JOURNAL OF COMBINATORIAL THEORY, Series A **43**, 91–97 (1986)

Regular Hypergraphs, Gordon's Lemma, Steinitz' Lemma and Invariant Theory

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Communicated by the Managing Editors

Received March 25, 1985

Let $D(n)(D(n, k))$ denote the maximum possible d such that there exists a d-regular hypergraph (d-regular k-uniform hypergraph, respectively) on n vertices containing no proper regular spanning subhypergraph. The problem of estimating $D(n)$ arises in Game Theory and Huckemann and Jurkat were the first to prove that it is finite. Here we give two new simple proofs that $D(n)$, $D(n, k)$ are finite, and determine $D(n, 2)$ precisely for all $n \geqslant 2$. We also apply this fact to Invariant Theory by showing how it enables one to construct an explicit finite set of generators for the invariants of decomposable forms.

1. INTRODUCTION

Suppose $n \geqslant 1$ and put $N = \{1, 2, 3,..., n\}$. A *(multi)-hypergraph* H on N is a multiset of elements of the power set $P(N)$, i.e., a collection of subsets of N, where the same subset can appear several times. The *degree* of a point $i \in N$ is $d_H(i) = \sum_{i \in S \subset N} f_H(S)$, where $f_H(S)$ is the number of occurrences of S in H. H is *d-regular* if $d_H(i) = d$ for all $i \in N$. We call H a k-hypergraph if $f_H(S) > 0 \Rightarrow |S| = k$. A *subhypergraph* G of H is a submultiset of H. H is *indecomposable* if it contains no proper nonempty regular subhypergraph. In this note we consider the maximum possible degree of regularity of regular indecomposable hypergraphs. More precisely, define for $n \geqslant 1$, $D(n) = \text{Max}\{d\colon H$ is a d-regular indecomposable hypergraph on $N\}$, and for $n \geqslant k \geqslant 1$ $D(n, k) = \text{Max}\{d\colon H$ is a d-regular indecomposable k-hypergraph on $N\}$, where it is understood that $D(n) = \infty$ is a possibility. Huckemann and Jurkat (cf. [5]) are the first to prove that $D(n)$ (and hence $D(n, k)$) is finite for all n. The problem of estimating $D(n)$ is considered by many people, since it has applications in Game Theory.

* Research supported in part by the Weizmann Fellowship for Scientific Research and by NSF under contract DMS-8406100.

0097-3165/86 $3.00

Huckemann, Jurkat and Shapley prove that $D(n) \leqslant (n+1)^{(n+1)/2}$ for all $n \geqslant 1$. (cf. [5]).

Our motivation to consider this problem comes from Invariant Theory. One of the fundamental theorems of Invariant Theory is Hilbert's Finiteness Theorem, that asserts that there exists a finite generating set for the invariants of forms of degree n in k variables. Hilbert's proof supplies an explicit finite set of generators only for the case $k=2$, i.e., the invariants of binary forms. For the general case, Popov [7], recently gives an explicit set of generators. His construction, however, involves several deep results from algebraic geometry. Here we obtain an elementary combinatorial construction for the case of decomposable forms, by showing how it follows from upper bounds for the $D(n, k)-s$. For more details, see [3].

As it frequently happens we learn (from P. Frankl) about the known results concerning $D(n)$ only after we have two new simple proofs that $D(n)$ is finite for all n. Our first proof is based on Gordon's lemma, and is very simple. It only shows, however, that $D(n)$ is finite and does not supply any bound. Our second proof is based on a geometric result, known in the Russian literature as Stienitz' Lemma. This proof supplies an explicit bound and, in fact, also enables us to prove an effective version of Gordon's lemma. The original proof of Huckemann, Jurkat and Shapley also supplies an explicit bound and is based on Hadamard bound for determinants. All these proofs together seem to reveal a close and interesting relationship between several geometric and combinatorial results.

Every upper bound for $D(n)$ is, of course, also an upper bound for $D(n, k)$. It is worthwhile, however, to find better upper bounds for the $D(n, k)-s$, since these supply more efficient sets of generators for the invariants of decomposable forms in k variables. Our second proof (as well as the methods described in [5]) supply upper bounds of the form $2^{cn \log n}$ for $D(n)$ and of the form $2^{c_k \cdot n}$ for $D(n, k)$. Obviously $D(n, 1)=1$ and we can apply known results from Graph Theory to show that

$$D(n, 2) \leqslant \begin{cases} n-1 & \text{for even } n, \\ 2 & \text{for odd } n. \end{cases}$$

This is sharp, provided loops are allowed.

As mentioned above, the finiteness of $D(n)$ has applications in Game Theory and in Invariant Theory. Another application is found very recently by the authors of [1], who used this fact to solve a conjecture of Erdös and V.T. Sós about simultaneously balancible sets.

2. Gordon's Lemma and the Finiteness of $D(n)$

The following lemma is due to Gordon. For its simple proof, see, e.g., [6].

LEMMA 2.1 (Gordon). *Let*

$$AX = 0 \tag{2.1}$$

be a homogeneous system of linear-equations in the variables x_i, where A is a matrix of integer coefficients. Let M denote the set of all solutions of (2.1) *over the non-negative integers. Then there exists a finite set of solutions $b_1,\dots, b_p \in M$ such that every solution $s \in M$ is a linear combination with non-negative integer coefficients of $b_1,\dots, b_p$.*

Note that the lemma gives no upper bound for p (in terms of A) and only guarantees that it is finite. In the next section we give such upper bound.

We now show that Lemma 2.1 implies that $D(n)$ is finite. Put $N = \{1, 2,\dots, n\}$. For each $S \subset N$ let x_S be a variable and consider the following system of equations in the variables $\{x_S: S \subset N\} \cup \{d\}$:

$$\sum_{i \in S} x_S - d = 0 \qquad i = 1,\dots, n. \tag{2.2}$$

Clearly the set of all solutions of (2.2) over the non-negative integers is precisely the set of all regular hypergraphs on N. Hence, by Lemma 2.1, there exists a finite generating set of solutions. One can easily check that every indecomposable regular hypergraph must belong to this set. Thus, there are only finitely many indecomposable regular hypergraphs on N, and $D(n)$ is finite, as needed.

It is worth noting that the finiteness of $D(n)$ follows similarly from the known fact that the set of all hypergraphs on N is a well quasi order (i.e., in any infinite sequence $\{G_i\}_{i=1}^{\infty}$ of hypergraphs on N, G_i is a sub-hypergraph of G_j for some $i < j$). This also implies that the set of all indecomposable regular hypergraphs on N is finite, and thus so is $D(n)$.

3. STEINITZ' LEMMA AND UPPER BOUNDS FOR $D(n)$, $D(n, k)$

Steinitz' lemma asserts that every sequence of m vectors of norm $\leqslant 1$ in $\mathbb{R}^n$ whose sum is the zero vector, can be rearranged such that all initial sums will have norm $\leqslant c(n)$, where $c(n)$ depends only on the dimension and not on the number of vectors. The estimate for $c(n)$ has been improved several times. The best-known upper bound is due to Sevast'yanov [8], (see also Bárány [2]), and it applies to any normed space. His result is the following:

LEMMA 3.1 (Sevast'yanov). *Let X be any normed n-dimensional space.*

Suppose $v_1, v_2, ..., v_m \in X$, $\|v_i\| \leqslant 1$ *and* $\sum_{i=1}^{m} v_i = 0$. *Then there is a permutation* π *on* $1, 2, ..., m$ *such that for all* $1 \leqslant j \leqslant m$.

$$\left\| \sum_{i=1}^{j} v_{\pi(i)} \right\| \leqslant n.$$

Lemma 3.1 supplies an upper bound for $D(n)$ as follows.

PROPOSITION 3.2. *For every* $n \geqslant 1$, $D(n) \leqslant \frac{1}{2} \cdot (2n+1)^n$.

Proof. Let H be a d-regular hypergraph on N, where $N = \{1, ..., n\}$ and $d > \frac{1}{2}(2n+1)^n$. We must show that H contains a proper regular subhypergraph. For $S \subset N$, let $f(S) = f_H(S)$ denote the number of occurrences of S in H. Let M be the sequence of $d + \sum_{S \subset H} f(S)$ vectors of length n, consisting of $f(S)$ copies of the characteristic vector of S (for all $S \subset N$), and d copies of the vector $(-1, -1, ..., -1)$. Clearly M has at least $2d$ vectors whose sum is the zero vector and each has sup-norm 1. By Lemma 3.1 one can rearrange these vectors such that all initial sums have sup-norm $\leqslant n$. Since there are $> (2n+1)^n$ such sums and the number of lattice points having sup-norm $\leqslant n$ is $(2n+1)^n$ some pair of initial sums coincide. Their difference is a nontrivial partial sum that vanishes. Since $(-1, -1, ..., -1)$ is the only vector in M having negative coordinates it must occur, say $d' < d$ times in this partial sum. The other vectors of this sum correspond to the set of edges of a proper d'-regular subhypergraph of H. This completes the proof. ∎

Similar to Proposition 3.2 is the following.

PROPOSITION 3.3. *For every* $n \geqslant k \geqslant 1$

$$D(n, k) < 2^n \cdot \binom{nk+n+1}{n} \simeq 2^{n(1+(k+1)H_2(1/(k+1)))}, \tag{3.1}$$

where H_2 *is the binary entropy function, i.e.,* $H_2(x) = -x \log_2 x - (1-x) \log_2(1-x)$.

Proof. Let H be a d-regular k-hypergraph on N, where $N = \{1, 2, ..., n\}$ and $d \geqslant 2^n \cdot \binom{nk+n+1}{n}$. Let M be the sequence of vectors of length n consisting of $f_H(S)$ copies of the characteristic vector of S (for all $S \subset N$), and of $d \cdot n$ copies of the vector $(-1/n, -1/n, ..., -1/n)$. The sum of vectors of M is 0 and each has l_1-norm $\leqslant k$. Hence, by Lemma 3.1, these vectors can be rearranged such that all initial sums have l_1-norm $\leqslant n \cdot k$. Note that each initial sum is a lattice point $+ r(1/n, 1/n, ..., 1/n)$ for some $0 \leqslant r < n$. One can easily check that the number of these is $\leqslant n \cdot 2^n \binom{nk+n+1}{n}$, and since the number of vectors in M is at least $d \cdot n + d \cdot n/k > n \cdot 2^n \binom{nk+n+1}{n}$, there is, as in the proof of Proposition 3.2, a nontrivial partial sum that vanishes. This implies that H contains a proper regular sub-hypergraph and completes the proof. It is worth noting that by using the methods described in [5] we can improve (3.1) to $D(n, k) \leqslant \binom{n}{k} \cdot k^{n/2}$. We conjecture that in fact $D(n, k) \leqslant n^{c(k)}$, where $c(k)$ depends only on k. ∎

The method used in the proofs of the last two propositions enables us to obtain an effective version of Gordon's lemma (Lemma 2.1), namely to give an explicit finite set of generators for the set of non-negative integer solutions of a homogeneous systems of linear equation. This is stated in the following Proposition whose proof, which is analogous to those of Propositions 3.2 and 3.3, is omitted.

PROPOSITION 3.4. *Let*

$$AX = 0 \tag{3.2}$$

be a homogeneous system of r linear equations in the t variables $x_1, x_2, \ldots, x_t$, *where* $A = (a_{ij})_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant t}$ *is a matrix of integer coefficients.*

Let M denote the set of all solutions of (3.2) *over the non-negative integers. Put* $a = \max\{|a_{ij}|:\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant t\}$ *and define* $B = \{(x_1, \ldots, x_t) \in M: \sum_{i=1}^{t} x_i \leqslant (2at+1)^t\}$. *Then every solution* $s \in M$ *is a linear combination with non-negative integer coefficients of the solutions in B.*

Note that we use here the sup-norm. Similar results with other norms can be formulated.

We conclude this section with the case $k = 2$ (multigraphs).

PROPOSITION 3.5.

$$D(n, 2) \leqslant \begin{cases} n-1 & \text{for even } n \\ 2 & \text{for odd } n. \end{cases}$$

Proof. Let H be a d-regular multigraph on n vertices. H has, possibly, loops. By a theorem of Taŝkinov [9], if H has at most $d-1$ bridges it contains a 2-regular factor ($=2$ regular spanning subgraph). Since the number of bridges is, clearly, at most $n-1$, we conclude that if $d \geqslant n$ H is not indecomposable. It remains to show that for odd n, every d regular multigraph on n vertices is not indecomposable for $d > 2$. However, since d must be even, such a graph always contains a 2-factor, by a theorem of Petersen (cf. [4]). This completes the proof. Note that if loops are allowed this result is best possible. Indeed, for even n, the graph obtained from a star with $n-1$ edges by adding $(n-2)/2$ loops at each endvertex is $(n-1)$-regular and indecomposable. For odd n, the cycle of length n is 2-regular and indecomposable. ∎

4. Invariant Theory of Decomposable Forms

In this section we explain very briefly the connection between $D(n, k)$ and Invariant Theory of decomposable forms. For more details see [3].

Let $f(\mathbf{x})$ be a decomposable homogeneous form of degree n in the k variables $x_1, x_2, ..., x_k$ over a field of characteristic zero, i.e.,

$$f(\mathbf{x}) = L_1(\mathbf{x})\, L_2(\mathbf{x}) \cdots L_n(\mathbf{x})$$

where

$$L_i(\mathbf{x}) = r_{i1}x_1 + r_{i2}x_2 + \cdots + r_{ik}x_k$$

$i = 1, 2, ..., n$. Let $N = \{1, 2, ..., n\}$ and let $\mathbb{R} = \{r_{ij}\}$, $i \in N$, $j = 1, 2, ..., k$. For $I = (i_1, i_2, ..., i_k) \in N^k$ such that $i_1 + i_2 + \cdots + i_k = n$ let $a_I = a_I(\mathbb{R})$ be the coefficient of $x_1^{i_1} \cdot x_2^{i_2} \cdots x_k^{i_k}$ in $f(\mathbf{x})$, i.e.,

$$f(\mathbf{x}) = \sum_{\substack{I \in N^k \\ i_1 + i_2 + \cdots + i_k = n}} a_I x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}.$$

A linear change of variables $\mathbb{C} = (c_{ij})$ is transformation from the variables $\mathbf{x} = (x_1, x_2, ..., x_k)$ to the variables $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_k)$ given by

$$\mathbf{x}^T = \mathbb{C}\bar{x}^T$$

such that $\det \mathbb{C} \neq 0$. Under a linear change of variables $L_i(x)$ is transformed to $\bar{L}_i(\bar{x}) = \bar{r}_{i1} \cdot \bar{x}_1 + \bar{r}_{i2}\bar{x}_2 + \cdots + \bar{r}_{ik}\bar{x}_k$ and $f(x)$ is transformed to

$$\bar{f}(\bar{x}) = \bar{L}_1(\bar{x})\, \bar{L}_2(\bar{x}) \cdots \bar{L}_n(\bar{x}) = \sum \bar{a}_I \bar{x}_2^{i_1} \bar{x}_2^{i_2} \cdots \bar{x}_k^{i_k}.$$

A non-constant polynomial $P(a_I)$ in the variables $\{a_I = a_I(\mathbb{R}),\ I \in N^k\}$ is an *invariant* if for all linear changes of variables $\mathbb{C}$

$$P(\bar{a}_I) = (\det \mathbb{C})^g\, P(a_I)$$

for some positive integer g.

For $i_1, i_2, ..., i_k$ distinct numbers from N the *bracket* (*of size* k) $B = [i_1, i_2, ..., i_k]$ is defined by

$$B = \det \begin{bmatrix} r_{i_1 1} & r_{i_1 2} \cdots r_{i_1 k} \\ r_{i_2 1} & r_{i_2 2} \cdots r_{i_2 k} \\ \vdots & \\ r_{i_k 1} & r_{i_k 2} \cdots r_{i_k k} \end{bmatrix}$$

A *bracket monomial* M is a product of brackets, i.e., $M = B_1 B_2 \cdots B_\beta$ where B_i is a bracket, $i = 1, 2, ..., \beta$. A bracket monomial M is *regular of degree* d if each number from N occurs in exactly d brackets of M.

Since a regular bracket monomial corresponds to a regular k-uniform hypergraph on the vertex set N whose edges correspond to the brackets of M, we have the following

PROPOSITION 4.1. *Every regular bracket monomial involving brackets of size k can be expressed as a product of regular bracket monomials each having degree at most $D(n, k)$.*

By Proposition 3.3 this supplies an explicit finite set of generators for the bracket monomials. In [3] it is shown that such a set provides an explicit set of generators for the invariants of decomposable forms of degree n in k variables.

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