2-factors in Dense Graphs

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Abstract

A conjecture of Sauer and Spencer states that any graph G on n vertices with minimum degree at least $\frac{2}{3}n$ contains any graph H on n vertices with maximum degree 2 or less. This conjecture is proven here for all sufficiently large n.

1 Introduction

All graphs considered here are finite and have no loops and no parallel edges. A **2-factor** of a graph G is a 2-regular spanning subgraph of G, that is, a spanning subgraph every connected component of which is a cycle. In the following discussion C_k will always denote a cycle of k vertices.

Corrádi and Hajnal [8] proved that any graph G on at least $3k$ vertices with minimum degree at least $2k$ contains k vertex disjoint cycles. In particular, a graph on $n = 3k$ vertices with minimum degree at least $\frac{2}{3}n$ contains a 2-factor consisting of k vertex disjoint triangles. It is easy to see that this is tight, as the complete 3-partite graph with vertex classes of sizes $k-1$, k and $k+1$ has minimum degree $\frac{2}{3}n - 1$ and does not contain k vertex disjoint triangles. The problem of determining the best possible minimum degree of a graph G that ensures it contains a 2-factor of a prescribed type has been considered by various researchers. Sauer and Spencer [10] proved that any graph with n vertices and minimum degree at least $\frac{3}{4}n$ contains any graph H on n vertices with maximum degree 2 or less. They conjectured that the same can be guaranteed by minimum degree at least $\frac{2}{3}n$:

Conjecture 1.1 ([10]) *Any graph G with n vertices and minimum degree at least $\frac{2}{3}n$ contains any graph H on n vertices with maximum degree 2 or less.*

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Note that any graph H with maximum degree 2 or less is contained in a graph (on the same number of vertices) consisting of vertex disjoint cycles and possibly either one isolated edge or one isolated vertex. Indeed, if H contains non-cycle components (which can be only isolated vertices or paths) whose total number of vertices is at least 3, these components can be joined to create a cycle. If H contains only 2 isolated vertices, these can be joined by an edge.

Here we prove Conjecture 1.1 for all sufficiently large n, as stated in the following:

Theorem 1.2 *There exists an integer N such that any graph G with $n > N$ vertices and minimum degree at least $\frac{2}{3}n$, contains any graph H on n vertices with maximum degree 2 or less.*

A much stronger conjecture is the one of El-Zahar [9]. This conjecture states, that in order for a graph G on n vertices to contain a 2-factor consisting of l vertex disjoint cycles C_{n_i} satisfying $\sum_{i=1}^{l} n_i = n$, minimum degree $\sum_{i=1}^{l} \lceil \frac{n_i}{2} \rceil$ suffices. Another generalization of Conjecture 1.1 is the one of Bollobás and Eldridge [5], which deals with subgraphs with a larger maximum degree. This conjecture states that if H_1 and H_2 are two graphs on n vertices whose maximum degrees are Δ_1 and Δ_2 respectively, where $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$, then the complete graph on n vertices contains edge disjoint copies of H_1 and H_2.

In the rest of the paper we present the proof of Theorem 1.2. The proof combines combinatorial and probabilistic arguments together with an asymptotic result proven in [2].

2 Outline and preliminaries

Here is an outline of the rest of the paper. We may always assume that H consists of vertex disjoint cycles, and possibly either one extra isolated edge or one extra isolated vertex. We consider several cases according to the sizes of the components of the graph H involved. The case of H consisting of triangles only was proven by Corrády and Hajnal in [8]. The case of H consisting of non-triangle cycles of a bounded size is treated in Section 4, and the case when H consists of sufficiently large cycles is treated in Section 5. The last two cases are joined together for dealing with all graphs H consisting of non-triangle components, and then this is joined together with the triangle case. The case when many but not all of the components are triangles requires special attention, and is treated in Section 3. The possibility of having an extra edge or vertex poses no real problem. The method of presentation used here is aimed at simplicity rather than the optimization of the constant N. To further simplify the presentation we omit all floor and ceiling signs whenever the implicit assumption that a certain quantity is integral makes no essential difference.

During the proof of the various cases we often encounter a need to treat some of the cycle components individually. This is done by the following lemma of Bondy:

Lemma 2.1 ([6]) *Any graph H on m vertices with minimum degree exceeding $\frac{m}{2}$ contains a cycle of length i for all $3 \leq i \leq m$.*

For the treatment of the non-triangle cases, and for the joining of the different cases, we need a way of splitting a graph into two subgraphs such that the relative minimal degree of each induced subgraph as well as that of the corresponding bipartite subgraph is not much lower than the original relative degree. For this we need the following lemma:

Lemma 2.2 ([7]) *Let*

$$F(n, \gamma, k, \delta) = \binom{n}{k}^{-1} \sum_{i=0}^{(\gamma-\delta)k} \binom{\gamma n}{i} \binom{(1-\gamma)n}{k-i}$$

denote the probability that a random subset of k elements out of n, γn of which are marked, contains at most $(\gamma - \delta)k$ marked elements, then:

$$F(n, \gamma, k, \delta) \leq e^{-2\delta^2 k}.$$

From this lemma we can prove the following result.

Lemma 2.3 *For any fixed $\epsilon > 0$, there exists a constant $N = N(\epsilon)$ such that if G is a graph on $n > N$ vertices with minimum degree at least αn, and k, l are two integers satisfying $k \geq \epsilon n$, $l \geq \epsilon n$ and $k+l = n$, then there exists a partition of the vertex group of G into two groups A, B of sizes k, l respectively, such that any vertex of G has at least $(\alpha - n^{-\frac{1}{3}})k$ neighbours in A and at least $(\alpha - n^{-\frac{1}{3}})l$ neighbours in B.*

Proof: Choose a partition of the vertices of G into groups A, B of sizes k, l randomly, each possible partition being equally likely. Let v be a vertex of G. By lemma 2.2, the probability that less than $(\alpha - n^{-\frac{1}{3}})k$ of its neighbours lie in A is at most $e^{-2n^{-\frac{2}{3}}k} \leq e^{-2\epsilon n^{\frac{1}{3}}}$. A similar estimate applies to the neighbours of v in B, so the probability that any of these is not satisfied for some vertex in G is at most $2ne^{-2\epsilon n^{\frac{1}{3}}}$. We can now choose N such that for $n > N$ this probability is less than 1, and the required partition exists. $\Box$

Since the following special case of Lemma 2.3 will be used extensively, for the simplicity of the presentation we state it separately:

Corollary 2.4 *For any fixed $\epsilon > 0$, $\alpha > \beta > 0$, there exists a constant $N = N(\epsilon, \alpha, \beta)$ such that if G is a graph on $n > N$ vertices of minimum degree at least αn, and k, l are two integers satisfying $k \geq \epsilon n$,*

$l \geq \epsilon n$ and $k + l = n$, then there exists a partition of the vertex group of G into two groups A, B of sizes k, l respectively, such that any vertex of G has at least βk neighbours in A and at least βl neighbours in B. $\square$

3 Graphs with many triangles

In this section we prove the following simple result.

Proposition 3.1 *If the minimum degree of a graph G on n vertices is at least $\frac{2}{3}n$, then G contains any graph H on n vertices with maximum degree 2 or less in which the non-triangle components occupy no more than $\frac{1}{3}n$ vertices.*

This proposition is of some interest in itself, as it is the only one proven for all n.

We need the following result of Corrády and Hajnal, mentioned in the introduction.

Lemma 3.2 ([8]) *Any graph G with $n = 3k$ vertices and minimum degree at least $2k$ contains k vertex disjoint triangles.*

We now prove Proposition 3.1, by induction on the number of vertices in the non-triangle components of H. Denote this number by k. As stated before, we may and will assume that all components are cycles except possibly a single edge or a single vertex.

The case $k \leq 2$: For $k = 0$, this is Lemma 3.2. For $k = 1$, remove a vertex from the graph. Since $n \equiv 1 \pmod 3$, the remaining subgraph still satisfies the conditions of Lemma 3.2, so all required $\frac{n-1}{3}$ triangles can be found. For $k = 2$, add a new vertex v_0 joining it to all other vertices of the graph. Apply Lemma 3.2, and then remove v_0 to obtain $\frac{n-2}{3}$ triangles plus an isolated edge.

The induction step: We prove the proposition for $k+3 \leq \frac{n}{3}$, provided it is known for k. Since $k+3 \geq 3$, there must be a non-triangle component which is a cycle, C. If C is a square, replace it by a triangle plus a vertex and use the induction hypothesis to find an appropriate subgraph. Now take the vertex, v_1, and find a triangle to which it is connected by at least 2 edges, forming a square. The average number of neighbours of v_1 in a triangle is at least $\frac{\frac{2}{3}n+1-k}{\frac{1}{3}(n-k)}$, and for $k < \frac{1}{3}n$ this is larger than 1, ensuring a triangle to which v_1 is connected by at least 2 edges. If C is a cycle C_m for $m > 4$, replace it by a triangle plus a C_{m-3} or an isolated edge if $m = 5$, and find the appropriate subgraph by the induction hypothesis. Now choose an edge from the C_{m-3} or take the isolated edge if $m = 5$, and find a triangle to which it is connected by four edges, forming the required C_m. The average number of edges from the two vertices v_1, v_2 of the edge to a triangle is at least $\frac{\frac{4}{3}n+2-2k}{\frac{1}{3}(n-k)}$, and for $k < \frac{1}{3}n$ this is larger than 3, ensuring a triangle to which the edge is connected four times.

This completes the proof of Proposition 3.1. □

4 2-factors with bounded non-triangle components

In this section we prove the following:

Proposition 4.1 *For every m there exists an $\eta = \eta(m) > 0$ and $N = N(m)$ such that any graph G with $n > N$ vertices and minimum degree $d \geq (\frac{2}{3} - \eta)n$ contains any 2-factor whose components are in the range $C_4, \ldots, C_m$.*

Most of the cycles required in the proposition are extracted from copies of the complete 3-partite graph on h-sized classes for h large enough, and in order to obtain these copies we need the following lemma:

Lemma 4.2 ([2]) *For every $\epsilon > 0$ and integers c, h there exists an $N = N(\epsilon, c, h)$ such that any graph with $n > N$ vertices and minimum degree $d \geq \frac{c-1}{c}n$ contains at least $(1-\epsilon)\frac{n}{ch}$ vertex disjoint copies of the complete c-partite graph with h vertices in each color class.*

Corollary 4.3 *For every $\eta > 0$ and integer h there exists an $N = N(\eta, h)$ such that any graph with $n > N$ vertices and minimum degree $d \geq (\frac{2}{3} - \eta)n$ contains at least $(1 - 10h\eta)\frac{n}{3h}$ vertex disjoint copies of $K_{h,h,h}$, the complete 3-partite graph with h vertices in each class.*

Proof: Set N to be $N(h\eta, 3, h)$ as in Lemma 4.2. For a graph G with $n > N$ vertices and minimum degree $d \geq (\frac{2}{3} - \eta)n$, add $l = 3\eta n$ new vertices $v_1, \ldots, v_l$ to the graph, joining them to all other vertices. Now use Lemma 4.2 to find (more than) $(1 - h\eta)\frac{n}{3h}$ copies of $K_{h,h,h}$, and discard the ones containing vertices from $v_1, \ldots, v_l$. □

To deal with the few remaining vertices in G, these will be grouped with other vertices selected in advance to ensure the matching of each vertex to a class of a copy of $K_{h,h,h}$ in a way that will enable the extraction of cycles. For this matching we will utilize the following simple consequence of Hall's theorem (see e.g. [4]):

Lemma 4.4 *If G is a bipartite graph on the vertex classes V_1, V_2 satisfying $|V_1| = |V_2| = t$ with minimum degree at least $\frac{1}{2}t$, G contains a perfect matching (i.e. it contains t vertex disjoint edges).*

To organize the components of the 2-factor into extractable groups, we need the following simple lemma:

Lemma 4.5 *If $n_1, \ldots, n_l$ are integers satisfying $1 \le n_i \le m$, than the index group $\{1, \ldots, l\}$ can be partitioned into subgroups $I_0, \ldots, I_j$ such that $\sum_{i \in I_0} n_i < 3mm!$, and each I_k, $0 < k \le j$, is a group of size $\frac{3m!}{s}$ for some $1 \le s \le m$ satisfying $n_i = s$ for all $i \in I_j$.*

Proof: For $j > 0$ we construct the groups I_j one by one. Once a new I_j can not be constructed, the remaining integers n_i contain less than $3m!$ 1's, $\frac{3m!}{2}$ 2's, and so on. Putting all those in I_0 yields the desired result. $\square$

We now prove Proposition 4.1. Set $h = m! - 1$, $\eta = \frac{1}{450hm!}$, and $N = \max\{\eta^{-1}3mm!, N_0, \frac{3m!}{3m!-1}N_1\} + 3mm!$ where $N_0 = N(\frac{14}{15m!}, \frac{2}{3} - 2\eta, \frac{2}{3} - 3\eta)$ as in Corollary 2.4, $N_1 = N(3\eta, h)$ as in Corollary 4.3.

Let G be a graph on $n > N$ vertices with minimum degree at least $(\frac{2}{3} - \eta)n$, and let H be a 2-factor on n vertices, all of whose components are cycles of sizes ranging from 4 to m. We use Lemma 4.5 to partition these into groups $I_0, \ldots, I_j$, the total size of the components in I_0 being less than $3mm!$ and each of the groups $I_1, \ldots, I_j$ being a group of cycles of equal size, whose total size is $3m!$.

We pick out the cycles in I_0 one by one, using Lemma 2.1 for that purpose. Denote the remaining induced subgraph of G by R. It is clear that R is a graph on $3jm!$ vertices, and that its minimum degree is at least $(\frac{2}{3} - 2\eta)n \ge (\frac{2}{3} - 2\eta)|R|$.

We now partition the vertex group of R using Corollary 2.4 into groups A, B of sizes $k = 3(m! - \frac{14}{15})j$, $l = \frac{14}{5}j$ respectively, so that each vertex of G has at least $(\frac{2}{3} - 3\eta)k$ neighbours in A and $(\frac{2}{3} - 3\eta)l$ neighbours in B.

We find in the induced subgraph on A $(1 - 30h\eta)\frac{k}{3h}$ copies of $K_{h,h,h}$ using Corollary 4.3, and keep j of them, adding the remaining $\frac{1}{5}j$ vertices to B. Denote this union by V, and denote the group of all color classes of the retained copies of $K_{h,h,h}$ by C. Note that by the choice of the parameters $30h\eta m! = \frac{1}{15}$, hence $(1 - 30h\eta)\frac{k}{3h} > j$, and hence the last step is possible.

Define a bipartite graph with color classes V, C as follows: $v \in V$ is connected to $c \in C$ iff the vertex v is connected to at least two vertices of the color class c. Note that $|V| = |C| = 3j$. We now claim that the minimum degree of this graph is at least $\frac{3}{2}j$: Each vertex $v \in V$ is connected to at least $\frac{3}{2}j$ vertices of C since by the choice of the parameters v has more than $(\frac{2}{3} - 3\eta - \frac{1}{15h})k > \frac{7}{12}k > \frac{3}{2}(1 + \frac{1}{h})jh$ neighbours in the set of all vertices of R in the classes $c \in C$. Each $c \in C$ is connected to at least $\frac{3}{2}j$ vertices of V since c contains at least 6 vertices of R, each of which is by the choice of the parameters connected to at least $(\frac{2}{3} - 3\eta)\frac{14}{5}j > \frac{21}{12}j$ vertices of V.

Therefore by Lemma 4.4, there is a perfect matching between C and V, and hence R contains j pairwise vertex disjoint copies of K, K being the graph on $3m!$ vertices consisting of a copy of $K_{h,h,h}$ on the color classes c_0, c_1, c_2 plus 3 extra vertices, denoted by v_0, v_1, v_2, such that for each $k \in \{0, 1, 2\}$, v_k is adjacent to two vertices in c_k. From these subgraphs all cycles in $I_1, \ldots, I_j$ are extracted as follows.

To extract the cycles of I_i for $1 \leq i \leq j$, we consider a copy of K. For each $k \in \{0, 1, 2\}$, we consider a triple $(x_0, x_1, x_2) \equiv (k, k+1, k+2) \pmod 3$. We now show how to extract $\frac{m!}{s}$ cycles of size s for each of these three triples.

For the first cycle, if s is even we use v_{x_0}, $\frac{s}{2}$ vertices from c_{x_0} including the two neighbours of v_{x_0}, and $\frac{s}{2} - 1$ vertices from c_{x_1}. If s is odd, $s \geq 5$, we use v_{x_0}, $\frac{s-1}{2}$ vertices from c_{x_0} including the two neighbours of v_{x_0}, $\frac{s-3}{2}$ vertices from c_{x_1}, and a single vertex from c_{x_2}. For each of the remaining $\frac{m!}{s} - 1$ cycles, if s is even we use $\frac{s}{2}$ vertices from c_{x_0} and c_{x_1}. If s is odd, we use $\frac{s-1}{2}$ vertices from c_{x_0} and c_{x_1}, and a single vertex from c_{x_2}.

It is easily seen that this process can be applied to each $k \in \{0, 1, 2\}$, thereby extracting all of the cycles in I_i from a copy of K. This means that all of the cycles in $I_1, \ldots, I_j$ are found in R. All components of H being found now, this ends the proof of Proposition 4.1. $\square$

5 2-factors with large components

The following is proved in this section:

Proposition 5.1 *For every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that any graph G with $n > N$ vertices and minimum degree $d \geq (\frac{1}{2} + \epsilon)n$ contains any 2-factor in which each component is of size at least N.*

Note that this proposition resembles the conjecture of El-Zahar, as it states that the required minimal degree of G is close to $\frac{1}{2}n$ as the components of the 2-factor get larger. The proof uses a "divide and conquer" approach similar to the one used in [1].

Given $\epsilon > 0$, set $\tilde{\epsilon} = min\{\frac{1}{3}, \frac{\epsilon}{2}\}$. Let G be a graph with n vertices, and let $n_1, \ldots, n_l$ be a partition of n with $n_i \geq N$, $N = N(\epsilon)$ to be determined later. Note that in particular $n > N$. We shall now prove that G contains the 2-factor whose components are $C_1, \ldots, C_l$, $|C_i| = n_i$, thereby proving Proposition 5.1.

Let n_1 be the largest n_i. There are two possible cases.

Case 1: If $n_1 \geq (1 - \tilde{\epsilon})n$ then by Lemma 2.1 for all $i > 1$ we can choose the cycles C_i one by one, and since the minimum degree will not decrease by more than $\tilde{\epsilon}n$ during this process, it will still be large enough after each stage to allow the process to end. Once it ends there is still a Hamilton cycle C_1 on the remaining vertices, by Dirac's Theorem (or by Lemma 2.1), completing the proof in this case.

Case 2: If for all n_i, $n_i < (1-\tilde{\epsilon})n$, then, since $\tilde{\epsilon} \leq \frac{1}{3}$, we can partition the index group $\{1, \ldots, l\}$ into two groups I, J such that

$$\sum_{i \in I} n_i \leq (1-\tilde{\epsilon})n \quad \text{and} \quad \sum_{j \in J} n_j \leq (1-\tilde{\epsilon})n.$$

Define

$$n_I = \sum_{i \in I} n_i \quad \text{and} \quad n_J = \sum_{j \in J} n_j.$$

For any graph H, define the normalized minimum degree $D(H)$ to be the minimum degree of H divided by the number of its vertices. By assumption, $D(G) \geq \frac{1}{2} + \epsilon$. According to Lemma 2.3 for $n > N_0$, $N_0 = N_0(\tilde{\epsilon})$, there exists a partition of the vertices of G into two groups G_I, G_J of cardinalities n_I and n_J respectively, such that the normalized minimum degrees of the induced subgraphs satisfy

$$D(G_I) \geq D(G) - n^{-\frac{1}{3}} \quad \text{and} \quad D(G_J) \geq D(G) - n^{-\frac{1}{3}}.$$

Now we can continue to apply the same splitting procedure to each of the subgraphs and its required partition as long as Case 2 holds. Since each time the number of vertices is reduced by at least a factor of $1-\tilde{\epsilon}$, and since the process terminates before $n \leq N$, it follows that for each final induced subgraph H, the appropriate $D(H)$ is at least

$$D(G) - \sum_{i=0}^{\infty} \left(\frac{(1-\tilde{\epsilon})^i}{N} \right)^{\frac{1}{3}} = D(G) - \frac{1}{1-(1-\tilde{\epsilon})^{\frac{1}{3}}} N^{-\frac{1}{3}}.$$

If $N > N_0$ is large enough, the last quantity is larger than $\frac{1}{2} + \tilde{\epsilon}$, and hence the required cycles can be found as in the first case, thereby proving Proposition 5.1. $\square$

6 The proof of the main result

We begin by combining the cases of the last two sections, incorporating also the possibility of a single edge or a single vertex.

Corollary 6.1 *There exist an $\eta > 0$ and an integer N such that any graph G on $n > N$ vertices with minimum degree at least $(\frac{2}{3} - \eta)n$ contains any graph H with maximum degree 2 or less containing no triangles.*

Proof: Set $m = N(\frac{1}{12})$ as in Proposition 5.1 and $\eta = \min\{\frac{1}{36}, \frac{1}{3}\eta_0\}$, $\eta_0 = \eta(m)$ as in Proposition 4.1. Set $N = \max\{2\eta^{-2}, \eta^{-1}(N_0 + 2), N_1\}$, where $N_0 = N(m)$ as in Proposition 4.1, and $N_1 = N(\eta, \frac{2}{3} - \eta, \frac{2}{3} - 2\eta)$

as in Corollary 2.4. We deal with three possible cases:

Case 1: If the total size of all components of size larger then m is less than ηn, we pick these out one by one using Lemma 2.1. Then the isolated edge or vertex, if needed, is picked out. By the choice of the parameters the remaining induced subgraph of G still satisfies the conditions of Proposition 4.1 required to find the remaining components.

Case 2: If the total size of all components in $C_4, \ldots, C_m$ is less than ηn, we pick these out one by one using Lemma 2.1, and then the isolated edge or vertex, if needed, is picked out. By the choice of the parameters the remaining subgraph satisfies the conditions of Proposition 5.1 required to find the remaining components.

Case 3: Let k be the total size of all components of size larger then m, and let l be $n - k$. If both k, l are at least ηn, we split G using Corollary 2.4 into two induced subgraphs A, B of sizes k, l, with minimum degrees at least $(\frac{2}{3} - 2\eta)k$, $(\frac{2}{3} - 2\eta)l$, respectively. A satisfies the conditions of Proposition 5.1 required to pick out all components of size larger than m. From B we pick the isolated edge or vertex if needed. The remaining induced subgraph still satisfies the conditions of Proposition 4.1 required to find the remaining components.

All possible cases being covered, this completes the proof of the corollary. $\square$

For finding subgraphs containing triangles, we have to pick most of those triangles out of a graph G of minimum degree a little less then $\frac{2}{3}|G|$. For this we use the following simple corollary of Lemma 3.2:

Corollary 6.2 *Any graph G on $n = 3k$ vertices with minimum degree at least $2k - \eta n$ contains at least $k - 2\eta n$ vertex disjoint triangles.*

Proof: Add $l = 3\eta n$ new vertices $v_1, \ldots, v_l$ to G, connecting them to all other vertices. Apply Lemma 3.2 to find $k + \eta n$ triangles, and discard the ones containing any of the vertices $v_1, \ldots, v_l$. $\square$

Proof of Theorem 1.2: Set $\lambda = \frac{1}{11}\eta$, η as in Corollary 6.1. Set $N = \max\{\lambda^{-1}N_0, N_1\}$, where N_0 is the integer of Corollary 6.1 and $N_1 = N(\lambda, \frac{2}{3}, \frac{2}{3} - \lambda^2)$ as in Corollary 2.4. Let G be a graph on $n > N$ vertices with minimum degree at least $\frac{2}{3}n$, and let H be a graph on n vertices with maximum degree 2 or less. We deal with three possible cases:

Case 1: If the total size of all non-triangle components in H is less than $\lambda n(< \frac{1}{3}n)$, the fact that G contains H is guaranteed by Proposition 3.1.

Case 2: If the total size of all triangle components is less than λn, we pick out the triangles one by one using Lemma 2.1. The remaining induced subgraph of G satisfies the conditions of Corollary 6.1 required to find all remaining components of H.

Case 3: Let k be the total size of all triangle components, and let l be $n-k$. If both k, l are at least λn, we split the vertex group of G using Corollary 2.4 into subgroups A, B of sizes k, l respectively, so that each vertex of G has at least $(\frac{2}{3}-\lambda^2)k$ neighbours in A and at least $(\frac{2}{3}-\lambda^2)l$ neighbours in B. Out of the induced subgraph of G on A we pick out all but less than $2\lambda^2 l < 2\lambda^2 n$ triangles using Corollary 6.2, and add the remaining vertices, denote their number by j, to B. Denote the induced subgraph on the union of B with these vertices by L. It is easily seen that $j \leq 6\lambda|B| \leq 6\lambda|L|$, and that by the choice of the parameters L has at least $N_0 + j$ vertices and minimum degree more than $(\frac{2}{3} - 5\lambda)|L| \geq (\frac{2}{3} - \eta)|L| + j$. We pick out from L all the remaining $\frac{1}{3}j$ triangles one by one using Lemma 2.1. The remaining induced subgraph of L satisfies then the conditions of Corollary 6.1, and hence contains the remaining components of H.

All cases being covered, this completes the proof of our theorem. $\square$

7 Concluding remarks

- It would be interesting to prove or disprove Conjecture 1.1 for all n. Since the proof given here depends heavily on Lemma 4.2, which is proved using the Regularity Lemma of Szemerédi [11], the bound it provides on the integer N is very large.

- The assertion of Conjecture 1.1 for 2-regular H with at most 3 connected components follows from the results in [9], [12]. Proposition 5.1 may be used to prove the following more general asymptotic result.

  Corollary 7.1 *For every integer l and every $\epsilon > 0$ there exists $N = N(l, \epsilon)$, such that any graph G with $n > N$ vertices and minimum degree at least $(\frac{1}{2} + \epsilon)n$ contains any 2-factor H with l components.*

  Proof: Set $N = l\epsilon^{-1}N_0(\frac{\epsilon}{l})$, N_0 being the function in Proposition 5.1. Suppose that G is a graph with $n > N$ vertices and minimum degree at least $(\frac{1}{2} + \epsilon)n$, and that $n_1, \ldots, n_l$ is a partition of n. For all $1 \leq j \leq l$ satisfying $n_j \leq \frac{\epsilon}{l}$n, apply Lemma 2.1 to pick a cycle of length n_j. Each remaining integer n_k satisfies $n_k > N_0$, and the remaining induced subgraph has the minimum degree required to apply Proposition 5.1 for finding all the remaining required cycles. $\square$

- The conjecture of El-Zahar implies in particular that a graph G with n vertices and minimum degree $\frac{1}{2}n$ contains any 2-factor consisting of even cycles only. In [3] it is proven that for any fixed integer h and any $\epsilon > 0$, there exist $N = N(h, \epsilon)$ such that any graph with $n = 2kh > N$ vertices and minimum degree at least $(\frac{1}{2} + \epsilon)n$ contains k vertex disjoint copies of $K_{h,h}$. Combining this with Proposition 5.1 in a similar manner as done in the proof of Corollary 6.1, we obtain that any graph G with $n > N_1(\epsilon)$ vertices and minimum degree at least $(\frac{1}{2} + \epsilon)n$ contains any bipartite graph H with n vertices and maximum degree 2 or less.

- Define the "El-Zahar Function" $\epsilon(\eta)$ as follows: $\epsilon(\eta) = \overline{\lim}_{n \to \infty} \epsilon_{\eta,n}$, $\epsilon_{\eta,n}$ being the minimum value required to ensure that any graph G with n vertices and minimum degree at least $\frac{1}{2}(1 + \epsilon_{\eta,n})n$ contains every 2-factor with no more than ηn odd cycles. The conjecture of El-Zahar [9], if true, implies that $\epsilon(\eta) = \eta$. The proof of Proposition 5.1, together with the result mentioned in the preceding remark, implies that $\epsilon(\eta)$ is bounded by a (smaller than 1) power of η. It would be interesting to prove a linear bound on $\epsilon(\eta)$, by an appropriate extension of the methods used here, or by some other means.

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References

[1] N. Alon, The Strong Chromatic Number of a Graph, *Random Structures and Algorithms* 3 (1992), 1–7.

[2] N. Alon and R. Yuster, Almost H-factors in Dense Graphs, *Graphs and Combinatorics* 8 (1992), 95–102.

[3] N. Alon and R. Yuster, H-factors in Dense Graphs, to appear.

[4] B. Bollobás, *Extremal Graph Theory*, Academic Press, 1978.

[5] B. Bollobás and S. E. Eldridge, Problem in: *Proc. Colloque Intern. CNRS* (J.-C. Bermond et al. eds.), 1978.

[6] J. A. Bondy, Pancyclic Graphs I, *J. Combinatorical Theory Ser. B* 11 (1971), 80–84.

[7] V. Chvátal, The Tail of the Hypergeometric Distribution, *Discrete Math.* 25 (1979), 285–287.

[8] K. Corrádi and A. Hajnal, On the Maximal Number of Independent Circuits in a Graph, *Acta Math. Acad. Sci. Hungar.* 14 (1963), 423–439.

[9] M. H. El-Zahar, On Circuits in Graphs, *Discrete Math.* 50 (1984), 227–230.

[10] N. Sauer and J. Spencer, Edge Disjoint Placements of Graphs, *J. Combinatorical Theory Ser. B* 25 (1978), 295–302.

[11] E. Szemerédi, Regular Partitions of Graphs, In: *Proc. Colloque Intern. CNRS* (J.-C. Bermond et al. eds.), 1978.

[12] Hong Wang, Three Independent Cycles of a Graph, to appear.