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The Maximum Number of Disjoint Pairs in a Family of Subsets

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Abstract. Let $\mathscr{F}$ be a family of 2^{n+1} subsets of a $2n$-element set. Then the number of disjoint pairs in $\mathscr{F}$ is bounded by $(1 + o(1))2^{2n}$. This proves an old conjecture of Erdös. Let $\mathscr{F}$ be a family of $2^{(1/(k+1)+\delta)n}$ subsets of an n-element set. Then the number of containments in $\mathscr{F}$ is bounded by $(1 - 1/k + o(1))\binom{|\mathscr{F}|}{2}$. This verifies a conjecture of Daykin and Erdös. A similar Erdös-Stone type result is proved for the maximum number of disjoint pairs in a family of subsets.

1. Introduction

Let $\mathscr{F}$ be a family of m distinct subsets of $X = \{1, 2, \ldots, n\}$. Let $d(\mathscr{F})$ $(c(\mathscr{F}))$ denote the number of disjoint (comparable, respectively) pairs in $\mathscr{F}$. That is:

$$d(\mathscr{F}) = |\{\{F, F'\}: F, F' \in \mathscr{F}, F \cap F' = \varnothing\}|$$

$$c(\mathscr{F}) = |\{(F, F'): F, F' \in \mathscr{F}, F \subset F'\}|.$$

Define

$$d(n, m) = \max\{d(\mathscr{F}): |\mathscr{F}| = m\},$$

$$c(n, m) = \max\{c(\mathscr{F}): |\mathscr{F}| = m\}.$$

Several years ago Erdös [4] raised the problems of determining or estimating $d(n, m)$. A similar problem for $c(n, m)$ is considered in [3].

Example 1.1. Let $X_1 \cup X_2 \cup \ldots \cup X_k$ be a partition of X, where $\lfloor n/k \rfloor \le |X_i| \le \lceil n/k \rceil$. Suppose $m \le k \cdot 2^{\lfloor n/k \rfloor}$ and let $\mathscr{A}_i$ be a collection of subsets of X_i with $\lfloor m/k \rfloor \le |\mathscr{A}_i| \le \lceil m/k \rceil$, $|\mathscr{A}_1| + |\mathscr{A}_2| + \cdots + |\mathscr{A}_k| = m$. Then

$$d(\mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \cup \mathscr{A}_k) \ge \left(1 - \frac{1}{k}\right)\binom{m}{2}.$$

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Example 1.2. Let $X_1, X_2, \ldots, X_k$ and m be as in Example 1.1. Let $\mathcal{B}_i$ be a collection of subsets of $X_1 \cup X_2 \cup \ldots \cup X_i$, each containing $X_1 \cup X_2 \cup \ldots \cup X_{i-1}$, where $\lfloor m/k \rfloor \le |\mathcal{B}_i| \le \lceil m/k \rceil$, $|\mathcal{B}_1| + |\mathcal{B}_2| + \cdots + |\mathcal{B}_k| = m$. Then

$$c(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_k) \ge \left(1 - \frac{1}{k}\right)\binom{m}{2}.$$

The following theorems show that these examples are essentially best possible.

Theorem 1.3. *For every positive integer k there exists a positive $\beta = \beta(k)$ such that if $m = 2^{(1/(k+1)+\delta)n}$ where $\delta > 0$ then*

$$d(n,m) < \left(1 - \frac{1}{k}\right)\binom{m}{2} + 0(m^{2-\beta\delta^2}).$$

Theorem 1.4. *For every positive integer k there exists a positive $\beta' = \beta'(k)$ such that if $m = 2^{(1/(k+1)+\delta)n}$ where $\delta > 0$ then*

$$c(n,m) < \left(1 - \frac{1}{k}\right)\binom{m}{2} + 0(m^{2-\beta'\delta^{k+1}}).$$

The case $k = 1$ of the above theorems was conjectured by Daykin and Erdös [7]. The general case settles a problem of Erdös [7], namely it shows that an Erdös-Stone type result holds. Also the case $k = 2$ implies another conjecture of Erdös in a much stronger form [4].

Our paper is organized as follows. In Section 2 we outline a direct probabilistic proof for the case $k = 1$ of both theorems.

In Section 3 a partition lemma for arbitrary families of sets is proved and applied to verify the theorems with a somewhat weaker estimate of the remainder term.

In Section 4 we combine the probabilistic approach with results of Bollobás, Erdös and Simonovits on supersaturated graphs and hypergraphs to prove both theorems in full strength.

In Section 5 we outline various generalizations dealing with the number of chains of given length, the number of pairwise disjoint r-tuples of sets, etc.

In the last section open problems and conjectures are mentioned.

2. The Basic Probabilistic Argument

Let $\mathcal{F}$ be a family consisting of $m = 2^{((1/2)+\delta)n}$ subsets of $X = \{1, 2, \ldots, n\}$, where $\delta > 0$. We claim that

$$d(\mathcal{F}) < m^{2-\delta^2/2}. \tag{2.1}$$

Note that this inequality, when applied to $\mathcal{F} \cup \{X - F \colon F \in \mathcal{F}\}$ shows that

$$c(\mathcal{F}) < 4m^{2-\delta^2/2}.$$

To prove (2.1) suppose it is false and pick independently t members $A_1, A_2, \ldots, A_t$ of $\mathcal{F}$ with repetitions at random, where t is a large positive integer, to be chosen later. We will show that with positive probability $|A_1 \cup A_2 \cup \ldots \cup A_t| > n/2$ and still this union is disjoint to more than $2^{n/2}$ distinct subsets of X. This contradiction will establish (2.1). In fact

$$Pr(|A_1 \cup A_2 \cup \ldots \cup A_t| \le n/2) \le \sum_{S \subset X, |S| \le n/2} Pr(A_i \subset S, i = 1, \ldots, t)$$
$$\le 2^n (2^{n/2}/2^{((1/2)+\delta)n})^t = 2^{n(1-\delta t)}. \tag{2.2}$$

Define

$$v(B) = |\{A \in \mathscr{F} : B \cap A = \varnothing\}|.$$

Clearly

$$\sum_{B \in \mathscr{F}} v(B) = 2d(\mathscr{F}) \ge 2m^{2-\delta^2/2}.$$

Let Y be a random variable whose value is the number of members $B \in \mathscr{F}$ which are disjoint to all the $A_i - s$ $(1 \le i \le t)$. By the convexity of z^t the expected value of Y satisfies

$$E(Y) = \sum_{B \in \mathscr{F}} (v(B)/m)^t = \frac{1}{m^t} \cdot m\left(\frac{\Sigma v(B)^t}{m}\right) \ge \frac{1}{m^t} \cdot m\left(\frac{2d(\mathscr{F})}{m}\right)^t > 2m^{1-t\delta^2/2}.$$

Since $Y \le m$ we conclude that

$$Pr(Y \ge m^{1-t\delta^2/2}) \ge m^{-t\delta^2/2}. \tag{2.3}$$

One can check that for $t = \lfloor 1 + 1/(\delta - \delta^2/4 - \delta^3/2) \rfloor$, $m^{1-t\delta^2/2} > 2^{n/2}$ and the right-hand side of (2.3) is greater than the right-hand side of (2.2). Thus, with positive probability, $|A_1 \cup A_2 \cup \ldots \cup A_t| > n/2$ and still this union is disjoint to more than $2^{n/2}$ members of F. This contradiction implies inequality (2.1), thus proving Theorems 1.3 and 1.4 for $k = 1$. □

3. A Partition Lemma for Families of Subsets

For $0 < \gamma < 1$ let $H(\gamma)$ denote the binary entropy, i.e., $H(\gamma) = \gamma \log_2(1/\gamma) + (1-\gamma)\log_2(1/(1-\gamma))$.

Lemma 3.1. *Let $\alpha, \beta, \gamma, \varepsilon, a$ be positive constants satisfying $0 < \alpha, \beta, \varepsilon < 1$, $1 < a < 2$, $H(\gamma) < \frac{1}{3}\log_2 a$. Let $\mathscr{F}$ be a family of a^n subsets of $X = \{1, 2, \ldots, n\}$. Then there exists a partition $\mathscr{F}_0 \cup \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_s$ of $\mathscr{F}$ and subsets $X_1, X_2, \ldots, X_s \subset X$ satisfying the following four conditions:*

(i) $|\mathscr{F}_0| < \varepsilon |\mathscr{F}|$.

(ii) $|\mathscr{F}_i| \ge \varepsilon \alpha^M |\mathscr{F}|$, *where* $M = \max\{t: 2^{((1-\beta)^t + H(\gamma))n} > \varepsilon \alpha^t a^n\}$.

(iii) $|F \cap (X - X_i)| \le \gamma |X - X_i|$ *holds for all* $F \in \mathscr{F}_i$.

(iv) *For every* $Y_i \subset X_i$ *satisfying* $|Y_i| \ge \beta |X_i|$

$$|\{F \in \mathscr{F}_i: |F \cap Y_i| < \gamma |Y_i|\}| < \alpha |F_i|.$$

Remark. One can easily check that if α, β, γ, ε and a are as above and $n > n_0(\alpha, \beta, \gamma, \varepsilon, a)$ then the number M defined in (ii) satisfies

$$M < |\log_2 \log_2 a^{1/2}/\log_2(1-\beta)|.$$

Note that this bound is independent of n.

Proof of Lemma 3.1. Suppose that disjoint subfamilies $\mathscr{F}_1, \mathscr{F}_2, \cdots, \mathscr{F}_{i-1}$ of $\mathscr{F}$, together with subsets $X_1, X_2, \ldots, X_{i-1}$ of X, satisfying (ii)–(iv) have already been defined. Set $\mathscr{G}_0 = \mathscr{F} - (\mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{i-1})$. If $|\mathscr{G}_0| < \varepsilon|\mathscr{F}|$, define $\mathscr{F}_0 = \mathscr{G}_0$ to complete the proof. Otherwise, define $Z_0 = X$. Clearly $(\mathscr{G}_0, Z_0)$ satisfy (iii) (as $(\mathscr{F}_i, X_i)$). Suppose that a pair $(\mathscr{G}_j, Z_j)$ satisfying (iii) has already been defined. If (iv) does not hold for this pair, then it fails for some $Y_j \subset Z_j$. In this case set $\mathscr{G}_{j+1} = \{G \in \mathscr{G}_j\colon |G \cap Y_j| < \gamma|Y_j|\}$ and $Z_{j+1} = Z_j - Y_j$. Clearly $(\mathscr{G}_{j+1}, Z_{j+1})$ satisfy (iii). Continue this procedure to obtain a pair $(\mathscr{G}_j, Z_j)$ satisfying (iv). (Since $|Z_j|$ is strictly decreasing during this procedure it must produce such a pair.) Set $\mathscr{F}_i = \mathscr{G}_j$, $X_i = Z_j$. To complete the proof we show that $\mathscr{F}_i$ satisfies (ii). Since $\mathscr{F}_i = \mathscr{G}_j$, we have $|X_i| \le n(1-\beta)^j$ and $|\mathscr{F}_i| \ge \alpha^j|\mathscr{G}_0| \ge \varepsilon\alpha^j|\mathscr{F}|$. Thus we must show that $j < M$. However, this follows from the definition of M and

$$\varepsilon\alpha^j|\mathscr{F}| \le |\mathscr{F}_i| \le 2^{|X_i|} \sum_{r \le \gamma|X - X_i|} \binom{|X - X_i|}{r} < 2^{n((1-\beta)^j + H(\gamma))}.$$

(Here we used a special case of Chernoff's inequality [6].) □

We now apply Lemma 3.1 to obtain a somewhat weaker version of Theorem 1.3. Let $\mathscr{F}$ be a family of $m = 2^{(1/(k+1)+\delta)n}$ subsets of $X = \{1, 2, \ldots, n\}$. Apply Lemma 3.1 to $\mathscr{F}$ with $a = 2^{(1/(k+1)+\delta)}$, with positive small constants $\varepsilon = \alpha$ and $\gamma > 0$ sufficiently small with respect to a, ε, δ and with $\beta = \gamma\delta/k$. Let $\mathscr{F}_i$ and X_i be the subfamilies of $\mathscr{F}$ and the subsets of X guaranteed by our Lemma. Since M is bounded by a function independent of n, (ii), (iii) imply that if $n \ge n_0$, $|X_i| > n\left(\frac{1}{k+1} + \frac{\delta}{2}\right)$. (Otherwise $|\mathscr{F}_i| \le 2^{|X_i|} 2^{H(\gamma)(n - |X_i|)}$, violating (ii).) We need the following simple observation.

Proposition 3.2. *If* $Z_1, \ldots, Z_{k+1} \subset X$, $|Z_i| \ge n\left(\frac{1}{k+1} + \frac{\delta}{2}\right)$, *then there are* $1 \le i < j \le k+1$ *with* $|Z_i \cap Z_j| \ge \frac{\delta}{k} n$.

Proof. If this is false then $|Z_i - (Z_1 \cup \ldots \cup Z_{i-1})| > n\left(\frac{1}{k+1} + \frac{\delta}{2}\right) - n\frac{(i-1)\delta}{k}$ and thus $|Z_1 \cup \ldots \cup Z_{k+1}| > n + \frac{(k+1)\delta}{2} n - \binom{k+1}{2}\frac{\delta}{k} \cdot n = n$, which is impossible. □

Therefore, among any $k+1$ X_i-s there are two, say $X_i, X_{i'}$, with $|X_i \cap X_{i'}| \ge \frac{\delta}{k} n$.

Suppose $|X_i \cap X_{i'}| \ge \frac{\delta}{k} n$. Applying (iv) twice we conclude that the number of disjoint pairs F, F' with $F \in \mathscr{F}_i$ and $F' \in \mathscr{F}_i'$ is $\le (1 - (1-\varepsilon)^2)|\mathscr{F}_i||\mathscr{F}_{i'}| \le 2\varepsilon|\mathscr{F}_i||\mathscr{F}_{i'}|$. (Indeed, at least $(1-\varepsilon)|\mathscr{F}_i|$ members of $\mathscr{F}_i$ contain a subset of at least $\frac{\gamma\delta}{k} n$ elements of $X_i \cap X_{i'}$ and each such subset has a nonempty intersection with at least $(1-\varepsilon)|\mathscr{F}_{i'}|$ members of $\mathscr{F}_{i'}$.)

Define a graph on the set of vertices $\mathscr{F}$ where $F, F' \in \mathscr{F}$ are joined iff $F \cap F' = \varnothing$, and $|X_j \cap X_{j'}| < \frac{\delta}{k} n$, where $F \in \mathscr{F}_j$, $F' \in \mathscr{F}_{j'}$. By Proposition 3.2 our graph contains no complete subgraph on $k+1$ vertices. Thus, by Turan's theorem, it has at most $\left(1 - \frac{1}{k}\right)\frac{m^2}{2}$ edges. This, (i), and the preceding discussion imply

$$d(\mathscr{F}) \le \left(1 - \frac{1}{k}\right)\frac{m^2}{2} + |\mathscr{F}_0|\,|\mathscr{F}| + \sum_{i,i'} 2\varepsilon|\mathscr{F}_i|\,|\mathscr{F}_{i'}| \le \left(1 - \frac{1}{k}\right)\frac{m^2}{2} + 3\cdot\varepsilon m^2.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$d(\mathscr{F}) \le \left(1 - \frac{1}{k} + o(1)\right)\binom{m}{2}.$$

This proves a weaker version of Theorem 1.3. A similar argument yields a proof of Theorem 1.4 with a somewhat weaker estimate of the remainder term. We omit the details. □

4. The Proof of the Erdös-Stone Type Results

In this section we prove Theorem 1.3 in detail and indicate how to prove Theorem 1.4 similarly. Suppose $k > 0$, and let $\mathscr{F}$ be a family of $m = 2^{(1/(k+1)+\delta)n}$ subsets of $X = \{1, 2, \ldots, n\}$, where $\delta > 0$. We must show that $d(\mathscr{F}) < \left(1 - \frac{1}{k}\right)\binom{m}{2} + O(m^{2-\beta\delta^2})$ for some $\beta = \beta(k) > 0$. Suppose

$$d(\mathscr{F}) = \left(1 - \frac{1}{k} + \varepsilon\right)\binom{m}{2}, \quad \text{where } \varepsilon = m^{-g}\left(< \frac{1}{k}\right).$$

Let G be a graph on the set of vertices $\mathscr{F}$ in which $A, B \in \mathscr{F}$ are adjacent iff $A \cap B = \varnothing$. Clearly G has $\left(1 - \frac{1}{k} + \varepsilon\right)\binom{m}{2}$ edges. Let t be a large integer, to be chosen later, and let $K = K_{(k+1)}(t)$ denote the complete $(k+1)$-partite graph with t vertices in each vertex class. Our proof is organized as follows. First we apply the so-called theory of supersaturated graphs to obtain a lower bound for the number of copies of K in G. Afterwards we use our probabilistic argument to obtain an upper bound for this number. Combining the two bounds we obtain the desired result. We begin with the following simple lemma.

Lemma 4.1. *For $s \le m$ the number of induced subgraphs of G on s vertices with* $\ge \left(1 - \frac{1}{k} + \frac{\varepsilon}{2}\right)\binom{s}{2}$ *edges is at least* $\frac{k\cdot\varepsilon}{2}\binom{m}{s}$.

Proof. The average number of edges in such a subgraph is $\left(1 - \frac{1}{k} + \varepsilon\right)\binom{s}{2}$. Thus, if N is the number of desired subgraphs then

$$N\binom{s}{2}+\left(\binom{m}{s}-N\right)\left(1-\frac{1}{k}+\frac{\varepsilon}{2}\right)\binom{s}{2}\geq\binom{m}{s}\left(1-\frac{1}{k}+\varepsilon\right)\binom{s}{2}.$$

The desired results follows. □

By a result of Bollobás, Erdös and Simonovits [2] if $0<\varepsilon<1/k$ then any graph on s vertices with $\geq\left(1-\frac{1}{k}+\frac{\varepsilon}{2}\right)\binom{s}{2}$ edges contains a copy of $K_{(k+1)}(t)$ for $t=\left\lceil\frac{\alpha\log s}{k\log 1/\varepsilon}\right\rceil$, where α is an absolute constant. Thus, each of our N subgraphs contains such a copy, and since every copy is obtained at most $\binom{m-(k+1)t}{s-(k+1)t}$ times the total number of K-s in G is at least $N\Big/\binom{m-(k+1)t}{s-(k+1)t}\geq\frac{k\cdot\varepsilon}{2}\left(\frac{m}{s}\right)^{(k+1)t}$.

Set $s=m^f$, where $0<f<1$ will be chosen later, to obtain:

Lemma 4.2. *G contains at least $\frac{k}{2}m^{(1-f)(k+1)t-g}$ copies of $K_{(k+1)}(t)$ where $t=\left\lceil\frac{\alpha f}{kg}\right\rceil$.* □

We now establish an upper bound for the number of K-s in G. Indeed, let us pick at random a class of t distinct members $A_1,\dots,A_t$ of $\mathscr{F}$. The probability that $|\bigcup_{i=1}^t A_i|\leq\frac{n}{k+1}$ is clearly bounded by

$$\sum_{S\subset X,|S|\leq n/(k+1)}\binom{2^{|S|}}{t}\Big/\binom{|\mathscr{F}|}{t}\leq 2^{n(1-\delta t)}.$$

Thus, if we choose at random $k+1$ such classes the probability that the cardinality of the union of at least one of these classes has size not exceeding $\frac{n}{k+1}$ is at most $(k+1)2^{n(1-\delta t)}$. However, this condition is necessary if these classes are the classes of vertices of a $K=K_{(k+1)}(t)$ in G. We thus proved the following.

Lemma 4.3. *G contains at most $(k+1)2^{n(1-\delta t)}\binom{m}{t}^{k+1}\frac{1}{(k+1)!}$ copies of $K_{(k+1)}(t)$.*

Combining Lemmas 4.2 and 4.3 we conclude that

$$m^{(1-f)(k+1)t-g}\leq 2^{n(1-\delta t)}m^{(k+1)t}.$$

Substitute $m=2^{((1/k+1)+\delta)n}$ to get

$$(1+(k+1)\delta)((1-f)(k+1)t-g)\leq t(k+1)(1+(k+1)\delta)+(1-\delta t)(k+1),$$

i.e.,

$$t\leq\frac{k+1+g(1+(k+1)\delta)}{(k+1)\delta-(k+1)f(1+(k+1)\delta)}.$$

Recall that $t=\left\lceil\frac{\alpha f}{kg}\right\rceil\geq\frac{\alpha f}{kg}$ and that we are still free to choose $0<f<1$. Choosing

$f = \frac{\delta k}{\alpha} \cdot \gamma$, where $\gamma < \frac{\alpha}{4k}$, this implies that for sufficiently small δ, $g \geq \frac{1}{5}\delta^2 \gamma$, i.e.,

$$d(\mathscr{F}) \leq \left(1 - \frac{1}{k} + m^{-(1/5)\delta^2\gamma}\right)\binom{m}{2}.$$

Since γ depends only on k, Theorem 1.3 follows. □

The proof of Theorem 1.4 is similar although slightly more complicated. Again, we begin with a family $\mathscr{F}$ of cardinality $m = 2^{(1/(k+1)+\delta)n}$ and assume

$$c(\mathscr{F}) = \left(1 - \frac{1}{k} + m^{-g}\right)\binom{m}{2}.$$

We let G denote the graph on the set of vertices $\mathscr{F}$ in which $A, B \in \mathscr{F}$ are joined iff $A \subset B$ or $B \subset A$. By [1] G contains at least $\gamma(k)m^{k+1-g}$ K_{k+1}-s each corresponding to a chain $A_1 \subset A_2 \subset \cdots \subset A_{k+1}$ of members of F, where $\gamma(k)$ is a constant dependent only on k. A straightforward averaging argument shows that there exists a partition $\mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{k+1}$ of $\mathscr{F}$, where $\left\lfloor \frac{m}{k+1} \right\rfloor \leq |\mathscr{F}_i| \leq \left\lceil \frac{m}{k+1} \right\rceil$ such that the number of chains $A_1 \subset A_2 \cdots \subset A_{k+1}$, where $A_i \in \mathscr{F}_i$ is $\geq \lambda(k) \cdot m^{k+1-g}$. Consider the $(k+1)$-uniform $(k+1)$-partite hypergraph H on the set of vertices $\mathscr{F}$ in which $(A_1, \ldots, A_{k+1})$ is an edge iff $A_i \in \mathscr{F}_i$ and $A_1 \subset \cdots \subset A_{k+1}$. By [5, Theorem 1**] if $g' > g$ and m is sufficiently large then H contains at least $\mu(k)m^{t(k+1)}m^{-g't^{k+1}}$ copies of K = the complete $(k+1)$-partite $(k+1)$-uniform hypergraph with t vertices in each vertex class. However, each copy of K corresponds to a collection of $k+1$ t-classes of members of $\mathscr{F}$, $\{A_1^i\}_{i=1}^t, \ldots, \{A_{k+1}^i\}_{i=1}^t$, where $A_j^i \subset A_{j+1}^l$ for $1 \leq i \leq t$, $1 \leq l \leq t$ and $1 \leq j \leq k$. On the other hand, our probabilistic argument easily supplies an upper bound of $p(k)\, m^{(k+1)t}\, 2^{n(1-\delta t)}$ for the number of these collections. Thus,

$$p(k) m^{(k+1)t}\, 2^{n(1-\delta t)} \geq \mu(k)\, m^{(k+1)t}\, m^{-g't^{k+1}}.$$

Substituting $m = 2^{(1/(k+1)+\delta)n}$ and choosing $t = \frac{2}{\delta}$ we obtain that $g' \geq v'(k)\delta^{k+1}$, i.e., $g \geq v(k)\delta^{k+1}$. This establishes Theorem 1.4. □

5. Generalizations

For $r \geq 2$ and for a family $\mathscr{F}$ of subsets of X define

$$d_r(\mathscr{F}) = |\{\{F_1, F_2, \ldots, F_r\}: F_1, \ldots, F_r \in \mathscr{F}, F_1 \cap \ldots \cap F_r = \varnothing\}|.$$

Thus $d_r(\mathscr{F})$ counts the number of r-tuples with empty intersection. (Note that $d_2(\mathscr{F}) = d(\mathscr{F})$.)

Theorem 5.1. *Suppose* $|\mathscr{F}| = m \geq 2^{((r-1/r)+\delta)n}$ *where* $\delta > 0$ *and* $r \geq 2$. *Then* $d_r(\mathscr{F}) = o\binom{m}{r}$ *(as* $n \to \infty, \delta, r$ *fixed.)*

Note that one can easily find an $\mathscr{F}$ of size $\simeq 2^{(1-1/r)n}$ with $d_r(\mathscr{F}) = \Omega_r(|\mathscr{F}|^r)$.

Outlined Proof. Apply Lemma 3.1 with sufficiently small constants $\alpha, \beta, \gamma, \varepsilon$, and with $a = 2^{1-1/r+\delta}$ to obtain subfamilies $\mathscr{F}_i$ of $\mathscr{F}$ with corresponding subsets X_i of X.

Clearly (as in Section 3) if n is sufficiently large then $|X_i| > \left(\frac{r-1}{r} + \frac{\delta}{2}\right)n$. Consequently, any r of the X_i-s have an intersection of size $\geq \frac{r\delta}{2} n$. If β is small enough r applications of condition (iv) of Lemma. 3.1 imply (as in Section 3) that the number of r-wise disjoint r-tuples of members of $\mathscr{F}$ is $\leq (1 - (1 - \varepsilon)^r)m^r + \varepsilon m^r$. Since ε is arbitrary the result follows. □

For $s \geq 2$ and a family $\mathscr{F}$ define $p_s(\mathscr{F}) = |\{\{F_1, F_2, \ldots F_s\}: F_i \in \mathscr{F}, F_i \cap F_j = \varnothing$ for $1 \leq i < j \leq s\}|$, i.e., $p_s(\mathscr{F})$ counts the number of pairwise disjoint s tuples. (Note that $p_2(\mathscr{F}) = d_2(\mathscr{F})$).

Theorem 5.2. *Suppose* $|\mathscr{F}| = m \geq 2^{((1/s)+\delta)n}$ where $\delta > 0$ and $s \geq 2$. *Then* $p_s(\mathscr{F}) = o\left(\binom{m}{s}\right)$ *(as $n \to \infty, \delta, s$ fixed).*

Again note that one can easily find an $\mathscr{F}$ of size $\simeq 2^{(1/s)n}$ with $p_s(\mathscr{F}) = \Omega_s(|\mathscr{F}|^s)$.

Outlined Proof. Apply Lemma 3.1 with sufficiently small α, β, γ, ε and with $a = 2^{((1/s)+\delta)}$ to get $\mathscr{F}_i$ and X_i. As before, if n is large enough $|X_i| > \left(\frac{1}{s} + \frac{\delta}{2}\right)n$. By Proposition 3.2 among any s of the X_i-s there are two, say X_i, $X_{i'}$, with $|X_i \cap X_{i'}| \geq \frac{\delta}{s-1} n$. Condition (iv) of Lemma 3.1 implies, again, the desired result. □

For $s \geq 2$ and a family $\mathscr{F}$ define $c_s(\mathscr{F}) = |\{(\mathscr{F}_1, F_2, \ldots, F_s): F_i \in \mathscr{F}, F_1 \subset F_2 \subset \ldots \subset F_s\}|$, i.e., $c_s(\mathscr{F})$ is the number of chains of s members of $\mathscr{F}$. (Note that $c_2(\mathscr{F}) = c(\mathscr{F})$).

Theorem 5.3. *Suppose* $|\mathscr{F}| = m \geq 2^{((1/s)+\delta)n}$ *where* $\delta > 0$ *and* $s \geq 2$. *Then* $c_s(\mathscr{F}) = o\left(\binom{m}{s}\right)$.

Here also there is an $\mathscr{F}$ of size $\simeq 2^{(1/s)n}$ with $c_s(\mathscr{F}) = \Omega_s(|\mathscr{F}|^s)$. The proof is similar to the previous ones. We omit the details.

Remark. Theorems 5.1, 5.2 and 5.3 can also be proved using probabilistic arguments, as in Section 2. Moreover, the probabilistic method supplies better estimates of the quantities discussed. However, since this involves somewhat tedious computations, we preferred presenting the proofs via Lemma 3.1.

6. Concluding Remarks and Open Problems

Recall that in Section 2 we proved that if $m = 2^{((1/2)+\delta)n}$ then

$$c(n, m) < 4m^{2-\delta^2/2}.$$

This inequality does not appear to be best possible, and in particular it does not seem to describe the asymptotic behavior of $c(n, m)$ for $m = 2^{(1/2)n} \cdot n^d$.

Example 6.1. Suppose $X = X_1 \cup X_2$, $|X_1| = |X_2| = n/2$. Define

$$\mathscr{F}_1^{(d)} = \{F \subset X: |F \cap X_2| \leq d\},$$
$$\mathscr{F}_2^{(d)} = \{F \subset X: |X_1 - F| \leq d\}.$$

Set $\mathscr{F} = \mathscr{F}^{(d)} = \mathscr{F}_1^{(d)} \cup \mathscr{F}_2^{(d)}$. Then

$$m = |\mathscr{F}| = 2^{(n/2)+1} \sum_{i=0}^{d} \binom{n/2}{i} = \Omega(2^{(n/d)} n^d) \text{ and } c(\mathscr{F}) \geq 2^{-2d-1}\binom{m}{2}.$$

This example disproves a conjecture of Erdös [4]. We suspect that the following is true.

Conjecture 6.2.

$$\lim_{d \to \infty} c(n, 2^{(n/2)} \cdot n^d)/(2^{(n/2)} \cdot n^d)^2 = 0.$$

Recall the definitions of $d_r(\mathscr{F})$, $p_s(\mathscr{F})$ and $c_s(\mathscr{F})$ given in Section 5. It would be interesting to describe the asymptotic behavior of these functions.

We conclude the paper noting that all our results remain true if we replace disjointness by having sufficiently small intersection. For example, our methods easily imply that if $\delta' < 2\delta$ and $|\mathscr{F}| > 2^{((1/2)+\delta)n}$ then

$$|\{\{F, F'\}: F, F' \in \mathscr{F}, |F \cap F'| < \delta' \cdot n\}| = o\binom{|\mathscr{F}|}{2}$$

as $n \to \infty$, $\delta' < 2\delta$ fixed.

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