cdot \frac{8}{7}q = \left(\frac{1}{2} + \frac{1}{30}\right)n,$$

implying the desired estimate, as

$$\frac{1}{30} > \frac{1}{16de^{4d^2}}.$$

Otherwise, by Lemma 7.5,

$$a(G) \geq p + \frac{1}{4dx}(4q - 3p) \geq p + \frac{p}{8dx} \geq p + \frac{p}{8de^{4d^2}} \geq \left(\frac{1}{2} + \frac{1}{16de^{4d^2}}\right)n.$$

□

7.2 Examples with no large acyclic subgraphs

In this subsection we prove Theorem 7.3. For convenience, we omit all floor and ceiling signs whenever these are not crucial. We need the following somewhat technical lemma.

Lemma 7.7 *Let m, i be integers, put $c = 64$ and suppose $1 \leq i \leq \frac{1}{2}\log_2 m$, and $ci < \sqrt{m}$. Then there is a bipartite graph H on the classes of vertices X and Y, where $|X| = m$ and $|Y| = \frac{m}{2^{i-1}}$, with at most $4cim$ edges, such that for every $X' \subset X$ and $Y' \subset Y$ satisfying $|X'| = \frac{m}{2^{i+1}}$ and $|Y'| = \frac{m}{2^i}$, the subgraph of H induced on $X' \cup Y'$ contains at least $\frac{cim}{2^{i+2}}$ ($> |X'| + |Y'|$) edges (and thus contains a cycle).*

Proof: Let H be the random graph obtained by picking each pair xy with $x \in X$, $y \in Y$ to be an edge, randomly and independently, with probability $\frac{ci2^i}{m}$ (note that by assumption this number is at most 1). The expected number of edges of H is $\frac{ci2^i}{m}|X| \cdot |Y| = 2cim$ and thus, by the standard estimates for binomial distributions (c.f., e.g., [9], Appendix A), with probability at least, say, 0.9, the number of edges of H is at most $4cim$.
Fix two subsets $X' \subset X$ and $Y' \subset Y$ satisfying $|X'| = \frac{m}{2^{i+1}}$ and $|Y'| = \frac{m}{2^i}$. The expected number of edges in the induced subgraph of H on $X' \cup Y'$ is

$$\frac{ci2^i}{m}|X'| \cdot |Y'| = \frac{cim}{2^{i+1}}.$$

By Theorem A.1.13 in Appendix A of [9], it follows that the probability that there are less than half of that many edges in this induced subgraph is at most

$$e^{-\frac{cim}{8 \cdot 2^{i+1}}} = e^{-\frac{cim}{2^{i+4}}}.$$

The number of possible sets X' is at most

$$\binom{m}{m/2^{i+1}} \leq 2^{H(\frac{1}{2^{i+1}})m} \leq 2^{\frac{2i+2}{2^{i+1}}m},$$

where here $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function, and we used the fact that for every $0 < x \leq 1/2$, $H(x) \leq -2x\log_2 x$.
The number of possible sets Y' is clearly at most

$$2^{|Y|} = 2^{\frac{m}{2^{i-1}}}.$$

It follows that the probability that there are subsets X' and Y' that violate the conclusion of the lemma is at most

$$2^{\frac{2i+6}{2^{i+1}}m} e^{-\frac{cim}{2^{i+4}}} = 2^{\frac{2i+6-8i\log_2 e}{2^{i+1}}m},$$

and this quantity is smaller than, say, 0.1.

Therefore, with positive probability H satisfies all the required properties, completing the proof. □

Using the lemma, we next prove the following statement that implies the assertion of Theorem 7.3.

Proposition 7.8 *For every integer $s \geq 3$ and every integer $m > m_0(s)$ which is divisible by 2^{s-2} there is a bipartite graph G with m vertices in each color class and with at most $4cs^2m$ edges which satisfies*

$$a(G) \leq m + \frac{m}{2^{s-2}}.$$

Proof: Let $A_2, A_3, \ldots, A_s, B_2, B_3, \ldots, B_s$ be pairwise disjoint sets, where for each i, $2 \leq i \leq s-1$, $|A_i| = |B_i| = \frac{m}{2^{i-1}}$, and where $|A_s| = |B_s| = \frac{m}{2^{s-2}}$. Put $A = \cup_{i=2}^{s} A_i$, $B = \cup_{i=2}^{s} B_i$ and note that $|A| = |B| = m$. Let H be a bipartite graph on the classes of vertices A and B whose set of edges is defined as follows.

- For each i, $2 \leq i \leq s-1$, let H_i be a copy of the graph H of Lemma 7.7, where the vertices of A play the role of those in X (in an arbitrary order) and the vertices of B_i plays the role of those in Y (in an arbitrary order). Let E_i denote the set of all edges of H_i.
- Similarly, for each i as above, let H'_i be a copy of the graph H of Lemma 7.7, obtained by substituting the vertices of B for those in X and the vertices of A_i for those in Y. Let E'_i denote the set of edges of H'_i.
- Finally, let H' be a copy of the graph of Lemma 7.7 with $i = 1$, where here A and B play the roles of X and Y, and let E' denote the set of edges of H'.

The set of edges E of H is simply the union of all sets above, that is: $E = E' \cup_{i=2}^{s-1} (E_i \cup E'_i)$. The number of edges of H clearly satisfies

$$|E| \leq |E'| + \sum_{i=2}^{s-1}(|E_i| + |E'_i|) \leq 4cm + 2\sum_{i=2}^{s-1} 4cim \leq 4cs^2m.$$

Let S be a set of vertices of H, and suppose the induced subgraph of H on S is acyclic. To complete the proof we have to show that $|S| \leq m + \frac{m}{2^{s-2}}$. Without loss of generality suppose that $|S \cap A| \leq |S \cap B|$. If $|S \cap A| \leq \frac{m}{2^{s-2}}$, then the desired estimate is trivial, as $|S \cap B| \leq |B| = m$. Otherwise we show that in fact $|S| \leq m$. Indeed, if $|S \cap A| \geq m/4$ then $|S \cap B| \leq m/2$ (since otherwise the induced subgraph of H' on S contains a cycle, and hence so does the induced subgraph of H on S). However, in this case $|S| \leq 2|S \cap B| \leq m$, as claimed. Therefore, we may assume that

$$\frac{m}{2^{s-2}} < |S \cap A| < \frac{m}{4}.$$

Let i be an integer so that

$$\frac{m}{2^{i+1}} \leq |S \cap A| \leq \frac{m}{2^i}.$$

Then $2 \leq i \leq s-2$. We claim that in this case $|S \cap B_i| \leq \frac{m}{2^i}$. To see this note that otherwise the induced subgraph of H_i on S contains a cycle, and hence so does the induced subgraph of H on S. Therefore, in this case $|B \setminus S| \geq \frac{m}{2^i} \geq |S \cap A|$, implying that $|S| \leq m$ in this case too, and completing the proof. □

By the last proposition, for every $d \geq 9 \cdot 256$, and all sufficiently large n divisible by $2^{\lfloor \frac{\sqrt{d}}{16} \rfloor - 1}$ there is a bipartite graph G on n vertices with average degree at most d, for which

$$a(G) \leq \frac{1}{2}n + \frac{n}{2^{\lfloor \frac{\sqrt{d}}{16} \rfloor - 1}}.$$

This clearly implies the assertion of Theorem 7.3, for an appropriately chosen constant b'.

7.3 Remarks

- The estimate in Theorem 7.2 can be easily improved for regular graphs. In [7] it is shown that for every d-regular bipartite graph G with n vertices

$$a(G) \geq \frac{1}{2}n + \frac{n}{2(d-1)^2}.$$

  Theorem 7.3 shows that such an estimate does not hold for non-regular bipartite graphs with average degree d.

- It will be interesting to close the gap between the bounds in Theorems 7.2 and 7.3.

8 The graph of diameters of a three-dimensional point set

The *Diameter graph* $D(X)$ of a finite set of points X in R^3 is the graph whose vertices are the points of X, where two are adjacent if and only if their distance is the diameter of X. It is well known that any such graph is 4-colorable, as proved by Eggleston [20] and Grünbaum [25], in response to a well known question suggested by Borsuk [14] whether any set in R^n can be partitioned into $n+1$ sets of smaller diameter.

Peter Brass [15] suggested that a stronger result may hold for any finite set of points X in R^3. Namely, he asked if it is true that $D(X)$ can always be colored properly by 4 colors so that two of the color classes are of size 1. This is partially supported by the known result that any two odd cycles in $D(X)$ have a common vertex [19]. A weaker conjecture he suggested is that there exists an absolute constant c such that for every finite set of points X in R^3 one can delete at most c points from $D(X)$ so that the resulting graph becomes bipartite. In this section we show that even this weaker conjecture is false.

Theorem 8.1 *For every positive integer s there is a finite set of points $X = X_s$ in R^3 that cannot be partitioned into three sets, A, B and C such that A is of size at most s and the diameter of each of the sets B and C is strictly smaller than that of X.*

To prove the theorem we construct appropriate finite sets of points in R^3 whose diameter graphs are related to the (abstract) graphs constructed in [39].

Proof: Let $S_0 = T_0$ be a set of $2s+1$ points forming the vertices of a regular $2s+1$-gon in the xy-plane, where the points lie on a circle centered at the origin and the diameter of S_0 is 1. Fix a small $\epsilon > 0$, and let

$$T_i = (1 - \binom{i+1}{2}\epsilon)S_0,$$

where $0 \leq i \leq k$. It is not difficult to check that the maximum distance between a point in T_i and a point in T_j is

$$1 - \frac{\binom{i+1}{2} + \binom{j+1}{2}}{2}\epsilon + O(\epsilon^2). \tag{8}$$

For $1 \leq i \leq k$, let S_i be a translate of T_i, where S_i lies on a plane parallel to the xy-plane at height $b_i\sqrt{\epsilon}$ for each odd i, and at height $-b_i\sqrt{\epsilon}$ for each even i, where the numbers b_i are chosen to be positive and such that the maximum distance between a point of S_i and a point of S_{i-1} is precisely 1. A simple computation based on (8) implies that $b_i = \lceil \frac{i}{2} \rceil + O(\epsilon)$ for each i. It thus follows that the maximum square of a distance between a point in S_i and a point in S_j, where i and j have different parities, is

$$(1 - \frac{\binom{i+1}{2} + \binom{j+1}{2}}{2}\epsilon + O(\epsilon^2))^2 + ((\lceil \frac{i}{2} \rceil + \lceil \frac{j}{2} \rceil)\sqrt{\epsilon} + O(\epsilon^{3/2}))^2$$
$$= 1 - \frac{(i-j)^2 - 1}{4}\epsilon + O(\epsilon^2).$$

This is 1, by the choice of the numbers b_i, if $|i - j| = 1$, and, assuming ϵ is sufficiently small, is strictly smaller than 1 if $|i - j| > 1$. If $i \equiv j$ $(mod\ 2)$ then, easily, the maximum distance between a point of S_i and a point of S_j is smaller than 1.

Suppose, now, that $k \geq s$, and let $X = X_{k,s} = \cup_{i=0}^{k} S_i$. By the discussion above, the diameter of X is 1. The graph $D(X)$ is the following graph. Its vertices are the $(k+1)(2s+1)$ points

$$\{a_j^{(i)} : 0 \leq i \leq k,\ 0 \leq j \leq 2s\},$$

where the points $\{a_j^{(i)} : 0 \leq j \leq 2s\}$ denote the points of S_i. The lower indices are chosen such that the points $a_0^{(0)}, a_1^{(0)}, \ldots, a_{2s}^{(0)}$ form a cycle in $D(X)$, in this order. Each set $\{a_j^{(i)} : 0 \leq j \leq 2s\}$, for $i \geq 1$, is an independent set, and $a_j^{(i)}$ is adjacent to $a_{j-1}^{(i-1)}$ and to $a_{j+1}^{(i-1)}$, where the lower indices are computed modulo $2s + 1$.

Since $k \geq s$, the above graph contains the following $2s + 1$ odd cycles:

$$C_j = a_j^{(0)}, a_{j+1}^{(1)}, a_{j+2}^{(2)}, \ldots, a_{j+s}^{(s)}, a_{j+s+1}^{(s-1)}, a_{j+s+2}^{(s-2)}, \ldots, a_{j+2s}^{(0)}, a_{j+2s+1}^{(0)} = a_j^{(0)},$$

where the lower indices are computed modulo $2s + 1$. Moreover, no vertex lies in more than two of the cycles C_j, $(0 \leq j \leq 2s)$. It follows that if we omit an arbitrary set of at most s vertices of the graph, at least one of the odd cycles C_j remains. Therefore, one cannot omit s vertices and transform $D(X)$ into a bipartite graph, completing the proof of the theorem. □

9 On ranks of perturbations of identity matrices

The main result of this section is motivated by a question of Moses Charikar concerning the maximum possible number of nearly orthogonal unit vectors in the d-dimensional Euclidean space. Although it seems that a related result can be obtained by a different technique, based on the linear programming method applied in Coding Theory, the result here applies to a more general situation, as it does not assume that the matrices considered are positive semi-definite, and the proof is simpler. We first describe the combinatorial result, and then comment on its relevance to the Johnson-Lindenstrauss Lemma [30].

Lemma 9.1 *Let $A = (a_{i,j})$ be an n by n real, symmetric matrix with $a_{i,i} = 1$ for all i and $|a_{i,j}| \leq \epsilon$ for all $i \neq j$. If the rank of A is d, then*

$$d \geq \frac{n}{1 + (n-1)\epsilon^2}.$$

In particular, if $\epsilon \leq \frac{1}{\sqrt{n}}$ then $d \geq n/2$.

Proof: Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A, then their sum is the trace of A, which is n, and at most d of them are nonzero. Thus, by Cauchy-Schwartz, $\sum_{i=1}^n \lambda_i^2 \geq d(n/d)^2 = n^2/d$. On the other hand, this sum is the trace of $A^t A$, which is precisely $\sum_{i,j} a_{i,j}^2 \leq n + n(n-1)\epsilon^2$. Hence $n + n(n-1)\epsilon^2 \geq n^2/d$, implying the desired result. □

Lemma 9.2 *Let $B = (b_{i,j})$ be an n by n matrix of rank d, and let $P(x)$ be an arbitrary polynomial of degree k. Then the rank of the n by n matrix $(P(b_{i,j}))$ is at most $\binom{k+d}{k}$. Moreover, if $P(x) = x^k$ then the rank of $(P(b_{i,j}))$ is at most $\binom{k+d-1}{k}$.*

Proof: Let $\mathbf{v_1} = (v_{1,j})_{j=1}^n, \mathbf{v}_2 = (v_{2,j})_{j=1}^n, \ldots, \mathbf{v_d} = (v_{d,j})_{j=1}^n$ be a basis of the row-space of B. Then the vectors $(v_{1,j}^{k_1} \cdot v_{2,j}^{k_2} \cdots v_{d,j}^{k_d})_{j=1}^n$, where $k_1, k_2, \ldots, k_d$ range over all non-negative integers whose sum is at most k, span the rows of the matrix $(P(b_{i,j}))$. In case $P(x) = x^k$ it suffices to take all these vectors corresponding to $k_1, k_2, \ldots, k_d$ whose sum is precisely k. □

Theorem 9.3 *Let B be an n by n real matrix with $b_{i,i} = 1$ for all i and $|b_{i,j}| \leq \epsilon$ for all $i \neq j$. If the rank of B is d, and $\frac{1}{\sqrt{n}} \leq \epsilon < 1/2$, then*

$$d \geq \Omega(\frac{1}{\epsilon^2 \log(1/\epsilon)} \log n).$$

Proof: We may and will assume that B is symmetric, since otherwise we simply apply the result to $(B + B^t)/2$ whose rank is at most twice the rank of B. If $\epsilon \leq 1/n^\delta$ for some fixed $\delta > 0$, the result follows by applying Lemma 9.1 to a $\frac{1}{\epsilon^2}$ by $\frac{1}{\epsilon^2}$ submatrix of B. Thus we may assume that $\epsilon \geq 1/n^\delta$ for some fixed, small $\delta > 0$. To simplify the presentation we omit all ceiling signs. Put $k = \frac{\log n}{2\log(1/\epsilon)}$ and note that by Lemma 9.2 the rank of the n by n matrix $(b_{i,j}^k)$ is at most $\binom{d+k}{k} \leq (\frac{e(k+d)}{k})^k$. On the other hand, by Lemma 9.1, the rank of this matrix is at least $n/2$. Therefore

$$\frac{\log n}{2\log(1/\epsilon)} \log(\frac{e(d + \frac{\log n}{2\log(1/\epsilon)})}{\frac{\log n}{2\log(1/\epsilon)}}) \geq \log(n/2),$$

implying that

$$d \geq (1 + o(1)) \frac{\log n}{2e \log(1/\epsilon)} (\frac{1}{\epsilon^2} - e),$$

where the $o(1)$ term tends to zero as δ tends to zero. This completing the proof. □

Remarks:

- The estimate in Theorem 9.3 is essentially optimal when ϵ is 1 over some small power of n, by known constructions of error-correcting codes.
- The Johnson-Lindenstraus Lemma, proved in [30], asserts that for any $\epsilon > 0$, any set A of n points in an Euclidean space can be embedded in an Euclidean space of dimension $k = c(\epsilon) \log n$ with distortion at most ϵ. That is, there is a mapping $f : A \mapsto R^k$ such that for any $a, b \in A$, the distance between $f(a)$ and $f(b)$ is at least the distance between a and b, and at most that distance multiplied by $1 + \epsilon$. The proof gives that $c(\epsilon) \leq O(\frac{1}{\epsilon^2})$. Theorem 9.3 shows that this

is nearly tight: $c(\epsilon)$ must be at least $\Omega(\frac{1}{\epsilon^2 \log(1/\epsilon)})$, even for embedding the set of points of a simplex. Indeed, if we have $n+1$ points in R^k, and the distance between any pair of distinct points among them is at least 1 and at most $1+\epsilon$, we can put one of the points, say P_0, at the origin, and shift all other points by at most ϵ making sure that their distance from P_0 is exactly 1. By the triangle inequality the distance between any pair of the shifted points is still $1 + O(\epsilon)$. Therefore, if v_i is the k-dimensional vector representing the i-th point, then the matrix $C = (v_i^t \cdot v_j)$ is an n by n matrix with all diagonal entries being 1, and all other entries being $1/2 + O(\epsilon)$. Moreover, the rank of this matrix is at most k. Therefore, the rank of $B = 2C - J$, where J is the all 1 n by n matrix, is at most $k+1$. By Theorem 9.3 this rank is at least $\Omega(\frac{1}{\epsilon^2 \log(1/\epsilon)} \log n)$, supplying the required lower bound for the dimension k.

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