PARTITIONING INTO GRAPHS WITH ONLY SMALL COMPONENTS

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Abstract. The paper presents several results on edge partitions and vertex partitions of graphs into graphs with bounded size components. We show that every graph of bounded tree-width and bounded maximum degree admits such partitions. We also show that an arbitrary graph of maximum degree four has a vertex partition into two graphs, each of which has components on at most 57 vertices. Some generalizations of the last result are also discussed.

1. Introduction

Graphs in this paper are simple, that is, without loops or multiple edges. The set of vertices of a graph G will be denoted by $V(G)$, and the set of edges of G will be denoted by $E(G)$. An *edge partition* of a graph G is a set $\{A_1, A_2, \ldots, A_k\}$ of subgraphs of G such that $\bigcup_{i=1}^{k} E(A_i) = E(G)$. Similarly, a *vertex partition* of G is a set $\{A_1, A_2, \ldots, A_k\}$ of induced subgraphs of G such that $\bigcup_{i=1}^{k} V(A_i) = V(G)$.

Observe that vertex coloring and edge coloring are special cases of partitions. More precisely, a *proper vertex k-coloring* is a vertex partition into k edgeless graphs, and a *proper edge k-coloring* is an edge partition into k matchings. Thus an edge partition or a vertex partition $\{A_1, A_2, \ldots, A_k\}$ may be viewed as an edge or vertex k-coloring,

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and the connected components of the subgraphs A_i may be referred to as *monochromatic components*. Note that a proper vertex (edge) coloring can be described as a vertex (edge) partition into graphs with only components of one vertex (at most two vertices). In this paper, we investigate the existence of vertex and edge partitions into graphs with only components of bounded size.

The *degree* of a vertex v of G, denoted by $d_G(v)$, is the number of edges incident with v. The maximum vertex degree of a graph G will be denoted by $\Delta(G)$. If $X \subseteq V(G)$ or $X \subseteq E(G)$, then $G(X)$ is the subgraph of G induced by X. If $F \subseteq E(G)$, then $G \backslash F$ is the subgraph obtained from G by deleting all edges in F. The *length* of a path is its number of edges.

Let k be a positive integer. A *k-tree* is a graph defined inductively as follows: A complete graph on k vertices is a k-tree. If G is a k-tree, and K is a subgraph of G that is a complete graph on k vertices, then a graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k-tree. Any subgraph of a k-tree is a *partial k-tree*. The *tree-width* of a graph G is zero if G is edgeless; otherwise it is the smallest integer k such that G is a partial k-tree. Nontrivial forests have tree-width 1, while every graph has some tree-width.

The first result of this paper, which is presented in Section 2, deals with graphs in which both maximum degree and tree-width are bounded. We show that each such graph has an edge partition and a vertex partition into two graphs whose components have bounded size. We also show there that, in general, bounding just one of maximum degree and tree-width is not sufficient to ensure the existence of such partitions.

Section 3 contains several lemmas useful later in the paper. In Sections 4 and 5, we investigate, respectively, vertex partitions and edge partitions where both the number of parts and maximum sizes of their components are bounded by functions of the maximum degree. In a previous paper [7], three of the authors in collaboration with Sanders showed that every graph of maximum degree three has an edge partition into two graph all of whose components are paths on at most seven edges. Later, Thomassen [13] proved a stronger result by showing that seven can be replaced by five, which is the best possible. Either of these two results can be stated in terms of vertex partitions of certain graphs of maximum degree four: line graphs of graphs with maximum degree three. In Section 4, we show a generalization of these results for all graphs with maximum degree four. More precisely, the following is an immediate corollary of Theorem 4.1, which appears in Section 4.

Corollary 1.1. *Every graph of maximum degree four has a vertex partition into two graphs each of which has only components on at most* 57 *vertices.* □

Note that, as demonstrated by K_5, the components in Corollary 1.1 need not be paths. Also, we do not believe that the bound in that corollary is the best possible.

In Section 6, we investigate partitions of planar graphs. We show that bounding the maximum degree of planar graph by six does not guarantee the existence of vertex or edge 2-coloring with bounded monochromatic components. We also show that, in general, planar graphs cannot be vertex 3-colored with all monochromatic components having bounded size.

2. Bounding Both Tree-Width and Maximum Degree

The main theorem of this section is based on the following result [4]. A *tree-partition* of a graph G is a pair (T, P) where T is a tree and P is a (disjoint) partition $\{P_t : t \in V(T)\}$ of $V(G)$ such that, for every pair of adjacent vertices u and v of G, either they are both contained in the same P_t, or there are two adjacent vertices s and t of T such that $u \in P_s$ and $v \in P_t$. The *width* of a tree-partition (T, P) is the maximum size of a P_t.

Proposition 2.1. *Every graph of maximum vertex degree* Δ *and tree-width* k *admits a tree-partition of width at most* $24k\Delta$. □

As a consequence of Proposition 2.1, we show that graphs of bounded tree-width and bounded maximum vertex degree can be vertex partitioned and edge partitioned into graphs whose connected components have bounded size.

Theorem 2.2. *Let* k *and* Δ *be positive integers, and let* G *be a graph whose tree-width is at most* k *and whose maximum vertex degree is at most* Δ. *Then* G *admits a vertex partition* $\{G_1, G_2\}$ *such that every connected component of* G_1 *and* G_2 *has at most* $24k\Delta$ *vertices, and* G *admits an edge partition* $\{H_1, H_2\}$ *such that every connected component of* H_1 *and* H_2 *has at most* $24k\Delta(\Delta + 1)$ *vertices.*

Proof. By Proposition 2.1, G has a tree-partition (T, P) of width at most $24k\Delta$ where $P = \{P_t : t \in V(T)\}$. Since T is a tree, it has a vertex partition $\{T_1, T_2\}$ such that neither T_1 nor T_2 has any edges. Let $G_i = \bigcup_{t \in V(T_i)} P_t$ for $i \in \{1, 2\}$. It is clear that $\{G_1, G_2\}$ is as described in Theorem 2.2.

Now we shall construct the edge partition $\{H_1, H_2\}$. Begin by choosing an arbitrary vertex t_0 of T. For each vertex t of T, let $h(t)$ denote the set of vertices s of T such that $s = t$ or s is a neighbor of t that is separated from t_0 by t. For each t, let $H(t)$ denote the subgraph of G that is induced by the edges with one endpoint in P_t and the other endpoint in P_s, for some $s \in h(t)$. Now let $H_i = \bigcup_{t \in V(T_i)} H(t)$ for $i \in \{1, 2\}$. Since P_t has at most $24k\Delta$ elements, each of which has at most Δ neighbors, the conclusion follows. □

It is natural to ask whether bounding just one of tree-width and maximum vertex degree suffices to ensure the existence of a vertex partition or an edge partition into two graphs with bounded size components. We show that, in general, the answer to this question is negative.

First, we consider graphs with bounded tree-width. Let S_n be a *star* on $2n$ vertices, that is, a tree with $2n - 1$ edges, all incident with the same vertex. Let F_n be a *fan* on $n^2 + n + 1$ vertices, that is, a graph obtained from a path on $n^2 + n$ vertices by adding a new vertex and joining it to all vertices of the path. Observe that, if n is a positive integer, the tree-width of S_n is one, and the tree-width of F_n is two. Yet it is clear that for every edge partition $\{G_1, G_2\}$ of S_n each of G_1 and G_2 is a star, and at least one of them has more than n vertices. Similarly, it is easy to show that, for every vertex partition $\{G_1, G_2\}$ of F_n, at least one of G_1 and G_2 has a connected component with more than n vertices. It is worth noting that the above examples have the smallest tree-width possible, for a graph of tree-width zero has no edges, and a graph of tree-width one is a forest, and hence has a vertex partition into two edgeless parts.

Now, we turn our attention to graphs with bounded maximum degree. Our example will be based on the following result of Erdős and Sachs [8]

Proposition 2.3. *For every integer $k \geq 2$ and every integer $g \geq 3$, there is a k-regular graph whose girth is g.* □

Let n be an integer exceeding two, and let G be a 4-regular graph of girth n. Then $|E(G)| = 2|V(G)|$ and hence, for every edge partition $\{A_1, A_2\}$ of G, at least one of A_1 and A_2 has a cycle. Since the girth of G is n, the monochromatic component containing such a cycle has at least n edges. For vertex partitions take H to be the line graph of G. Then H is 6-regular and every vertex 2-coloring of H results in a monochromatic component with at least n vertices.

The following question remains open.

Question 2.4. *Is there a number n such that every graph of maximum degree five can be vertex 2-colored so that each monochromatic component has at most n vertices?*

3. Lemmas

We start with a lemma from [9], which is an improved version of a result of [1] (see also [2], page 61).

Lemma 3.1. *Let $(V_1, V_2, \dots, V_n)$ be a partition of $V(G)$. Suppose $|V_i| \geq 2\Delta(G)$, for all i. Then there is an independent set W of vertices such that $W \cap V_i \neq \emptyset$, for all i.* □

The following is the key lemma, which will be used in proving the main results.

Lemma 3.2. *Let d be an integer exceeding two and let G be a graph with $\Delta(G) \leq d$. Let $\{A, B\}$ be a partition of $V(G)$ and let $\{B_1, B_2, \dots, B_t\}$ be a partition of B. Suppose*

- (i) $\Delta(G(A)) \leq 1$;
- (ii) $\Delta(G(B)) \leq d - 2$;
- (iii) *each $G(B_i)$ is either a cycle or a path; and*
- (iv) *there is a number $r \geq 1$ such that for each $i \in \{1, 2, \dots, t\}$ and each $v \in B_i$ with $d_{G(B_i)}(v) = 2$, there are at most r components of $G(A)$ that contain neighbors of v.*

Then there is a set $W \subseteq B$ such that every component of $G(B_i - W)$, where $1 \leq i \leq t$, and every component of $G(A \cup W)$ has at most $K = (12r + 6)d - (18r + 27)$ vertices.

Proof. For each component C of $G(A)$, let $N(C)$ be the set of vertices that are not in C but are adjacent to some vertices in C. From (i) it is clear that C has at most two vertices and thus $|N(C)| \leq \max\{\Delta(G), 2\Delta(G) - 2\} \leq 2d - 2$.

Without loss of generality, we may assume that $|B_i| > K$ for $i = 1$, $2, \dots, s$, and $|B_i| \leq K$ for all other i not exceeding t. We will use Lemma 3.1 to break each B_i, for $i \in \{1, 2, \dots, s\}$ into paths.

Let $k = (2r + 1)d - (3r + 4)$. Then $d > 2$ and $r \geq 1$ imply that $K > 2k > 0$. For each $i \leq s$, let B'_i be the set of vertices v of B_i with $d_{G(B_i)}(v) = 2$. Since $2k > 0$, each $|B'_i|$ can be expressed as $2kp_i + q_i$, where p_i and q_i are nonnegative integers and $q_i < 2k$. From (iii) we know that $|B'_i| \geq |B_i| - 2$, which implies $|B'_i| \geq K - 1 \geq 2k$ and thus $p_i > 0$. Choose mutually disjoint subsets V_{i1}, V_{i2}, $\dots$, V_{ip_i} of B'_i such that each $G(V_{ij})$ is a path on $2k$ vertices. Let U be the union of V_{ij} for all $j \leq p_i$ and $i \leq s$. Then define a graph H on U such that two vertices are adjacent if either they both belong to $N(C)$ for some component C of $G(A)$, or they are adjacent in G and they are not contained in the same B_i. We will refer the two kinds of edges as, respectively, the first and second type.

We first show that $\Delta(H) \leq k$. Let $u \in U$. From (iv) it is clear that u is contained in at most r sets $N(C)$. Since $|N(C)| \leq 2d-2$, for all C, we conclude that u is incident in H with at most $r(2d-3)$ edges of the first type. In addition, from (ii) it follows that u is incident in H with at most $(d-2)-2 = d-4$ edges of the second type. Therefore, $\Delta(H) \leq r(2d-3)+(d-4) = k$.

Now, from Lemma 3.1 we deduce that H has an independent set W that meets every V_{ij}. Without loss of generality, we may assume that each V_{ij} contains precisely one vertex in W. We need to show that W satisfies the conclusion of Lemma 3.2. It is clear from the construction of H that every edge of $G(W)$ is an edge of some $G(B'_i)$, which means every component of $G(W)$ is a subgraph of some $G(B'_i)$. Since each $G(V_{ij})$ is a path on $2k \geq 2$ vertices and $|W \cap V_{ij}| = 1$, we conclude from (iii) that each component of $G(W)$ may have at most two vertices. We also observe from the construction of H that $N(C) \cap W$ has at most one vertex, for every component C of $G(A)$. Thus each component of $G(A \cup W)$ consists of at most two vertices in W and, by (iv), at most $2r$ components of $G(A)$. It follows that each component of $G(A \cup W)$ has at most $4r+2 \leq K$ vertices, as $d > 2$ and $r \geq 1$. For each $i \leq s$, the graph $G(B_i - W)$ may have only two kinds of components: a path P that is cut off from $G(B_i)$ by two vertices $w_1, w_2 \in W$ (say $w_1 \in V_{i1}$ and $w_2 \in V_{i2}$), or a path P that is cut off from $G(B_i)$, when $G(B_i)$ is a path, by a single vertex $w \in W$ (say $w \in V_{i1}$). In the first case, $|V(P)| \leq (|V_{i1}|-1)+q_i+(|V_{i2}|-1) \leq 3(2k-1) = K$. In the second case, $|V(P)| \leq q_i+(|V_{i1}|-1) \leq 2(2k-1) \leq K$. The lemma follows. □

4. Vertex partitions

Our first main result of this paper is the following theorem. Let $f(0) = 1$, let $f(1) = f(2) = 2$, and let $f(\Delta) = 12\Delta^2 - 36\Delta + 9$ for all $\Delta \geq 3$.

Theorem 4.1. *Every graph with maximum degree Δ can be vertex $\lceil(\Delta+2)/3\rceil$-colored such that each monochromatic component has at most $f(\Delta)$ vertices.*

In order to prove this theorem, we need the following result of Lovász [11].

Lemma 4.2. *Let G be a graph and let k_1, k_2, ... , k_m be nonnegative integers with $k_1 + k_2 + \cdots + k_m \geq \Delta(G) - m + 1$. Then $V(G)$ can be partitioned into V_1, V_2, ... , V_m so that $\Delta(G(V_i)) \leq k_i$, for all $i \in \{1, 2, \ldots, m\}$.* □

Proof of Theorem 4.1. The result is clear when $\Delta \leq 2$. Thus we may assume that $\Delta > 2$. Let $h = \lceil (\Delta + 2)/3 \rceil$. Then $\Delta \leq 3h - 2$. By taking $k_1 = 1$, $k_2 = \Delta - 2$, and $m = 2$, we deduce from Lemma 4.2 that $V(G)$ can be partitioned into A and B such that $\Delta(G(A)) \leq 1$ and $\Delta(G(B)) \leq \Delta - 2$. Next, by taking $k_1 = k_2 = \cdots = k_{h-1} = 2$ and $m = h - 1$ we deduce from Lemma 4.2 that B can be partitioned into $V_1, V_2, \ldots, V_{h-1}$ such that $\Delta(G(V_i)) \leq 2$ for all i. Let $C_1, C_2, \ldots, C_t$ be all components of $G(V_1), G(V_2), \ldots,$ and $G(V_{h-1})$. Clearly, each C_i is either a cycle or a path. Let $B_i = V(C_i)$ for all i. Then G, A, B, and B_i, where $1 \leq i \leq t$, satisfy the assumptions of Lemma 3.2 with $d = \Delta$ and $r = \Delta - 2$. Let $W \subseteq B$ be chosen as in Lemma 3.2. Observe that, for each i, every component of $G(V_i - W)$ is a component of some $C_j - W$. Therefore, the h-coloring $(A \cup W, V_1 - W, V_2 - W, \ldots, V_{h-1} - W)$ satisfies the conclusion of Theorem 4.1. □

The next theorem says that if we can use a few more colors, then we can make the size of the monochromatic components independent of Δ.

Theorem 4.3. *For any positive $\epsilon < 3$, there is a number N_ϵ for which every graph G can be vertex $\lceil (\Delta(G) + 2)/(3 - \epsilon) \rceil$-colored so that each monochromatic component has at most N_ϵ vertices.*

Proof. Let $\Delta = \Delta(G)$ and $h = \lceil (\Delta + 2)/(3 - \epsilon) \rceil$. Then $\Delta \leq (3-\epsilon)h - 2$. Let $p = \lceil 1/\epsilon \rceil$ and let $N_\epsilon = f(3p - 2)$. Since f is a non-decreasing function, we have

$$f(t) \leq N_\epsilon, \quad \text{for all} \;\; t \leq 3p - 2. \tag{1}$$

Consequently, by Theorem 4.1, the result holds when $\Delta \leq 3p-2$. Next, we consider the case when $\Delta \geq 3p - 1$. Since $h > 0$, we may assume $h = px+y$ for some nonnegative integers x and y with $1 \leq y \leq p$. Then $\Delta-(x+1)+1 \leq [(3-1/p)(px+y)-2]-x < (3p-2)x+(3y-2)$. It follows from Lemma 4.2 that $V(G)$ can be partitioned into $V_1, V_2, \ldots, V_{x+1}$ such that $\Delta(G(V_i)) \leq 3p - 2$, for $i \leq x$, and $\Delta(G(V_{x+1})) \leq 3y - 2$. Let $h_i = p$, when $i \leq x$, and $h_{x+1} = y$. By Theorem 4.1, each $G(V_i)$ can be vertex h_i-colored so that its monochromatic components have size at most $f(3h_i - 2)$, which, by (1), is at most N_ϵ. Therefore, G can be vertex h-colored, where $h = h_1 + h_2 + \cdots + h_{x+1} = px + y$ and all monochromatic components have size at most N_ϵ. □

5. Edge partitions

For line graphs, both Theorem 4.1 and Theorem 4.3 can be improved. These improvements, which are stated in terms of edge partitions, appear as Theorem 5.1 and Theorem 5.5 below. The following is an improvement of Theorem 4.1 for line graphs. Let us define $g(0) = 0$, $g(1) = 1$, and $g(\Delta) = 60\Delta - 63$ for all $\Delta \geq 2$.

Theorem 5.1. *Every loopless graph G with maximum degree Δ can be edge $\lceil (\Delta+1)/2 \rceil$-colored so that each monochromatic component has at most $g(\Delta)$ edges.*

We first present two lemmas, which will play the role that Lemma 4.2 had for vertex partitions. The first of these, which is stated below, is an easy consequence of Lemma 1 in [3].

Lemma 5.2. *Every loopless graph G has a set A of edges such that $\Delta(G \backslash A) < \Delta(G)$ and each component of $G(A)$ is a path of length at most two.* □

The next lemma is a reformulation of a well-known result of Petersen [4], which states that every even regular graph has a 2-factor.

Lemma 5.3. *Let d be an integer and let G be a loopless graph with $\Delta(G) \leq 2d$. Then $E(G)$ can be partition into F_1, F_2, ... , F_d such that $\Delta(G(F_i)) \leq 2$ for all i.*

Proof. It is well known that G has a $2d$-regular supergraph H. Then the lemma follows from repeatedly applying Petersen's result to H. □

Proof of Theorem 5.1. Let H be the line graph of G. We prove the theorem by applying Lemma 3.2 to H. Let $h = \lceil (\Delta+1)/2 \rceil$. Then $2h-2 \leq \Delta \leq 2h-1$. Consequently, $\Delta(H) \leq 4h-4$. Let $d = 4h-4$. We first examine the case when $d \leq 2$. Clearly, $d \leq 2$ implies $h \leq 1$, which in turn implies $\Delta \leq 1$ and thus every component of G is either K_1 or K_2. In this case, the theorem obviously holds because $E(G)$ can be $\lceil (\Delta+1)/2 \rceil = 1$ colored and each monochromatic component of G has at most $\Delta = g(\Delta)$ edges.

Now, we assume that $d > 2$. Let $A \subseteq E(G)$ be chosen as in Lemma 5.2 and let $B = E(G) - A$. Then $\Delta(G(B)) \leq 2h-2$. It follows that $\Delta(H(A)) \leq 1$ and $\Delta(H(B)) \leq 2(2h-3) = d-2$. Next, by applying Lemma 5.3 to $G(B)$, we conclude that B can be partitioned into F_1, F_2, ... , F_{h-1} such that $\Delta(G(F_i)) \leq 2$ for all i. Let C_1, C_2, ... , C_t be all components of $G(F_1)$, $G(F_2)$, ... , $G(F_{h-1})$. Then each C_i is either a cycle or a path. Let $B_i = V(C_i)$, for all i. It follows that $(B_1, B_2, \ldots, B_{h-1})$ is a partition of B and each $H(B_i)$ is either a cycle or a path. Since a line graph does not have induced $K_{1,3}$, we conclude that each $x \in B$ can be adjacent in H to vertices in at most two components of $H(A)$. Therefore, G, A, B, B_i $(1 \leq i \leq h-1)$ satisfy the assumptions of Lemma 3.2 with $r = 2$.

Let $W \subseteq B$ be chosen as in Lemma 3.2. Notice that, for any $X \subseteq E(G)$, the graph $G(X)$ is connected if and only if $H(X)$ is connected. In addition, for each i, every component of $G(F_i - W)$ is a component of some $C_j - W$. It follows that each component of $G(A \cup W)$ and $G(F_i - W)$ $(1 \leq i \leq h-1)$ has at most $30d - 63 = 60(2h-2) - 63 \leq 60\Delta - 63$ edges. Therefore, the h-coloring $(A \cup W, F_1 - W, F_2 - W, \dots, F_{h-1} - W)$ satisfies the conclusion of Theorem 5.1. □

One of the authors, in collaboration with others, proved in [3] the following:

Theorem 5.4. *There is an absolute constant $c > 0$ such that for every Δ-regular graph G and every $\sqrt{\Delta} > k \geq 2$ the edges of G can be colored with $(k+1)\Delta/(2k) + c\sqrt{k\Delta \log \Delta}$ colors so that each monochromatic component is a path of length at most k.* □

Theorem 5.4 has the following corollary, which may be viewed as an improvement of Theorem 4.3 for line graphs.

Theorem 5.5. *For any $\epsilon > 0$ there is a number N_ϵ for which every loopless graph G with maximum degree Δ can be edge $\lceil((1+\epsilon)\Delta + 1)/2\rceil$-colored such that each monochromatic component has at most N_ϵ edges.*

Proof. Let c be the number from Theorem 5.4, let G be a graph with maximum degree Δ, let ϵ be a positive number, and let ϵ_0 be a positive number satisfying $\epsilon_0 \leq \epsilon$, $\epsilon_0 \leq 1/2$, and $\epsilon_0\sqrt{\log(1/\epsilon_0)} \leq \epsilon/(8c)$. Without loss of generality, we may assume that G is Δ-regular, since every graph of maximum degree Δ is a (not necessarily spanning) subgraph of a Δ-regular graph. Let $N_\epsilon = g(\epsilon_0^{-4})$ where g is the function defined immediately before Theorem 5.1. If $\Delta \leq \epsilon_0^{-4}$, then the conclusion follows from Theorem 5.1; hence we may assume that $\Delta > \epsilon_0^{-4}$. Upon applying Theorem 5.4 with $k = \epsilon_0^{-2}$, we conclude that the edges of G can be colored using at most $\Delta/2 + \Delta\epsilon_0^2/2 + c\epsilon_0\sqrt{\Delta \log \Delta}$ colors so that each monochromatic component is a path on at most ϵ_0^{-2} edges. Clearly, the size of the monochromatic components satisfies the conclusion of the theorem. The following computation gives the desired bound on the number of colors used. Note that the first inequality uses the fact that $(\log x)/x$ is decreasing when $x > e$.

$$\begin{aligned}\frac{(k+1)\Delta}{2k} + c\sqrt{k\Delta\log\Delta} &= \frac{\Delta}{2} + \frac{\Delta\epsilon_0^2}{2} + c\epsilon_0^{-1}\Delta\sqrt{\frac{\log\Delta}{\Delta}} \\ &\leq \frac{\Delta}{2} + \Delta\left(\frac{\epsilon_0^2}{2} + 2c\epsilon_0\sqrt{\log\frac{1}{\epsilon_0}}\right) \\ &\leq \frac{\Delta}{2} + \Delta\left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) \\ &\leq \frac{(1+\epsilon)\Delta + 1}{2}.\end{aligned}$$

□

We note that Theorem 5.5 cannot be strengthened to allow $\epsilon = 0$. More precisely, we have the following:

Remark 5.6. *For each integer n there is a graph G such that every edge $\lceil(\Delta(G)+1)/2\rceil$-coloring of G results in a monochromatic component with more than n vertices.*

Proof. Let $\Delta = 2n+1$, and let G be a Δ-regular graph of girth $n+1$, whose existence is guaranteed by Proposition 2.3. Suppose that G_1, G_2, ... , G_m is an edge partition of G such that $m \leq \lceil(\Delta(G)+1)/2\rceil$ and every monochromatic component has at most n edges. Since the girth of G exceeds n, each monochromatic component is acyclic and has at most $n+1$ vertices. Hence, the number of components of G_i is at least $|V(G_i)|/(n+1)$ and so $|E(G_i)| \leq |V(G_i)| - |V(G_i)|/(n+1) \leq |V(G_i)|(1 - 1/(n+1))$. Therefore

$$\begin{aligned}|E(G)| &\leq \left\lceil\frac{\Delta+1}{2}\right\rceil |V(G)|\frac{n}{n+1} \\ &= (n+1)|V(G)|\frac{n}{n+1} \\ &< |V(G)|\frac{\Delta}{2}.\end{aligned}$$

This is impossible since G is Δ-regular. □

6. Planar graphs

In Section 2, we showed that for every integer n, there is a 4-regular graph G and a 6-regular graph H such that every edge 2-coloring of G and every vertex 2-coloring of H results in a monochromatic component containing a cycle of length at least n. However, graphs G and H are nonplanar whenever $n > 3$. In the first part of this section, we show that for every integer n there is a planar graph G of maximum degree six such that every edge 2-coloring and every vertex 2-coloring results in a monochromatic component with at least n vertices.

A graph is a *near-triangulation* if it is a plane graph whose every face, except possibly for the infinite face, is a triangle. For a positive integer n, let T_n be the graph whose vertices are the triples of nonnegative integers summing to n, with an edge connecting two triples if they agree in one coordinate and differ by one in each of the other two coordinates. The graph T_n may be viewed as embedded in the plane whose equation in $\mathbb{R}^3$ is $x + y + z = n$ where the name of each vertex forms its coordinates, and edges are straight line segments. The graph T_5 is illustrated in Figure 1. It is clear that each T_n is a near-triangulation with no vertices of degree exceeding six. The next theorem states that it is impossible to find a vertex partition or an edge partition of T_n into two graphs neither of which has connected components with more than n vertices.

Theorem 6.1. *If $\{G_1, G_2\}$ is a vertex partition or an edge partition of T_n, then at least one of G_1 and G_2 has a connected component with more than n vertices.*

Before addressing the proof of the theorem, we need a few definitions. Let G be a near-triangulation and let v_1, v_2, and v_3 be three distinct vertices in the cycle C that bounds the infinite face. Then v_1, v_2, and v_3 induce a partition of C into paths P_1, P_2, and P_3 such that, for each $i \in \{1, 2, 3\}$, P_i avoids v_i and has the other two members of $\{v_1, v_2, v_3\}$ as endvertices. A *connector* of G with respect to $\{v_1, v_2, v_3\}$ is a connected subgraph H of G such that, for each $i \in \{1, 2, 3\}$, the set $V(H) \cap V(P_i)$ is not empty.

The part of Theorem 6.1 that speaks of vertex partitions follows immediately from the following two results of [10].

FIGURE 1. T_5

Proposition 6.2. *Let G be a near-triangulation and let v_1, v_2, and v_3 be distinct vertices in the cycle bounding the infinite face of G. For every vertex partition $\{G_1, G_2\}$ of G there is a connector H of G with respect to $\{v_1, v_2, v_3\}$ that is a subgraph of G_1 or of G_2.* □

Proposition 6.3. *If H is a connector of T_n with respect to $(0, 0, n)$, $(0, n, 0)$, and $(n, 0, 0)$, then H has more than n vertices.* □

The part of Theorem 6.1 on edge partitions follows immediately from Proposition 6.3 and the edge version of Proposition 6.2, which is stated and proved below.

Proposition 6.4. *Let G be a near-triangulation and let v_1, v_2, and v_3 be distinct vertices in the cycle bounding the infinite face of G. For every edge partition $\{G_1, G_2\}$ of G there is a connector H of G with respect to $\{v_1, v_2, v_3\}$ that is a subgraph of G_1 or of G_2.*

Proof. We will apply Proposition 6.2 to the graph G' obtained from G in the following process. Let the vertex set of G' be $V(G) \cup E(G)$ with two such vertices being joined by an edge if and only if one of them is an edge e and the other is either a vertex of G incident with e, or an edge of G that shares a common vertex and a common finite face with e. Alternatively, G' may be viewed as obtained from G by subdividing each of its edges once, and adding new edges incident with the new vertices so that each of the finite faces of G becomes subdivided into four triangular faces. For example, if $G = T_n$, then G' is isomorphic to T_{2n}.

Note that each of v_1, v_2, and v_3 lies in the cycle bounding the infinite face of G'. For each $i \in \{1, 2\}$, let $V_i = V(G) \cup E(G_i)$ and let G'_i be the subgraph of G' induced by V_i. Then $\{G'_1, G'_2\}$ is a vertex partition of G'. Upon applying Proposition 6.2 to G', we conclude that there is a connector H' of G' with respect to v_1, v_2, and v_3 that is a subgraph of G'_1 or of G'_2. Without loss of generality, we may assume that H' is a connected component of G'_1. Let H be a subgraph of G induced by those vertices of H' that are also vertices of G. We shall prove that H is a connector of G with respect to v_1, v_2, and v_3. Let P_1, P_2, and P_3 be the paths that partition the cycle bounding the infinite face of G as described in the definition immediately preceding Proposition 6.2, and let P'_1, P'_2, and P'_3 be the corresponding paths in G'. Then, for each $i \in \{1, 2, 3\}$, the vertex set of P'_i is the union of $V(P_i)$ and $E(P_i)$.

Suppose H avoids one of the paths P_i for some $i \in \{1, 2, 3\}$. But H', being a connector, has a vertex e in P'_i, which must be an edge of P_i. Let v be a vertex that is incident in G with e. Then, clearly, $v \in V(P_i)$. Since all vertices of G are in V_1, which induces G'_1, and H' is a connected component of G'_1, we conclude that v is a vertex of H', and hence also of $P_i \cap H$; a contradiction.

It remains to show that H is connected. Let u and v be two vertices of H. Then, as H' is connected, it contains a path P from u to v. Take the list of consecutive vertices of P and modify it as follows: Between every two consecutive vertices of P that are both edges of G insert the vertex of G that is incident with both edges. Since V_1 contains all vertices of G, and G'_1 is induced by V_1, the modified list consists of vertices of H'. The same list, when interpreted in G, alternates vertices and edges with two consecutive entries being incident. Since all vertices of G that appear in the list are in H', and hence in H, the list forms a walk in H that begins in u and ends in v. It follows that H is a connector of G with respect to v_1, v_2, and v_3, as required. □

Finally, we address vertex partitions of arbitrary planar graphs. The well-known Four Color Theorem states that every planar graph has a vertex 4-coloring so that all monochromatic components have exactly one vertex. Theorem 6.1 implies that the size of monochromatic components cannot be bounded if only two colors are used for coloring a planar graph. The following natural question arises:

Question 6.5. *Is there a number c such that every planar graph has a vertex 3-coloring so that each monochromatic components has at most c vertices?*

Next, we show that the answer to this question is negative. For a positive integer n, let U_n be the graph consisting of n pairwise disjoint copies of F_n, which was defined in Section 2 as a fan on $n^2 + n + 1$ vertices, and let U'_n be the graph obtained from U_n by adding one vertex and joining it to all other vertices.

Theorem 6.6. *Let n be a positive integer and let $\{A_1, A_2, A_3\}$ be a vertex partition of U'_n. Then at least one of A_1, A_2, and A_3 has a component with more than n vertices.*

Proof. Let v denote a vertex of U'_n that is adjacent to all other vertices. Without loss of generality, we may assume that $v \in V(A_3)$. If A_3 meets each component of U_n, then the conclusion follows. Hence we may assume that some component of U_n, which is a fan F_n, meets only A_1 and A_2. The conclusion follows from the discussion in Section 2. □

Note that the maximum degree of the graphs U'_n grows with n. Indeed, we do not know whether the graphs U'_n in Theorem 6.6 can be replaced by graphs whose maximum degree is bounded by a universal constant.

We close the paper with analogs of some results and questions for the class of planar graphs to larger classes of graphs. A graph G is a *minor* of graph H if G can be obtained from a subgraph of H by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every member H of $\mathcal{G}$ all minors of H are also in $\mathcal{G}$. It is clear that the class of planar graphs is minor-closed. The following theorem speaks of vertex 4-coloring of graphs in minor-closed classes.

Theorem 6.7. *Let $\mathcal{G}$ be a minor-closed class of graphs other than the class of all graphs, and let Δ be a positive integer. Then there is a number c depending only on $\mathcal{G}$ and Δ such that every graph G in $\mathcal{G}$*

with maximum degree at most Δ admits a vertex 4-coloring with each monochromatic component having at most c vertices.

Proof. It is shown in [5] that there is a number a, depending only on $\mathcal{G}$, such that every graph in $\mathcal{G}$ can be vertex 2-colored so that each monochromatic subgraph has tree-width at most a. This, together with Proposition 2.1, easily implies the conclusion. □

Note that Theorem 6.6 implies that the number of colors in Theorem 6.7 cannot be reduced from four to three, even for the class of planar graphs. However, it is not known whether the number c in Theorem 6.7 can depend only on $\mathcal{G}$.

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