Explicit construction of linear sized tolerant networks

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Abstract

For every $\epsilon > 0$ and every integer $m > 0$, we construct explicitly graphs with $O(m/\epsilon)$ vertices and maximum degree $O(1/\epsilon^2)$, such that after removing any $(1 - \epsilon)$ portion of their vertices or edges, the remaining graph still contains a path of length m. This settles a problem of Rosenberg, which was motivated by the study of fault tolerant linear arrays.
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1. Introduction

What is the minimum possible number of vertices and edges of a graph G, such that even after removing all but ϵ portion of its vertices or edges, the remaining graph still contains a path of length m? This problem arises naturally in the study of fault tolerant linear arrays, (see [18]). The vertices of G represent processing elements and its edges correspond to communication links between these processors. If $p, 0 < p < 1$ is the failure rate of the processors, it is desirable that after deleting any p portion of the vertices of G, the remaining part still contains a (simple) path (= linear array) of length m. Similarly, if $\mu, 0 < \mu < 1$ denotes the failure rate of the communication links, it is required that after deleting any μ portion of the edges of G the remaining part still contains a relatively long path. The objective is to construct such graphs G with a small number of vertices and edges, since these will give rise to efficient networks. Some variants of this problem are discussed in [14,9,19]. In this note we prove the following result.

Theorem 1.1. *For every $\epsilon > 0$ and every integer $m \geqslant 1$ there is a graph G, which can be explicitly constructed, with $O(m/\epsilon)$ vertices and maximum degree $O(1/\epsilon^2)$, such that even after deleting all but ϵ-portion of its vertices or all but ϵ-portion of its edges, the remaining graph still contains a path of length m.*

This settles the problem raised by Rosenberg [18], and is also related to the study of size Ramsey numbers (see [10,4]). We note that the edge version of the above theorem for $\epsilon = \frac{1}{2}$ was proved by Beck [4], and his proof can be modified to give the general case, without explicit construction. Our main contribution here is to obtain an explicit construction by combining the arguments of Beck [4] with some of the eigenvalues technique of [1–3] and the recent construction of expanders given by Lubotzky et al. [16].

2. The proof of the main result

For a graph $H = (V, E)$ and $S \subseteq V$, let $N_H(S)$ denote the set of all neighbors in H of vertices of S. Pósa [17] proved the following useful lemma which provides a relation between the expanding properties of a graph and the size of the longest path it contains. (See also [15] for a simpler proof.)

Lemma 2.1. *Let $H = (V, E)$ be a nonempty graph. If $|N_H(S) - S| \geqslant 2|S| - 1$ for every vertex subset $S \subseteq V$ of cardinality $|S| \leqslant k$, then H contains a path of length $3k - 2$.*

The following simple lemma is an old folklore result (see, e.g. [4]):

Lemma 2.2. *Any graph G on n vertices with average degree d contains an induced subgraph H such that for every vertex-set S of H the number of edges incident to vertices in S is at least $d|S|/2$.*

Next we need a relation between the eigenvalues of a graph G and the density of its induced subgraphs. Let $A = A_G$ be the n by n adjacency matrix of a d-regular graph $G = (V, E)$ on $V = \{1, 2, \ldots, n\}$. Clearly d is the largest eigenvalue of A and its eigenvector is the all 1 vector. Suppose that the absolute value of any other eigenvalue of A is at most λ. For $S \subseteq V$ let $e(S, \bar{S})$ denote the number of edges of G between vertices of S and vertices of $\bar{S} = V - S$, and let $e(S)$ denote the number of edges that join two vertices of S.

Lemma 2.3. *In the above notation, for every subset $S \subseteq V$ of cardinality $|S| = \alpha n$*

$$|e(S) - \tfrac{1}{2} d\alpha^2 n| \leqslant \tfrac{1}{2} \lambda \alpha (1 - \alpha) \cdot n. \tag{2.1}$$

We note that the term $\frac{1}{2} d\alpha^2 n$ is roughly the expected number of edges in an induced subgraph of G of size $\alpha \cdot n$. Thus, for small λ, every such induced subgraph has about the same number of edges.

Proof. Define a vector $f : V \to R$ by $f(i) = -1/|S|$ if $i \in S$ and $f(i) = 1/(n - |S|)$ if $i \notin S$. Since $\sum_{i=1}^n f(i) = 0$, i.e. f if orthogonal to the eigenvector of the largest eigenvalue of A, we conclude that $|(Af, f)| \leqslant \lambda(f, f)$, where $(,)$ is the usual scalar product. One can easily check that

$$(Af, f) = 2 \sum_{ij \in E} f(i) \cdot f(j) = d \sum_{i=1}^n f^2(i) - \sum_{ij \in E} (f(i) - f(j))^2.$$

For the specific f defined above

$$\sum_{i=1}^n f^2(i) = \frac{1}{|S|} + \frac{1}{n - |S|} \quad \text{and} \quad \sum_{ij \in E} (f(i) - f(j))^2 = e(S, \bar{S}) \left(\frac{1}{|S|} + \frac{1}{n - |S|} \right)^2.$$

Thus

$$\left| e(S, \bar{S}) \left(\frac{1}{|S|} + \frac{1}{n - |S|} \right)^2 - d \left(\frac{1}{|S|} + \frac{1}{n - |S|} \right) \right| \leqslant \lambda \left(\frac{1}{|S|} + \frac{1}{n - |S|} \right),$$

which implies, by substituting $|S| = \alpha n$, that

$$|e(S, \bar{S}) - d\alpha(1 - \alpha)n| \leqslant \lambda \alpha (1 - \alpha) \cdot n. \tag{2.2}$$

Since G is a d-regular graph,

$$2e(S) + e(S, \bar{S}) = d|S| = d\alpha n,$$

i.e.

$$e(S) = \tfrac{1}{2} d\alpha n \tfrac{1}{2} e(S, \bar{S}).$$

This and inequality (2.2) imply inequality (2.1). This completes the proof. $\square$

Proof of Theorem 1.1. Lubotzky et al. [16] showed that if p and q are primes congruent to 1 mod 4, with p a quadratic non-residue mod q, then there is an explicitly constructed $d = p + 1$ regular graph G with $n = q(q^2 - 1)/2$ vertices, such that the absolute value of each of its eigenvalues but the first is at most $\lambda = 2\sqrt{d - 1}$. We next show that for properly chosen p and q, G satisfies the assertion of Theorem 1.1. We first consider the case of deleting vertices. Suppose we delete all but a set V of $\epsilon \cdot n$ vertices of G. By Lemma 2.3 the induced subgraph of G on V contains at least $\frac{1}{2} d\epsilon^2 n - \frac{1}{2}\lambda\epsilon(1 - \epsilon)n$ edges, i.e. has average degree at least $\epsilon d - \lambda(1 - \epsilon)$. By Lemma 2.2 this graph contains an induced subgraph H in which every vertex set of cardinality x hits at least $\frac{1}{2}(\epsilon d - \lambda(1 - \epsilon)) \cdot x$ edges. Let S be an arbitrary vertex subset of H, of cardinality $x = \beta n \leqslant \alpha n$, where $\alpha < \frac{1}{9}\epsilon$ will be chosen later. We next show that for a properly chosen d:

$$|N_H(S) - S| > 2|S|. \tag{2.3}$$

Indeed, otherwise, if $T = N_H(S) - S$, then $|S \cup T| \leqslant 3x$ and there are in H (and hence in G) at least $\frac{1}{2}(\epsilon d - \lambda(1 - \epsilon)\beta) \cdot n$ edges joining vertices of $S \cup T$. However, by Lemma 2.3

$$e(S \cup T) \leqslant \tfrac{1}{2} d9\beta^2 n + \tfrac{1}{2}\lambda \cdot 3\beta(1 - 3\beta) \cdot n$$

and therefore the inequality

$$\epsilon d - \lambda(1 - \epsilon) \leqslant 9\beta d + 3\lambda(1 - 3\beta)$$

must hold. Since $\lambda \leqslant 2\sqrt{d - 1}$ this implies

$$d \leqslant \frac{2\sqrt{d - 1}(4 - \epsilon - 9\beta)}{(\epsilon - 9\beta)}. \tag{2.4}$$

Hence if we choose d such that

$$d > 4 \cdot \left(\frac{4 - \epsilon - 9\alpha}{\epsilon - 9\alpha} \right)^2 \tag{2.5}$$

then (2.4) is violated for all $\beta \leqslant \alpha$ and hence (2.3) holds. By Lemma 2.1 we conclude that H contains a path of length $3\lfloor \alpha n \rfloor - 2$. Therefore if we choose e.g. $\alpha = \frac{1}{18}\epsilon$ and we choose the primes p and q in the construction of G such that

$$d = p + 1 > 4\left(\frac{8}{\epsilon} \right)^2 \quad \text{and} \quad n = \frac{q(q^2 - 1)}{2} \geqslant \frac{6}{\epsilon}(m + 5) \tag{2.6}$$

we conclude that even if we delete all but $\epsilon \cdot n$ vertices of G, the remaining part still contains a path of length m. This completes the proof for the case of deleting vertices. The case of deleting edges is somewhat simpler. Indeed, if we delete all but an ϵ-portion of the edges we are left with a graph of average degree $\epsilon \cdot d$. This graph contains, by Lemma 2.2, an induced subgraph H in which any set of x vertices hits at least $\epsilon d \frac{1}{2} x$ edges. Hence if S is any vertex subset of cardinality $\beta n \leqslant \alpha n$ of H, where $\alpha < \frac{1}{9}\epsilon$, and if d satisfies $d > 4\left(\frac{3 - 9\alpha}{\epsilon - 9\alpha} \right)^2$ one can check, as before, that $|N_H(S) - S| > 2|S|$. Thus, by Lemma 2.1, H contains a path of length $3\lfloor \alpha n \rfloor - 2$. It is easy to check that the previous choice of p, q given in (2.6) suffices to guarantee a path of length m in this case, as well. (In fact, here a slightly smaller d is enough.) By the standard results about the distribution of primes (see e.g. [8]), there is a choice for p and q for which (2.6), as well as the estimates $n = O(m/\epsilon)$ and $d = O(1/\epsilon^2)$ hold. This completes the proof. $\quad\square$

3. Related problems

3.1. Size Ramsey numbers

The size Ramsey number $\lambda_s(G)$ of a graph G is the least number of edges in a graph H with the property that any two coloring of the edges of H contains a monochromatic copy of G. Size Ramsey numbers were first considered in [10], and several results on them can be found also in [6,12,13]. Beck's result [4], mentioned in Section 1, resolves the problem raised by Erdös of estimating the size Ramsey numbers for paths. Beck's construction, however, is probabilistic.

Typically, explicit constructions are much more difficult to find than random ones for Ramsey type problems (see [13]). Our construction supplies an explicit example showing that the size Ramsey number for paths is linear.

3.2. Fault tolerant graphs for bounded degree trees

A natural extension of Theorem 1.1. is obtained by replacing the requirement for paths of length m by a requirement for all trees of maximum degree k and size m. Beck [4] proved, without an explicit construction, that there exists a graph G with $O(k \cdot m \cdot (\log m)^{12})$ edges, such that any set of half of its edges contains every tree of size m and maximum degree k. Very recently, Friedman and Pippenger [11] gave, for every $\epsilon > 0$ and $m, k \geqslant 2$, an explicit construction of a graph G with $O(mk^2/\epsilon)$ vertices and maximum degree $O(k^2/\epsilon^2)$ such that any set of an ϵ-portion of its edges contains every tree of size m and maximum degree k. Their construction is based on an interesting generalization, of Pósa's theorem (Lemma 2.1) from paths to trees. Some other related results can be found in [5,7].

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