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A Simple Proof of the Upper Bound Theorem

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Let $c_i(n, d)$ be the number of i-dimensional faces of a cyclic d-polytope on n vertices. We present a simple new proof of the upper bound theorem for convex polytopes, which asserts that the number of i-dimensional faces of any d-polytope on n vertices is at most $c_i(n, d)$. Our proof applies for arbitrary shellable triangulations of $(d-1)$ spheres. Our method provides also a simple proof of the upper bound theorem for d-representable complexes.

1. Introduction

Let $c_i(n, d)$ be the number of i dimensional faces of a cyclic d-polytope on n vertices. In this note we present a simple proof of the *upper bound theorem* (*UBT*) *for convex polytopes*, which asserts that the number of i-dimensional faces of any d-polytope on n vertices is at most $c_i(n, d)$. We will consider only simplicial polytopes, since it is well known that it suffices to prove the UBT in this case ([8, p. 80], [13, sect. 2.5]).

The UBT was conjectured by Motzkin in 1957 [14], and proved by McMullen in 1970 ([12], [13, chp. 5]). Another proof was given by Bondesen and Brønsted [2]. Stanley [15] proved that the assertion of the UBT holds also for triangulations of $(d-1)$-spheres (see also [7], [10] and [16]).

McMullen's proof uses a fundamental result of Bruggesser and Mani [3], which asserts that the boundary complex of a convex polytope is *shellable*. This notion is crucial also here, and in fact our proof applies for arbitrary shellable triangulations of $(d-1)$-spheres.

Our method supplies also a simple proof of a theorem conjectured by Katchalski and Perles and proved independently by Eckhoff [5] and by the second author [9]. This theorem asserts that if $\mathcal{K}$ is a family of n convex sets in $\mathbb{R}^d$ and $\mathcal{K}$ has no intersecting subfamily of size $d+r+1$, then the number of intersecting k-subfamilies of $\mathcal{K}$ for $d < k \leq d+r$ is at most

$$\sum_{i=0}^{d} \binom{n-r}{i} \cdot \binom{r}{k-i}.$$

Equality holds, e.g. if $\mathcal{K} = \{K_1, \ldots, K_n\}$ where $K_1 = K_2 = \cdots = K_r = \mathbb{R}^d$ and $K_{r+1}, \ldots, K_n$ are hyperplanes in general position in $\mathbb{R}^d$.

2. On the Number of Elementary Collapses

We begin with a combinatorial lemma. Equivalent formulations of it were proved by Frankl [6] and by the second author [9], and a generalization was proved by the first author [1]. Here we present a short proof, following the approach of [1].

Lemma 2.1. *Suppose $n > 1$, $N = \{1, 2, \ldots, n\}$ and $1 \leq s \leq m \leq n$. For $1 \leq i \leq h$ let A_i and B_i be subsets of N that satisfy*

$$|A_i| \leq s \quad and \quad |B_i| \geq m, \qquad for \quad 1 \leq i \leq h. \tag{2.1}$$

$$A_i \subset B_i, \qquad for \quad 1 \leq i \leq h. \tag{2.2}$$

$$A_i \not\subset B_j, \qquad for \quad 1 \leq i < j \leq h. \tag{2.3}$$

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Then

$$h \leq \binom{n-m+s}{s}.$$

PROOF. Clearly we may assume that $|A_i| = s$ for $1 \leq i \leq h$. Let $V = \mathbb{R}^{n-m+s}$ be the $(n-m+s)$-dimensional real space and let $v_1, v_2, \ldots, v_n$ be vectors in general position in V (i.e. every set of $\leq n-m+s$ of them is linearly independent). Let $\bigwedge V$ denote the exterior algebra over V, equipped with the usual wedge product $\wedge$ (see [4] or [11] for general information on exterior algebra).

For $1 \leq i \leq h$ define $y_i = \bigwedge_{j \in A_i} v_j \in \bigwedge^s V$ and $\bar{y}_i = \bigwedge_{k \in N \setminus B_i} v_k$. By (2.1), (2.2) and the general position of the v_js,

$$y_i \wedge \bar{y}_i \neq 0, \qquad \text{for} \quad 1 \leq i \leq h. \tag{2.4}$$

By (2.3)

$$y_i \wedge \bar{y}_j = 0, \qquad \text{for} \quad 1 \leq i < j \leq h. \tag{2.5}$$

To complete the proof we show that the set $\{y_i: 1 \leq i \leq h\}$ is linearly independent in $\bigwedge^s V$ and thus $h \leq \dim(\bigwedge^s V) = \binom{n-m+s}{s}$. Indeed, suppose this is false and let

$$\sum_{i \in I} c_i y_i = 0 \tag{2.6}$$

be a linear dependence, with $c_i \neq 0$ for $i \in I$. Put $j = \max\{i: i \in I\}$. Combining (2.5) and (2.6) we obtain $0 = (\sum_{i \in I} c_i y_i) \wedge \bar{y}_j = c_j \cdot y_j \wedge \bar{y}_j$, which, together with (2.4), supplies the contradiction $c_j = 0$.

A face S of a simplicial complex C is *free* if S is contained in a unique maximal face M of C. The operation of deleting S and all faces that contain it is an *elementary-collapse*. If the size of S is s and the size of M is m, it is called an *elementary-(s, m)-collapse*. A *collapse process* on C is a sequence $C = C_0 \supset C_1 \supset \cdots \supset C_t$ of simplicial complexes such that for $1 \leq i \leq t$ C_i is obtained from C_{i-1} by an elementary-collapse.

The following lemma plays a crucial role in our proofs.

LEMMA 2.2. Let s, m be nonnegative integers, $s \leq m$, and let C be a simplicial complex on n vertices. The number of all elementary-(s', m')-collapses, with $s' \leq s$ and $m' \geq m$, in any collapse process on C, is at most $\binom{n-m+s}{s}$.

PROOF. Let $C = C_0 \supset C_1 \supset \cdots \supset C_t$ be a collapse process on C. Let S_i and M_i be the free face and the maximal face, corresponding to the ith elementary-collapse, $1 \leq i \leq t$. Let $(A_j, B_j)_{j=1}^h$ be the subsequence of $(S_i, M_i)_{i=1}^h$ consisting of those pairs (S_i, M_i) with $|S_i| \leq s$ and $|M_i| \geq m$. One can easily check that A_i, B_i $(1 \leq i \leq h)$ satisfy the hypotheses of Lemma 2.1. Therefore $h \leq \binom{n-m+s}{s}$.

3. The Upper Bound Theorem for Shellable Spheres

Let C be a triangulation of a $(d-1)$-sphere on a set $N = \{1, 2, \ldots, n\}$ of n vertices. Let $f = f_i(c)$ be the number of i-dimensional faces of C, $0 \leq i \leq d-1$, and put $f_{-1} = 1$. The h-vector $(h_0, \ldots, h_d)$ of C is defined by the equations

$$f_j = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i, \qquad -1 \leq j \leq d-1. \tag{3.1}$$

(See, e.g. [15] or [13, chp. 5] where $g_k^{(d)}$ is used for h_{k+1}).

As is well known ([8, sect. 9.2], [13, p. 171]) C satisfies the Dehn–Somerville equations that can be written as

$$h_i = h_{d-i}, \qquad 0 \leq i \leq [d/2]. \tag{3.2}$$

For $0 \leq i \leq d$ define $\tilde{h}_i = \sum_{j=0}^{i} h_j$. Thus

$$h_i = \tilde{h}_i - \tilde{h}_{i-1}. \tag{3.3}$$

Substituting (3.2) and then (3.3) in (3.1) one can express every f_j as a linear combination of $\{\tilde{h}_i: 0 \leq i \leq [d/2]\}$ with *nonnegative* coefficients as follows:

$$\begin{aligned} f_j &= \sum_{i=0}^{[d/2]} \left[\binom{d-i-1}{d-j-2} - \binom{i}{d-j-2} \right] \tilde{h}_i, && \text{for odd } d \\ f_j &= \sum_{i=0}^{[d/2]} \left[\binom{d-i-1}{d-j-2} - \binom{i}{d-j-2} \right] \tilde{h}_i + \binom{d/2}{d-j-1} \tilde{h}_{d/2}, && \text{for even } d. \end{aligned} \tag{3.4}$$

It is well known (see, e.g. [13, p. 172]) that for the cyclic d-polytope with n vertices (or any neighbourly d-polytope with n vertices) $h_i = \binom{n-d+i-1}{i}$, $0 \leq i \leq [d/2]$, and thus $\tilde{h}_i = \binom{n-d+i}{i}$, $0 \leq i \leq [d/2]$. In view of (3.4), in order to prove the UBT it is enough to show that

$$\tilde{h}_i \leq \binom{n-d+i}{i}. \tag{3.5}$$

We proceed to show that (3.5) is satisfied by any $(d-1)$-shellable sphere.

For $F \subset N$ let $\bar{F}$ denote the set of all subsets of F. C is *shellable* if its maximal faces are all of dimension $d-1$ and can be ordered $F_1, F_2, \ldots, F_t$ so that for

$$1 \leq k \leq t-1 \quad \bar{F}_k \cap \left(\bigcup_{i=k+1}^{t} \bar{F}_i \right) = \bigcup_{j=1}^{s_k} \bar{G}_j^k,$$

where G_j^k are $s_k \geq 1$ distinct faces of C of dimension $d-2$. In this case define, for $0 \leq i \leq t$, $C_i = \bigcup_{k=i+1}^{t} \bar{F}_k$ (thus $C_t = \varnothing$). For $1 \leq i \leq t-1$ put $S_i = F_i \setminus \bigcap_{j=1}^{s_i} G_j^i$ and define $S_t = \varnothing$. One can easily check that S_i is a free face of C_{i-1} and C_i is obtained from C_{i-1} by deleting S_i and all faces that contain it, i.e. by an elementary $(|S_i|, d)$-collapse.

Let g_i denote the number of elementary (i, d)-collapses in the shelling of C. (i.e. $g_i = |\{k: s_k = i\}|$). It is well-known (see [13, p. 175]) that $g_i = h_i$, $0 \leq i \leq d$. (Indeed, the number of j-faces deleted in an elementary (i, d)-collapse is $\binom{d-i}{j+1-i}$ and since $C_t = \varnothing$

$$f_j = \sum_{i=0}^{j+1} \binom{d-i}{j+1-i} g_i, \qquad \text{for } -1 \leq j \leq d-1.$$

Therefore $g_0, \ldots, g_d$ satisfy the defining equations (3.1) for the h_is and hence $g_i = h_i$ for $0 \leq i \leq d$.) Since $\tilde{h}_i = \sum_{j=0}^{i} g_j$ is just the number of all elementary (i', d)-collapses with $i' \leq i$ in the collapse process $C = C_0 \supset C_1 \supset \cdots \supset C_t$ on C, Lemma 2.2 implies (3.5). This proves the UBT for shellable triangulations of spheres. Bruggesser and Mani [3] proved that the boundary complex of any convex polytope is shellable, and thus the UBT for convex polytopes follows.

4. d-Representable Complexes

Let C be a simplicial complex on the vertex set N. C is *d-representable* if there exists a family $\mathcal{K} = \{K_1, \ldots, K_n\}$ of convex sets in $\mathbb{R}^d$ such that $S \in C$ iff $\bigcap_{i \in S} K_i \neq \varnothing$. C is *d-collapsible* if there exists a collapse process $C = C_0 \supset C_1 \supset \cdots \supset C_t$ in which every elementary-collapse is of type (d, m) for some $m \geq d$ and C_t has no faces of size $\geq d$. In this case, let h_i denote the number of elementary—$(d, d+i)$—collapses in the process $(i \geq 0)$. Clearly in each such collapse precisely $\binom{i}{j+1-d}$ j-dimensional faces of C were deleted for $d-1 \leq j \leq d+i-1$. Let f_j denote the number of j-dimensional faces of C. Suppose $f_{d+r}=0$ and put $\tilde{h}_i = \sum_{j=i}^{r} h_j (0 \leq i \leq r)$. (Thus $\tilde{h}_{r+1}=0$.) Clearly for $d \leq j \leq d+r-1$

$$f_j = \sum_{i=j+1-d}^{r} h_i \binom{i}{j+1-d} = \sum_{i=j+1-d}^{r} (\tilde{h}_i - \tilde{h}_{i+1}) \binom{i}{j+1-d} = \sum_{i=j+1-d}^{r} \tilde{h}_i \binom{i-1}{j-d}.$$

By Lemma 2.2 $\tilde{h}_i \leq \binom{n-i}{d}$ for $i \geq 0$. Therefore we have:

THEOREM 4.1. *For $d \leq j \leq d+r-1$ let f_j denote the number of faces of dimension j of a d-collapsible complex on n vertices. If $f_{d+r}=0$ then*

$$f_j \leq \sum_{i=j+1-d}^{r} \binom{n-i}{d} \binom{i-1}{j-d} \left(= \sum_{i=0}^{d} \binom{n-r}{i} \binom{r}{j+1-i} \right).$$

This theorem was first proved in [9]. By a fundamental result of Wegner [17], every d-representable complex is d-collapsible, and thus the assertion of Theorem 4.1 holds for d-representable complexes.

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