Finding disjoint paths in expanders deterministically and online

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Abstract

We describe a deterministic, polynomial time algorithm for finding edge-disjoint paths connecting given pairs of vertices in an expander. Specifically, the input of the algorithm is a sufficiently strong d-regular expander G on n vertices, and a sequence of pairs s_i, t_i, $(1 \leq i \leq r)$ of vertices, where $r = \Theta(\frac{nd \log d}{\log n})$, and no vertex appears more than $d/3$ times in the list of all endpoints $s_1, t_1, \ldots, s_r, t_r$. The algorithm outputs edge-disjoint paths $Q_1, \ldots, Q_r$, where Q_i connects s_i and t_i. The paths are constructed online, that is, the algorithm produces Q_i as soon as it gets s_i, t_i and before the next requests in the sequence are revealed. This improves in several respects a long list of previous algorithms for the above problem, whose study is motivated by the investigation of communication networks. An analogous result is established for vertex disjoint paths in blow-ups of strong expanders.

1 Introduction

Given an undirected graph $G = (V, E)$ and a set of pairs s_i, t_i, $1 \leq i \leq r$, we are interested in finding r edge-disjoint paths $Q_1, \ldots, Q_r$, where Q_i connects s_i and t_i. This problem received a considerable amount of attention, motivated by a variety of communication contexts including the study of communication systems and distributed-memory computer architectures.

The corresponding decision problem for general graphs G is NP-complete, although it is in P for every fixed r, as shown by Robertson and Seymour in [18]. If G is a sufficiently strong bounded degree expander on n vertices, then it is known that edge-disjoint paths always exist and can be found efficiently provided r is not too large (and no vertex appears too many times as an s_i or a t_i). This has first been shown by Peleg and Upfal in [17] for $r = \Theta(n^{\epsilon})$ where $\epsilon > 0$ depends on the expansion properties of the graph. Improvements by Broder, Frieze and Upfal [6], [7], Frieze [8], Leighton and Rao [12] and Leighton, Lu, Rao and Srinivasan [13],[14] showed that if the expansion properties of the graph are sufficiently strong, then the paths always exist even if r is $\Theta(\frac{n}{\log^c n})$ where c depends on

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the expansion properties but is always at least 2. Finally Frieze [9] showed that disjoint paths always exist and can be found in randomized polynomial time even if $r = \Theta(\frac{n}{\log n})$. This is optimal, up to the dependence of the constant factor on the degree of regularity of the graph and its expansion, as the distance between each pair of points s_i, t_i may be $\Omega(\log n)$. The problem of finding the paths by a deterministic, efficient algorithm remained open. In the present paper we present such a deterministic algorithm, and also further improve the lower bound for the number of paths r, making it optimal, up to a constant factor, as a function of the number of vertices of G as well as its degree of regularity. More crucially, our algorithm works in the online setting, that is, it produces the i-th path as soon as its endpoints s_i, t_i are presented.

Our algorithm can also be used to improve a related result, by Kleinberg and Rubinfeld [11]. In this paper the authors present a simple online algorithm which is allowed to reject some of the pairs and manages to connect at least a fraction of $\Omega(\frac{1}{\log n \log\log n})$ of the maximum number of pairs that can be connected offline by disjoint paths in an n vertex expander. Applying our algorithm to d-regular expanders on n vertices, we can connect online any desired set of $\Theta(nd/\log n)$ pairs (in which no vertex appears as an endpoints more than $\Theta(d)$ times). This is trivially within a factor $\Theta(\frac{1}{\log n})$ of the optimum (as the total number of edges is $nd/2$). The algorithm is polynomial, though it is less efficient than the one of [11].

Finally, we describe a result for vertex disjoint paths, providing a variant of non-blocking networks with optimal parameters.

1.1 The main results

Our results hold for sufficiently strong expanders, the precise definition is given below. We note that there are many explicit constructions of graphs with these expansion properties. In particular, the assertion of both theorems below hold for Ramanujan graphs of sufficiently large (constant) degree, as well as for an appropriate (fixed) power of any bounded degree expander. It is convenient to state the theorems in terms of a game played by two players, an *Adversary* and a *Router*. The game, which we call the *Routing Game* for G is played on a d-regular graph $G = (V, E)$ on n vertices, and proceeds in rounds. In each round of the game, the adversary picks a pair of vertices s_i, t_i of G, and the router has to provide a path Q_i from s_i to t_i. The game is played for r rounds. The router wins if she manages to produce r edge disjoint paths during these r rounds. Our main result asserts that if G is a sufficiently strong expander, then the router wins the game even if the number of rounds r is as large as possible (up to a constant factor), as long as no vertex is chosen by the adversary as an s_i or a t_i too many times. Moreover, she can produce the required paths in deterministic polynomial time. Note that the paths are determined online and once a path is produced, it cannot be modified.

Throughout the paper we use the following definition, in which $d > 60$.

Definition 1.1 *A graph $G = (V, E)$ is a d-regular* very strong expander *on n vertices if it satisfies the following:*
(i) The average degree in any subgraph of G on at most $n/10$ vertices is at most $d/6$.
(ii) The average degree in any subgraph of G on at most $n/2$ vertices is at most $2d/3$.

In the following theorem and throughout the paper, all logarithms are in base 2.

Theorem 1.1 *Let $G = (V, E)$ be a very strong d-regular expander on n vertices, and suppose $r = \frac{nd}{150 \log n}$. Then there is a deterministic, polynomial time algorithm that enables the router to win the routing game with r rounds on G, provided no vertex is chosen by the adversary as an s_i or a t_i more than $d/3$ times.*

It is easy to see that the value of r is the largest possible, up to an absolute constant factor, as very strong expanders can contain $n/2$ disjoint pairs of vertices s_i, t_i so that for all i the distance between s_i and t_i is at least, say, $\log n/10$. It is also obvious that the adversary can trivially win in $d+1$ rounds if he is allowed to choose the same vertex as an s_i or t_i more than d times, hence the $d/3$ limit is also optimal, up to a constant factor.

Definition 1.2 *An (n, d, λ)-graph is a d-regular graph G on n vertices so that the absolute value of every nontrivial eigenvalue of the adjacency matrix of G is at most λ.*

This notion was introduced by the first author in the 80s, motivated by the fact that if λ is much smaller than d, then these graphs behave in many respects like random d-regular graphs.

Theorem 1.2 *There is an absolute constant $c > 0$ so that the following holds. Let $G = (V, E)$ be an (n, d, λ)-graph where $d > 8\lambda$, and put $r = c\frac{nd \log(d/4\lambda)}{\log n}$. Then there is a deterministic polynomial time algorithm that enables the router to win the routing game for r rounds on G provided the adversary does not choose any vertex as an s_i or a t_i more than $d/3$ times.*

The known explicit constructions of Ramanujan graphs given in [15], [16] supply infinite families of $(n, d, 2\sqrt{d-1})$-graphs. For such graphs, the number of rounds above is $r = \Theta(\frac{nd \log d}{\log n})$. Note that this is optimal up to a constant factor for any d-regular graph, as it is not difficult to show that any d-regular graph on n vertices contains $\lfloor n/2 \rfloor$ disjoint pairs s_i, t_i so that for every i the distance between s_i and t_i is at least $\frac{\log n}{\log(d-1)} - 1$.

Besides the above two results we describe an analogous result for vertex disjoint paths on blow-ups of very strong expanders. This result is also optimal in the relevant parameters, up to absolute constant factors.

1.2 Overview

Here is a rough description of the procedure that enables the router to win the game on a very strong expander G for r rounds. Throughout the game, the router maintains a dense subgraph G' of G that does not share any edge of the disjoint paths Q_i she has already constructed. As a dense subgraph of a very strong expander, the subgraph G' also enjoys strong expansion properties and thus has a logarithmic diameter. In addition, the router maintains a collection $\mathcal{P}$ of edge-disjoint paths from all the vertices that may still appear as endpoints in the future requests of the adversary, to the vertices of the subgraph G'. The paths in $\mathcal{P}$ and the paths Q_i used to satisfy the previous requests do not share any edges. When a new request s_ℓ, t_ℓ appears, the router uses the paths in $\mathcal{P}$ that connect s_ℓ

and t_ℓ to G' together with a short path in G' between their endpoints, to produce the required path Q_ℓ. She then omits the edges of Q_ℓ from G' and updates this subgraph to keep it dense. In addition she updates the collection of paths $\mathcal{P}$ by solving an appropriate network flow problem.

The detailed proofs require to establish several properties of expanders, and combine them with Menger's Theorem as well as with some spectral techniques.

1.3 Organization

The next section contains the main lemmas needed for the proofs. Theorem 1.1 is proved in Section 3, and Theorem 1.2 is proved in Section 4. A result for vertex disjoint paths is described in Section 5, and the final section contains some concluding remarks.

2 Lemmas

In order not to overload the paper with parameters, we have chosen to describe the results for specific values of (some of) the constants. The analysis can be easily modified to deal with other values. It will be convenient to assume, throughout the paper, that $d > 60$. We further assume, whenever this is needed, that the numbers of vertices n, n' of our graphs are sufficiently large, and omit all floor and ceiling signs whenever these are not crucial.

We need the following additional definition, which is similar to Definition 1.1, with somewhat different parameters and with the regularity assumption replaced by one on the minimum degree.

Definition 2.1 *A graph $G' = (V', E')$ is a d'-*strong expander *on n' vertices if its minimum degree is at least d' and it satisfies the following:*
(i) The average degree in any subgraph of G' on at most $n'/10$ vertices is at most $2d'/9$.
(ii) The average degree in any subgraph of G' on at most $n'/2$ vertices is at most $8d'/9$.

Our first simple lemma asserts that if one deletes a relatively small number of edges from a very strong expander, then the resulting graph contains a large subgraph with high minimum degree.

Lemma 2.1 *Let $G = (V, E)$ be a very strong d-regular expander on n vertices. Let $W \subset E$ be a set of edges, where $|W| \leq \frac{nd}{150}$. Let Dense(G, W) be the subgraph of G obtained from $G - W$ by repeatedly omitting from $G - W$ vertices of degree smaller than $3d/4 + 2$ as long as there are such vertices. Then the subgraph Dense(G, W) has more than $n - \frac{15|W|}{d}$ vertices.*

Proof: Suppose this is false. Then during the process of producing Dense(G, W) from G all the vertices in a set X of exactly $k = \frac{15|W|}{d} \leq \frac{n}{10}$ vertices have been omitted. Each vertex of X is connected in $G - W$ by at most $\frac{3d}{4} + 1$ edges to $V - X$, and hence in G the number of edges between X and $V - X$ is at most $|W| + k(\frac{3d}{4} + 1)$. But since $k \leq \frac{n}{10}$ this number has to be at least $k\frac{5d}{6}$, by property (i) in the definition of a very strong expander. We conclude that

$$k\frac{5d}{6} \leq |W| + k(\frac{3d}{4} + 1),$$

implying that $k(\frac{d}{12}-1) \leq |W|$. As $d > 60$, $k\frac{d}{15} < k(\frac{d}{12}-1) \leq |W|$, it follows that $k < \frac{15|W|}{d}$, contradicting the choice of k and completing the proof. ■

The above lemma will be applied repeatedly in our algorithm, we thus state the following simple corollary.

Corollary 2.2 *Let $G = (V, E)$ be a very strong expander. Starting with $V_1 = V, E_1 = E$ and $G_1 = G = (V_1, E_1)$ consider a process of ℓ steps, where in step number i we choose a set of edges W_i in E_i, and define $G_{i+1} = (V_{i+1}, E_{i+1}) = Dense(G_i, W_i)$. If $\sum_{i=1}^{\ell} |W_i| \leq \frac{nd}{150}$ then the graph $G_{\ell+1}$ has more than*

$$n - \frac{15\sum_{i=1}^{\ell} |W_i|}{d}$$

vertices. ■

Note that the conclusion holds even if during the above process we also delete, in each step i, some additional edges of G besides W_i, as long as these extra edges lie outside V_i, as these do not change the graph G_i. Although this fact is obvious we mention it explicitly as indeed additional edges will be omitted when we apply the corollary in the next section.

The following fact is an immediate consequence of the definitions.

Lemma 2.3 *Any subgraph of minimum degree at least $d' = \frac{3d}{4}$ of a very strong d-regular expander is a d'-strong expander.* ■

We also need the following property of strong expanders.

Lemma 2.4 *For every integer d', any d'-strong expander G' on n' vertices has diameter at most $\frac{2}{3}\log n' + 14$ $(< \log n')$.*

Proof: Fix a vertex v of G'. For each $i \geq 0$, let V_i denote the set of all vertices within distance i from v. Suppose that $|V_{i+1}| \leq \frac{n'}{10}$. Then the number of edges in the induced subgraph of G' on V_{i+1} is at most $\frac{1}{9}d'|V_{i+1}|$, since the average degree in this induced subgraph is at most $\frac{2}{9}d'$. On the other hand, these edges include all edges that touch vertices in V_i. As the average degree of the induced subgraph of G' on V_i is at most $\frac{2}{9}d'$ and the minimum degree in G' is at least d', there are at least $\frac{8}{9}d'|V_i|$ edges touching vertices of V_i. It thus follows that

$$\frac{1}{9}d'|V_{i+1}| \geq \frac{8}{9}d'|V_i|,$$

implying that $|V_{i+1}| \geq 8|V_i|$. By the last inequality $|V_s| \geq \frac{n'}{10}$ for $s = \frac{1}{3}\log(n'/10)$.

Repeating the computation above for sets of size bigger than $n'/10$ but smaller than $n'/2$ we conclude, by part (ii) of the definition of a strong expander, that if $|V_{i+1}| \leq \frac{n'}{2}$ then $\frac{4}{9}d'|V_{i+1}| \geq \frac{5}{9}d'|V_i|$, implying that $|V_{i+1}| \geq \frac{5}{4}|V_i|$. It follows that for s as above, $|V_{s+8}| > \frac{n'}{2}$, that is, for any vertex v of G', more than half the vertices are within distance $s + 8 < \frac{1}{3}\log n' + 7$ of v. This clearly implies that the distance between any two vertices of G' is at most $2s + 16 < \frac{2}{3}\log n' + 14 < \log n'$, where the last inequality follows from the fact that n' is sufficiently large. This completes the proof. ■

The final ingredient we need is the following somewhat technical lemma, that will play a crucial role in the proof of the main results.

Lemma 2.5 *Let $G = (V, E)$ be a d-regular very strong expander on n vertices, and let $\tilde{G} = (V, \tilde{E})$ be a subgraph of G. Let $G' = (V', E')$ be a subgraph of $\tilde{G}$ of minimum degree at least $d' = \frac{3d}{4}$ on a set V' of n' vertices, and let V" be a subset of V' satisfying $|V"| \geq \frac{9n}{10}$. Let S be a sequence of vertices of $\tilde{G}$ so that each vertex appears in S at most $\frac{d}{3}$ $(= \frac{4}{9}d')$ times. Suppose that $\tilde{G}$ contains a collection of $|S|$ edge disjoint paths $\mathcal{P} = \{P_s, s \in S\}$, where for each $s \in S$, P_s starts in s and ends at a vertex of G' (and may possibly be of length 0 in case $s \in V'$.) Then there is a collection $\mathcal{P}' = \{P'_s, s \in S\}$of edge disjoint paths in $\tilde{G}$, where for each s, P'_s starts at s and ends at V".*

Remark: Note that in the above lemma the elements s in the subscript of each path P_s are elements of the sequence S, and the same vertex of G may appear more than once as such an s. Although this notation may be a bit confusing, it is convenient to use it here.

Proof: We apply Menger's Theorem (or, equivalently, the max-flow min-cut and the integrality theorems for network flows). We may and will assume that the paths in $\mathcal{P}$ contain no edges of G', and that only the last vertex of each path in $\mathcal{P}$ lies in V'. Indeed, if this is not the case for some path P_s, simply replace it by its initial segment until the first vertex of V' along the path.

Let H be the graph obtained from $\tilde{G}$ by adding to it two new vertices, x and y. The vertex x is joined to each vertex $v \in V$ that appears in S by m_v parallel edges, where m_v is the number of times v appears in the sequence S. The vertex y is joined by $|S|$ parallel edges to each vertex of V". Our objective is to show that there are $|S| = \sum m_v$ edge disjoint paths in H from x to y. By Menger's Theorem, (c.f., e.g., [19]), this is equivalent to showing that any cut separating x and y contains at least $|S|$ edges. Let (X, Y) be such a cut, where $x \in X, y \in Y$ and X and Y form a partition of the vertex set of H into two disjoint parts. Since there are $|S|$ parallel edges joining y to each vertex of V", it follows that if some vertex of V" lies outside Y then the size of the cut is at least $|S|$. We thus may and will assume that $V" \subset Y$, implying that X contains at most $n' - \frac{9n}{10} \leq \frac{n'}{10}$ vertices of V'.

Let p denote the number of paths P_s in $\mathcal{P}$ so that the cut (X, Y) either contains the edge xs corresponding to this path, or contains any other edge of the path. Then $\mathcal{P}$ contains $|S| - p$ additional paths, and all their endpoints belong to X. We claim that each vertex $v \in V'$ is the end vertex of at most $\frac{7}{9}d'$ of these paths. Indeed, by assumption, there are at most $\frac{d}{3} \leq \frac{4}{9}d'$ trivial paths, of length 0, that end (and start) at v. In addition, there can be at most $\frac{d}{4} = \frac{d'}{3}$ nontrivial paths that end at v, since each such path contains an edge of G incident with v which does not lie in the graph G', and there are at most $d/4$ such edges, as the degree of v inside G' is at least $d' = \frac{3d}{4}$. This proves the claim, and implies that the number of vertices of V' that belong to X is at least

$$\frac{|S| - p}{7d'/9}.$$

Since this number is at most $n'/10$, as mentioned above, and as by Lemma 2.3 G' is a d'-strong expander, it follows that the number of edges of G' from X to Y is at least $|X|\frac{7d'}{9} \geq |S| - p$. Altogether this shows that the size of (X, Y) is at least $|S|$, completing the proof. ∎

3 The algorithm

In this section we prove Theorem 1.1. It is convenient to consider the following equivalent formulation of the routing game between the two players, the adversary and the router. The game starts with a sequence S of length $nd/3$ in which each of the n vertices of the d-regular very strong expander $G = (V, E)$ appears $d/3$ times. In round number i the adversary picks two vertices s_i, t_i from S, hands them to the router and omits them from S. The router then has to construct a path Q_i from s_i to t_i, making sure that it does not share any edge of the previous paths Q_j. Our objective is to show that the router can produce such paths, in deterministic polynomial time, for $r = \frac{nd}{150 \log n}$ steps.

In order to achieve this goal, the router maintains, during the game, the set of paths Q_j constructed so far, as well as a subgraph of G and a collection of paths from the remaining vertices of S to this subgraph. More precisely we assume that after the end of the first $\ell - 1$ rounds the router has already constructed $\ell - 1$ edge disjoint paths $Q_1, Q_2, \ldots, Q_{\ell-1}$, where Q_i is a path from s_i to t_i. In addition she maintains a subgraph $G_\ell = (V_\ell, E_\ell)$ of G and a collection $\mathcal{P}_\ell = \{P_s : s \in S_\ell\}$ where $S_\ell = S - \{s_1, t_1, \ldots, s_{\ell-1}, t_{\ell-1}\}$ } of edge disjoint paths, so that the following conditions hold:

(i) The minimum degree of G_ℓ is at least $\frac{3d}{4} + 2$.

(ii) The number of vertices of G_ℓ is at least $n - \frac{15(\ell-1)\log n}{d}$ (which exceeds $9n/10$ provided $\ell \leq r = \frac{nd}{150 \log n}$.) In fact, G_ℓ is the graph Dense(G, W_ℓ), for some explicit set W_l of no more than $(\ell - 1)\log n$ edges.

(iii) For each $s \in S_\ell$ the path P_s in the collection $\mathcal{P}_\ell$ is a path from s to the graph G_ℓ, such that the paths in $\mathcal{P}_\ell$ are edge-disjoint from each other and from each of $Q_1, \ldots, Q_{\ell-1}$ and contain no edges of G_ℓ.

(iv) G_ℓ contains no edge of any of the paths $Q_1, \ldots, Q_{\ell-1}$.

At the beginning $\ell = 1$, there are no paths Q_i, $S_1 = S$, $\mathcal{P}_1$ is a collection of $|S| = n\frac{d}{3}$ trivial paths (each of length 0), $G_1 = (V_1, E_1)$ is simply $G = (V, E)$ and $W_1 = \emptyset$. Trivially this satisfies all the requirements above. Suppose, now, by induction, that the router has managed to compute paths $Q_1, \ldots, Q_{\ell-1}$, a collection of paths $\mathcal{P}_\ell$, a set of edges W_ℓ and a subgraph G_ℓ so that all requirements above hold. The adversary now picks a pair of vertices s_ℓ, t_ℓ that lie in S_ℓ, presents them to the router and omits them from S_ℓ to get $S_{\ell+1} = S_\ell - \{s_\ell, t_\ell\}$. The router first constructs the path Q_ℓ as follows. The collection $\mathcal{P}_\ell$ contains disjoint paths $P_1' = P_{s_\ell}$ and $P_2' = P_{t_\ell}$ from s_ℓ and t_ℓ, respectively, to the set of vertices of G_ℓ. By Lemma 2.3 and Lemma 2.4, the diameter of G_ℓ is at most $\log n$, hence there is a path Q_ℓ' of length at most $\log n$ between the two endpoints of the paths P_1' and P_2', and this path consists of edges of G_ℓ, and thus does not intersect any of the paths $Q_1, \ldots, Q_{\ell-1}$, by item (iv) above. The union of the paths P_1', Q_ℓ' and P_2' is a (not necessarily simple) path between s_ℓ and t_ℓ and hence contains the edges of such a simple path, which will be Q_ℓ. Note that this path can be constructed efficiently, the most difficult computational task here is that of finding the short path Q_ℓ', which is easy.

The router can now generate the graph $G_{\ell+1}$. Set $W_{\ell+1} = W_\ell \cup Q_\ell'$, and then use Lemma 2.1 to define $G_{\ell+1} = \text{Dense}(G, W_{\ell+1})$ ($= \text{Dense}(G_\ell, Q_\ell')$). By Lemma 2.1 and Corollary 2.2 the number of vertices of $G_{\ell+1}$ is at least $n - \frac{15\ell \log n}{d} \geq \frac{9n}{10}$, showing that items (i) and (ii) above (with $\ell + 1$ replacing

ℓ) hold for $G_{\ell+1}$. Furthermore, $G_{\ell+1}$ is a subgraph of G_ℓ and is edge-disjoint from Q'_ℓ, so $G_{\ell+1}$ satisfies (iv), with $\ell - 1$ replaced by ℓ.

We now show that (iii) is satisfied, with ℓ replaced by $\ell + 1$. Let $\mathcal{P}'_\ell = \mathcal{P}_\ell - \{ P_{s_\ell}, P_{t_\ell} \}$. The graph $\tilde{G}$ consisting of all edges of the paths in $\mathcal{P}'_\ell$ together with all edges of the graph G_ℓ besides those of Q'_ℓ, its subgraph $G' = G_\ell - E(Q'_\ell)$ (which has minimum degree at least $\frac{3d}{4} + 2 - 2 = \frac{3d}{4}$) and the set $V" = V_{\ell+1}$ satisfy the conditions in Lemma 2.5 with respect to $S_{\ell+1}$. Thus, there is a collection $\mathcal{P}_{\ell+1}$ of edge-disjoint paths in $\tilde{G}$ from the vertices of $S_{\ell+1}$ to $V"$. Without loss of generality each of these paths contains only one vertex of $V"$- its end vertex, and hence it contains no edge of $G_{\ell+1}$. This shows that item (iii) holds, with ℓ replaced by $\ell + 1$. The whole procedure can be performed in polynomial time, as the only nontrivial algorithmic task is that of solving an appropriate network flow problem for finding the paths in the collection $\mathcal{P}_{\ell+1}$. This completes the proof of the theorem. ∎

4 An improved bound for (n, d, λ)-graphs

There are many known explicit constructions of graphs that satisfy the assumptions in Theorem 1.1. Recall the definition of an (n, d, λ)-graph (Definition 1.2). By a lemma in [2] (see also [4], Chapter 9), the average degree of any subgraph of such a graph on a set of bn vertices is at most $bd + (1 - b)\lambda$. Therefore, if $\lambda \leq \frac{2d}{27}$ then any such graph is a very strong expander. It is known that if d is fixed and n is large, then λ cannot be much smaller than $2\sqrt{d-1}$, and the Ramanujan graphs of [15], [16] are infinite families of explicit $(n, d, 2\sqrt{d-1})$-graphs for any d which is $p + 1$ for a prime p congruent to 1 modulo 4. Thus there are explicit constructions satisfying the assumptions in Theorem 1.1.

In fact, one can use any family of expanders to get examples satisfying the assumptions of Theorem 1.1. Indeed, it is well known (see [1]) that a family of d-regular graphs (with a loop in each vertex) is a family of expanders (that is, any set X of at most half the vertices has at least $c|X|$ neighbors outside the set, where the same $c > 0$ applies to all members of the family), if and only if there is a fixed $\lambda < d$ so that every graph G in the family is an (n, d, λ)-graph for $n = |V(G)|$. Raising the graphs to an appropriate power raises d and λ to the same power, and we can thus get a family of (n, D, μ)-graphs with $\mu \leq D^{1-\epsilon}$.

Moreover, using (n, d, λ)-graphs one can in fact slightly improve the result in Theorem 1.1 getting an extra $\log(d/4\lambda)$ factor in the number of rounds in the routing game, as stated in Theorem 1.2. This follows by repeating the reasoning of the proof of Theorem 1.1, replacing Lemma 2.4 by the following result (whose proof is similar to that of a lemma proved in [3]).

Lemma 4.1 *Let G' be a subgraph of an (n, d, λ)-graph G, where $\lambda < d/4$, in which the minimum degree is at least $3d/4$. Then the diameter of G' is at most $\frac{\log n}{\log(d/4\lambda)}$.*

Remark: The constant 4 can be easily reduced, we make no attempt to optimize it here.

Proof: Let X be a set of vertices of G', let B denote the set of all non-neighbors of the vertices of X in G', and assume that $|B| \geq n/2$. Then each vertex of X has (in G) at most $d/4$ neighbors in B. By Theorem 9.2.4 in [4] this implies that

$$|X|(\frac{d}{4})^2 \leq \lambda^2|B|(n - |B|)/n < \lambda^2(n - |B|).$$

Thus $n - |B|$, which is the number of vertices within distance 1 (in G') from X is at least $|X|(\frac{d}{4\lambda})^2$. By a repeated application of the above it follows that for every vertex v the number of vertices of G' which are within distance at most $\frac{\log(n/2)}{2\log(d/4\lambda)}$ from v exceeds $n/2$, implying the desired result. ■

Theorem 1.2 can be proved by repeating the proof of Theorem 1.1, using the fact that each of the paths Q'_ℓ will now be of length at most $\frac{\log n}{\log(d/4\lambda)}$. We omit the details.

5 Vertex disjoint paths

Finding vertex disjoint paths between pairs of vertices is more difficult than finding edge disjoint ones. Indeed, if G is any d-regular graph then obviously it cannot contain nontrivial $d+1$ vertex disjoint paths starting at a vertex and its d neighbors. Thus, when we deal with vertex disjoint paths we have to assume that not too many paths start at a neighborhood of a given vertex. Since we are interested in the online problem, we consider the Vertex Routing Game, which is the vertex disjoint analog of the routing game discussed in the introduction. In each round of this game, played on a graph G, the adversary presents a pair of vertices s_i, t_i, and the router has to provide a path between them, keeping all paths produced vertex disjoint. Obviously the adversary is not allowed to use a vertex as an s_i or a t_i in case this vertex has already been used by the router, and in addition, he has to be restricted to avoid some trivial cases like the one mentioned above. It is easy to deduce a result for this game from our results for the edge disjoint game, as follows.

Theorem 5.1 *Let $G = (V, E)$ be a d-regular very strong expander on n vertices, and let H be the d-blow-up of G, that is, the graph obtained by replacing each vertex v of G by an independent set I_v of d vertices, and each edge uv of G by a complete bipartite graph with vertex classes I_u, I_v. Put $r = \frac{nd}{150\log n}$. Then the router can win the vertex routing game played on H for r rounds, provided none of the vertices s_i, t_i given by the adversary is equal to a vertex used already by the router for previous paths, and no set I_v contains more than $d/3$ of them. Moreover, the router wins by a polynomial time deterministic algorithm.*

This is optimal, up to a constant factor, as the distance between each s_i and t_i may be $\Omega(\log n)$ and the total number of vertices of H is nd. The $d/3$ requirement is also essentially optimal (up to a constant factor) by the comment preceding the theorem, and the assumption that all vertices s_i, t_i differ from previously used vertices is, of course, necessary for getting vertex disjoint paths. The proof is simple: one uses the strategy for edge-disjoint paths on the original graph G and replaces them online by vertex disjoint paths in H in the obvious way. If we start with a Ramanujan graph G, then the bound for r can be increased by another $\Theta(\log d)$ factor.

6 Concluding remarks

We have given a deterministic, polynomial time algorithm for finding edge-disjoint paths in expanders online. For a sufficiently strong d-regular expander on n vertices G (for example, a bounded degree Ramanujan graph) and a sequence of pairs s_i, t_i, $(1 \le i \le r)$ of vertices of G, where $r = \Theta(\frac{nd\log d}{\log n})$, and no vertex appears more than $d/3$ times in the list of all endpoints $s_1, t_1, \ldots, s_r, t_r$, the algorithm outputs edge-disjoint paths $Q_1, \ldots, Q_r$, where Q_i connects s_i and t_i. This improves the previously best known result on the problem, due to Frieze [9], in several respects:

- The number of pairs r is bigger by a $\log d$ factor than the one given in [9], it is best possible up to a constant factor, and matches the bound obtained for r for random graphs by Frieze and Zhao [10].
- The algorithm is deterministic, whereas the one given in [9] is randomized.
- Most crucially, the algorithm is online, that is, the path Q_i is produced as soon as s_i, t_i are given, and before the subsequent requests s_j, t_j are revealed.

The result can be applied to a related problem considered by Kleinberg and Rubinfeld in [11], where one is allowed to reject some of the requested paths. This improves the approximation guarantee of [11] from $\Omega(\frac{1}{\log n \log\log n})$ to $\Omega(\frac{1}{\log n})$ of the optimum. Although the algorithm is more complicated than that of [11], and not all the paths it produces are necessarily short, it has another advantage that it can decide arbitrarily which paths it rejects and which paths it accepts (and its approximation guarantee is 1 if the total number of paths requested is $O(\frac{nd}{\log n})$).

We have also shown that blow-ups of sufficiently strong expanders enjoy a similar property with respect to vertex disjoint paths. This is related to the notion of non-blocking networks, such as the one constructed in [5]. Although it does not ensure short paths and the routing algorithm is more complicated than the one of [5], the extra property we have here is that there are no pre-specified distinguished sets of vertices of size $O(n/\log n)$ each, which are specified at the beginning as inputs and outputs. Indeed, one can route here a path between any desired pair of vertices as long as the total number of paths is at most $O(n/\log n)$ (which is optimal) and the endpoints satisfy some minimal requirement (which is also essentially optimal.)

It will be interesting to find a version of our algorithm in which all paths are guaranteed to be of logarithmic length. This remains open.

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