Incentives in Online Auctions via Linear Programming

Niv Buchbinder[1], Kamal Jain[2], and Mohit Singh[3]

[1] Open University, Israel niv.buchbinder@gmail.com ⋆
[2] Microsoft Research, Redmond, WA, USA. kamalj@microsoft.com
[3] McGill University, Montreal. mohitsinghr@gmail.com ⋆⋆

Abstract. Online auctions in which items are sold in an online fashion with little knowledge about future bids are common in the internet environment. We study here a problem in which an auctioneer would like to sell a single item, say a car. A bidder may make a bid for the item at any time but expects an immediate irrevocable decision. The goal of the auctioneer is to maximize her revenue in this uncertain environment. Under some reasonable assumptions, it has been observed that the online auction problem has strong connections to the classical secretary problem in which an employer would like to choose the best candidate among n competing candidates [HKP04]. However, a direct application of the algorithms for the secretary problem to online auctions leads to undesirable consequences since these algorithms do not give a fair chance to every candidate and candidates arriving early in the process have an incentive to delay their arrival.

In this work we study the issue of incentives the online auction problem where bidders are allowed to change their arrival time if it benefits them. We derive incentive compatible mechanisms where the best strategy for each bidder is to first truthfully arrive at their assigned time and then truthfully reveal their valuation. Using the linear programming technique introduced in Buchbinder et al [BJS10], we first develop incentive compatible mechanisms for a variant of the secretary problem. We then show that the new mechanisms for the secretary problem can be used as a building block for a family of incentive compatible mechanisms for the online auction problem which perform well under different performance criteria. In particular, we design a mechanism for the online auction problem which is incentive compatible and is $3/16 \approx 0.187$-competitive for revenue, and a (different) mechanism that is $\frac{1}{2\sqrt{e}} \approx 0.303$-competitive for efficiency.

1 Introduction

Online auctions in which items are sold in an online fashion with little knowledge about future bids are common in the modern environment. Consider a problem in

⋆ This work was done while at Microsoft Research, New England.
⋆⋆ Part of the work done while at Microsoft Research, New England.

which a seller would like to put his car, a Honda civic in an excellent condition, on an auction[4]. As a first step he publishes an advertisement for the car, and defines a time frame for the sale. Assume that at future time t a potential buyer reads the advertisement, and would like to participate in the auction. The potential buyer has her value v_i for the car. However, her knowledge about the values of other potential buyers is very limited. Therefore, a reasonable assumption for her is that other buyers evaluating the car similarly to her. In particular, she believes that in a random subset of k potential buyers her value is the highest with probability $1/k$. Based on her beliefs she may now choose to arrive at any time $t' \geq t$ and then report some value v_i', possibly different than v_i if it benefits her.

Consider next the seller side of the story. The seller's knowledge about values of the potential buyers is also very limited. In particular, different people may value his Honda civic very differently. A natural model that captures such limited knowledge is an adversarial setting in which the set of values buyers have for the car are chosen arbitrarily, but that the arrival times of the buyers is a random permutation. The seller would like to design a mechanism which is incentive compatible and achieves good performance. In this work, we devise mechanisms for such an auction scenario where for any bidder, bidding truthfully and arriving at their assigned time maximizes its expected profit. Moreover, these mechanisms perform well under the criteria of both efficiency and revenue as compared to the offline VCG mechanism that sells the item to the highest bidder but charges a price of the second highest bidder [Vic61].

1.1 Auction Model

We model the online auction problem as the following mechanism design question. An auctioneer would like to sell a single item to a collection of n bidders $C = \{1, 2, \ldots, n\}$. Each bidder i has an arrival time $a_i \in [0, T]$ and a valuation v_i both of which are private information. The information given to the mechanism is only the number of bidders and the time horizon. The bidder may arrive at any time $t_i \geq a_i$. When the bidder arrives, it bids b_i for the item which may be distinct from her valuation v_i. The mechanism must then make a decision of whether to allocate the good to the bidder and at what price. All allocation decisions are irrevocable. We assume that the utility function for bidder i is the quasilinear function $v_i - p_i$ where p_i is the price faced by bidder i.

Now, we explain how the valuation and the arrival times are selected. First, an adversary chooses a set of arrival times $\{a_1, a_2, \ldots, a_n\}$ and assigns them adversarially. Then it chooses a set of values $\{v_1, v_2, \ldots, v_n\}$ and the values are matched with the arrival times using a random permutation. From the above model, each bidder makes the following reasonable assumption.

Assumption 11 *Each bidder believes that if all the bidders are sorted by their valuations then each permutation of bidders is equally likely.*

[4] The first author of this paper owns a Honda civic 2004 that he would like to sell shortly. The rest of the details may be fictional.

Unconditionally, if all the bidders are sorted by their valuations then each permutation of bidders is equally likely. What the above assumption states is that any bidder, **conditioned on her information**, still believes the above claim. Informally, this means that each bidder believes her valuation (or any other bidder's valuation) is equally likely to be the jth largest valuation for any j. Observe that the assumption is inherently ordinal and we contrast it with typical assumptions in such scenarios where it assumed that valuations are drawn independently from a fixed distribution.

We evaluate any mechanism by two criteria, *efficiency* and *revenue*. We define the outcome of a mechanism to be efficient if it allocates the good to the highest bidder and efficiency of a mechanism to be the probability with which the outcome is efficient. The revenue of a mechanism is defined to be the expected price charged by the mechanism. In the spirit of online algorithms, we compare its performance to the offline VCG mechanism that sells the item to the highest bidder but charges a price of the second highest bidder [Vic61]. We are interested in designing Bayesian incentive compatible mechanisms which ensure truthfulness with respect to both the arrival time as well as the bid. In particular, we design a mechanism where for any bidder arriving truthfully on their assigned time maximizes the expected profit given their beliefs and that other bidders are also truthful. Moreover, we also show that reporting the true valuation for the item is a dominant strategy for the bidders.

We note that ensuring truthfulness with respect to valuation is a well understood phenomenon in an offline setting [Vic61] and generalizes easily to our online model as well. The main contribution of our paper is to design an incentive compatible mechanism where arriving at their assigned time is a dominant strategy.

1.2 Results

We design a family of incentive compatible mechanisms where each mechanism in the family gives a different efficiency and revenue. Specifically, we prove the following main theorem.

Theorem 1. *For any* $0 \leq \tau \leq 1$*, there exists a incentive compatible online auction mechanism that is:*

- $\frac{\tau}{4} + \frac{\tau}{2} \ln \frac{1}{\tau}$*-competitive for efficiency.*
- $\frac{\tau}{2} - \frac{\tau^2}{3}$*-competitive for revenue.*

In particular, there exists a mechanism that is $3/16 \approx 0.187$*-competitive for revenue, and a (different) mechanism that is* $\frac{1}{2\sqrt{e}} \approx 0.303$*-competitive for efficiency.*

Our results are illustrated in Figure 1. The dashed lines mark the interesting values of τ which define a set of Pareto optimal mechanisms with respect to efficiency and revenue.

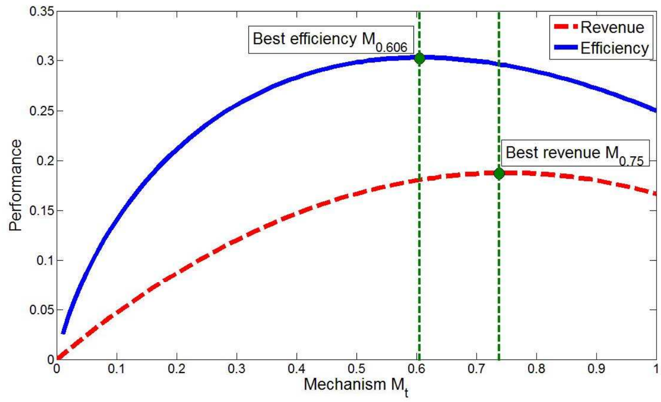

Fig. 1. The performance of the online auction mechanism as a function of τ.

Techniques and connections with secretary problems Our results are closely related to better understanding of variants of the secretary problem. In the classical secretary problem an employer would like to choose the best candidate among n competing candidates. The candidates are assumed to arrive in a random order. The secretary problem as well as many variants of it have been studied extensively in the past (See Section 1.3 for more details). Our auction mechanism is based on designing an underlying mechanisms for a variant of the secretary problem where we want to ensure that the probability that the mechanism selects the ith candidate is at least the probability of selecting the $i+1$th candidate for each i, where probability is taken over all permutations. This property in a secretary mechanism captures the inherent combinatorial structure of the auction problem where any bidder would not delay her arrival since the probability of acceptance decreases over time. We also modify our performance goals in secretary problem to mimic the goals of the auction problem. The goal of efficiency of a mechanism in the auction setting corresponds to maximizing the probability of accepting the best candidate. The other goal of maximizing revenue corresponds to maximizing the probability of hiring the best candidate while having the second best candidate appear before the best candidate. For formal definitions of the secretary model see Section 2. To obtain such mechanisms for the secretary problem, we use a recently introduced linear programming technique by Buchbinder et al [BJS10]. We remark that a different incentive compatible mechanism was designed in [BJS10]. The definition of incentive compatible in [BJS10] is slightly stronger and the requirement is that the probability of hiring in each position is the same (compared with non-increasing as here). Also, the objective there only corresponds to efficiency, while our objective function is either efficiency of revenue (or some combination). This weaker assumption allows us to design better (in terms of performance) mechanisms for the settings studied in this paper.

Finally, we believe that our novel truthfulness assumption that each bidder believes "her valuation is as good as anyone else" is very reasonable in many scenarios of lack of information, and may be useful in designing mechanisms for various other settings.

1.3 Previous Results

Recently, there has been significant work on using generalizations of secretary problems as a framework for online auctions [HKP04,Kle05,BIKK07,BIK07,BIKK08]. Incentives issues in online mechanisms have been studied in several models [LN00,HKP04,AAM03]. These works designed mechanisms where incentive issues were considered for both value and time strategies. The closest to our model is a model studied in Hajiaghayi et al[HKP04]. They studied a similar model in which an item is sold online. Bidders in their model have arrival and departure time, and the item must be allocated to a bidder by their reported departure time. The main difference of their model from our model is that they make the assumption that bidders do not receive any utility from the item if they get the item outside their arrival/departure interval. This makes the design easier since bidders who arrive early have no incentive to delay their arrival later than their departure time since they will get no utility. We believe that our model captures a significantly broader class of online auctions.

The secretary problem is a well-studied problem introduced by Gardner [Gar60]. We refer the reader to the survey by Ferguson [Fer89] on the history of the problem. For our results on the secretary problem, we use the linear programming technique introduced by Buchbinder et al [BJS10] who apply the technique to the secretary problem and some of its generalizations.

2 Secretary Problem and Linear Programming

In this section, we give new mechanisms for variants of the secretary problem which form the basis for the mechanisms for the online auction problem. In the secretary problem we have a set of candidates $C = \{1, 2, \ldots, n\}$ that arrive in a random order. There is total order $\mathcal{R}$ over the set of candidates which specifies the quality of the candidates with respect to each other. The rank of the candidate is the position of the candidate in the total order $\mathcal{R}$. After interviewing a candidate, the mechanism designer learns her rank in relation to the candidates that have already been interviewed. The mechanism designer then has to take an irrevocable decision whether to hire the interviewed candidate. We study two objectives which the mechanism designer needs to maximize. The first, which we call *efficiency*, is the probability of hiring the best candidate. This goal closely relates to efficiency in the online auction scenario. The second objective, which we call *revenue*, is the probability of the event of hiring the best candidate while having the second best candidate appear before the best candidate. This objective is closely related to the revenue in the auction model. Since we want to map mechanisms for the secretary problem to incentive compatible mechanisms

$$
\begin{array}{lll}
(P) \text{ (Efficiency) max} & \frac{1}{n}\cdot\sum_{i=1}^{n} f_i & \\
\quad\text{(Revenue) max} & \frac{1}{n(n-1)}\cdot\sum_{i=1}^{n}(i-1)\cdot f_i & \\
\text{s.t.} & & \\
\forall\, 1\le i\le n & f_i \le i\cdot p_i & \\
\forall\, 1\le i\le n & f_i \le 1-\sum_{j=1}^{i-1} p_i & \\
\forall\, 1\le i\le n-1 & p_i \ge p_{i+1} & \\
\forall\, 1\le i\le n & f_i\ge 0, p_i\ge 0 &
\end{array}
$$

$$
\begin{array}{lll}
(D) & \min\ \sum_{i=1}^{n} x_i & \\
\text{s.t.} & & \\
\forall\, 2\le i\le n-1 & \sum_{j=i+1}^{n} x_j - z_i + z_{i-1} \ge i y_i & \\
 & \sum_{j=2}^{n} x_j - z_1 \ge y_1 & \\
 & z_{n-1}\ge n y_n & \\
\text{(Efficiency)} & \forall\, 1\le i\le n & x_i+y_i\ge \frac{1}{n} \\
\text{(Revenue)} & \forall\, 1\le i\le n & x_i+y_i\ge \frac{i-1}{n(n-1)} \\
\forall\, 1\le i\le n & x_i\ge 0, y_i\ge 0, z_i\ge 0 &
\end{array}
$$

Fig. 2. (P) is an LP for Maximizing efficiency/revenue with $p_i \ge p_{i+1}$. (D) is the corresponding dual LP of (P)

for the online auction problem, we want the following property to be satisfied by the secretary mechanisms. For any position $1 \le i \le n-1$, probability that a candidate is selected at position i is more than the probability a candidate is selected at position $i+1$. The above property will be crucial in establishing incentive compatibility of mechanisms for the online auction problem and therefore, we call an interview mechanism *incentive compatible* if it satisfies the above mentioned property.

In this section, we give incentive compatible mechanisms for the secretary problem and prove the following Theorem 2.

Theorem 2. *There is a mechanism $\mathcal{M}_\tau$ for each $0 \le \tau \le 1$ which is incentive compatible. The mechanism picks the best candidate with probability $\frac{\tau}{4}+\frac{\tau}{2}\ln(1/\tau)$ (efficiency) and picks the best candidate and the second best candidate appeared before the first with probability $\frac{\tau}{2}-\frac{\tau^2}{3}$ (revenue). In particular, there exists a mechanism that is $3/16 = 0.1875$-competitive for revenue, and a (different) mechanism that is 0.303-competitive for efficiency and these are optimal.*

The proof the theorem follows from the mapping feasible mechanisms for the secretary problem to feasible solutions to a linear program and then optimizing the desired objective function of efficiency or revenue. This follows the technique introduced by Buchbinder et al[BJS10]. We state the following two lemmas which will prove Theorem 2.

Lemma 1. ***(Mechanism to LP solution)*** *Let π be any incentive compatible mechanism for the secretary problem. Let p_i^π denote the probability of selecting the candidate at position i and f_i^π denote the probability of selecting the candidate* *at position i given that the best candidate is at position i. Then (p^π, f^π) is a feasible solution to the linear program (P). Moreover the efficiency of π is at least $\frac{1}{n} \cdot \sum_{i=1}^n f_i^\pi$ and the revenue is at least $\frac{1}{n(n-1)} \cdot \sum_{i=1}^n (i-1) \cdot f_i^\pi$.*

Proof. We first show that the solution (p^π, f^π) is a feasible solution to the linear program (P). The first two set of constraints are satisfied follows from Lemma 3.1 from Buchbinder et al [BJS10]. The last set of constraints is satisfied since π is incentive compatible for delay only strategies. Thus, probability that π of accepting a candidate at position i must be decreasing function of i.

Lemma 1 shows that the optimal solution to (P) is an upper-bound on the performance of the mechanism. The following lemma shows that every LP solution actually corresponds to a mechanism which performs as well as the objective value of the solution.

Lemma 2. ***(LP solution to Mechanism)*** *Let (p_i, f_i) for $1 \leq i \leq n$ be any feasible LP solution to (P). Then there is a mechanism π with efficiency $\frac{1}{n} \sum_{i=1}^n f_i$ and revenue $\frac{1}{n(n-1)} \cdot \sum_{i=1}^n (i-1) \cdot f_i$.*

Proof. Consider the mechanism π defined as follows. Let $r_i = \lfloor \frac{ip_i}{1-\sum_{j=1}^{i-1} p_j} \rfloor$. Then the mechanism selects the candidate at position i with probability 1 if the rank of i^{th} candidate among the candidates $1, \ldots, i$ is less than or equal to r_i. If the rank of the candidate i is $r_i + 1$ then it selects the candidate with probability $\frac{ip_i}{1-\sum_{j=1}^{i-1} p_j} - r_i$. A simple calculation shows that the probability the mechanism accepts the i^{th} candidate is exactly p_i and probability of selecting the best candidate given that it is best over all, f_i^π, is at least f_i. Hence, the efficiency of the mechanism is at least $\frac{1}{n} \sum_{i=1}^n f_i$. Moreover, the mechanism is incentive compatible for delay only strategies since $p_i \geq p_{i+1}$ for each i.

Let f_{ij}^π denote the probability that the mechanism accepts the candidate at position i given that it is the best and the j^{th} candidate is the second best. Then we have the following claim.

Claim. For each $i > 1$: $f_i^\pi = \frac{1}{i-1} \sum_{j=1}^{i-1} f_{i,j}^\pi$.

Proof. First, by the definition of f_i^π and $f_{i,j}^\pi$,

$$f_i^\pi = \frac{1}{n-1} \left[\sum_{j=1}^{i-1} f_{i,j}^\pi + \sum_{j=i+1}^{n} f_{i,j}^\pi \right]$$

We claim that for each $j > i$, $f_{i,j}^\pi = f_i^\pi$. The reason is that given that when $j > i$ is second best the probability the algorithm accepts i only depends on the first i numbers and **doesn't** use their values at all. The argument follows since the first $i-1$ numbers forms a random permutation. Thus we get:

$$f_i^\pi = \frac{1}{n-1} \sum_{j=1}^{i-1} f_{i,j}^\pi + \frac{n-i}{n-1} \cdot f_i^\pi$$

and so we get our claim.

Using this claim it is easy to derive the lemma since the total revenue of the mechanism is:

$$\frac{1}{n(n-1)}\sum_{i=2}^{n}\sum_{j=1}^{i-1} f^{\pi}_{i,j} = \frac{1}{n(n-1)}\sum_{i=1}^{n}(i-1)\cdot f^{\pi}_{i} \geq \frac{1}{n(n-1)}\sum_{i=1}^{n}(i-1)\cdot f_{i}$$

since $f^{\pi}_i \geq f_i$ for each i.

Thus solving the primal program we can derive a family of mechanisms for the problem. The mechanisms are parameterized by a real number $0 \leq \tau \leq 1$ and are as follows.

Incentive Compatible Mechanism $\mathcal{M}_\tau$:

- Let $0 \leq \tau \leq 1$. For each $1 \leq i \leq n$, while no candidate is selected, do
  - If $1 \leq i \leq \tau n$, select the i^{th} candidate with probability $\frac{i}{2\tau n - i + 1}$ if she is the best candidate so far.
  - If $\tau n < i \leq n$, select the i^{th} candidate if she is the best candidate so far.

The following claim shows that each of the mechanisms $\mathcal{M}_\tau$ is incentive compatible.

Lemma 3. *For each $1 \leq i \leq n-1$, we have $p_i \geq p_{i+1}$.*

Proof. A simple calculation shows that $p_i = \frac{1}{2\tau n}$ for each $1 \leq i \leq \tau n$ and $p_i = \frac{\tau n}{2i(i-1)}$ for each $\tau n < i \leq n$ and hence the claim holds.

By selecting τ, we obtain mechanisms with different values of efficiency and revenue. A simple calculation then yields the following lemma about the performance of the mechanisms.

Lemma 4. *The mechanism $\mathcal{M}_\tau$ for any $0 \leq \tau \leq 1$,*

- ***(Efficiency)*** *Picks the best candidate with probability $\frac{\tau}{4} + \frac{\tau}{2}\ln(1/\tau)$.*
- ***(Revenue)*** *Picks the best candidate when the second best candidate appeared before the first with probability $\frac{\tau}{2} - \frac{\tau^2}{3}$.*

Optimizing for τ, the best efficiency of $\frac{1}{2\sqrt{e}}$ is obtained when $\tau = \frac{1}{\sqrt{e}}$ while the best revenue of $\frac{3}{16}$ is obtained when $\tau = \frac{3}{4}$. Moreover, all values of τ are in the range $[\frac{1}{\sqrt{e}}, 3/4] = [0.606, 0.75]$ results in a mechanism with efficiency and revenue that is Pareto optimal.

We also show that the mechanism for efficiency and revenue are optimal by giving dual solutions to the dual linear program (D) of the corresponding value.

Lemma 5. *Let π be any mechanism which is incentive compatible. Then the efficiency of π cannot be better than $\frac{1}{2\sqrt{e}}$ and the revenue of π cannot be better than $3/16$.*

Proof. We give two dual solutions to the linear program (D) in figure 2 where the corresponding constraint for the efficiency and revenue are present. Observe that each dual solution is an upper bound on performance of any mechanism.

Efficiency Let $\tau = \frac{1}{\sqrt{e}}$. Let $x_i = 0$ and $y_i = \frac{1}{n}$ and $z_i = i\tau \sum_{j=(\tau n+1)}^{n} \frac{1}{j} - \frac{i(i+1)}{2n}$ for $1 \leq i \leq \tau n$ and $x_i = \frac{1}{n}(1 - \sum_{j=i}^{n-1} \frac{1}{j})$ and $y_i = \frac{1}{n} \sum_{j=i}^{n-1} \frac{1}{j}$, $z_i = 0$ for $\tau n < i \leq n$. We now show that the above dual solution is feasible and has an objective value of $\approx \frac{1}{2\sqrt{e}}$. A simple calculation shows that $x_i, y_i, z_i \geq 0$ for each $1 \leq i \leq n$. We now calculate the objective value before verifying all the constraints.

$$s = \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{i=\tau n+1}^{n} (1 - \sum_{j=i}^{n-1} \frac{1}{j})$$

$$= \frac{1}{n}(n - \tau n - \sum_{j=\tau n+1}^{n-1} \sum_{i=\tau n+1}^{j} \frac{1}{j}) = \frac{1}{n}(n - \tau n - \sum_{j=\tau n+1}^{n-1} (1 - \frac{\tau n}{j}))$$

$$= \frac{1}{n}(1 + \tau n \ln \frac{n}{\tau n}) \approx \tau \ln \frac{1}{\tau} = \frac{1}{2\sqrt{e}}$$

Observe that the constraint $x_i + y_i \geq \frac{1}{n}$ is satisfied at equality for each $1 \leq i \leq n$.

We now verify the constraint $\sum_{j=i+1}^{n} x_j - z_i + z_{i-1} \geq iy_i$. For $1 \leq i \leq \tau n$, observe that $z_i = is + \frac{i(i+1)}{2n}$. Thus we have

$$\sum_{j=i+1}^{n} x_j - z_i + z_{i-1} = s - z_i + z_{i-1} =$$

$$= s - s + \frac{i(i+1) - (i-1)i}{2n} = \frac{i}{n} = iy_i$$

as required. For $\tau n + 2 \leq i \leq n - 1$, we have

$$\sum_{k=i+1}^{n} x_k - z_i + z_{i-1} = \frac{1}{n} \sum_{k=i+1}^{n} (1 - \sum_{j=k}^{n-1} \frac{1}{j}) = \frac{i}{n} \sum_{j=i}^{n-1} \frac{1}{j} = iy_i$$

The constraints for boundary cases $i = \lceil t \rceil$ and $i = n$ can be verified similarly.

Revenue Due to lack of space we defer the proof to the full version of the paper.

3 The Online Auction Mechanism

Given the family of mechanisms in Section 2, we design mechanisms for the online auction problem which prove Theorem 1. The family of mechanisms, parameterized by parameter τ is given below. The mechanism A_τ selects two random permutations π_1 and π_2 on bidders. The permutation π_1 is used to break ties among bidders who arrive at the same time and the permutation π_2 is used to break ties among bidders who have the same valuation.

Auction mechanism A_τ: Let B_t be the set of agents arriving at time t.

- Order the bidders in B_t by permutation π_1. Use the valuation to define the ranks of the bidders while breaking ties according to permutation π_2.
- Feed the bidders one-by-one according to their order in B_t along with their rank to the mechanism M_τ in Section 2.
- If the mechanism decides to accept the bidder then allocate the item to that bidder.
- Set the price p for the bidder to be the highest value of any bid that arrived prior to this bidder.

Observe that the mechanism indeed satisfies the online requirement of allocating the item and setting a price for it immediately at the arrival time of the bidder. We now prove that for every τ, the mechanism A_τ given above is incentive compatible.

Lemma 6. *For any $0 \leq \tau \leq 1$, the mechanism A_τ is incentive compatible for both valuation and time arrival.*

Proof. First, we prove that the online mechanism is incentive compatible for valuation. This follows simply since the price a bidder has to pay, in case she wins the item, is the maximum price seen so far by the mechanism and is independent of her bid. Moreover, the mechanism gives the item only to the person with the highest valuation so far, therefore the mechanism is incentive compatible for valuation.

We now show that the mechanism is incentive compatible for time strategies. For simplicity, we assume that no two bidders arrive at the same time. We prove that for any bidder, conditioned on her beliefs, the expected utility of the bidder is a decreasing function of the position. Thus, the bidder has no incentive to delay her arrival time.

Consider a bidder with valuation v. Let S be a random variable of the values of the $n-1$ bidders (except the bidder we currently consider) arranged according to their arrival time. For each i, let S_i be the first i values in S, and let $v(S_i)$ be the maximal value in S_i. Let X_i be the indicator random variable that the bidder that arrived at the ith position is assigned the item. First observe that the mechanism allocates the item only to the highest bid seen so far and thus $X_i = 0$ if v is not the highest until the ith position. Therefore, the expected profit of the bidder had she arrived just before the ith arrival time is $E[(v - v(S_{i-1})) \cdot X_i]$. We next prove that the expected profit for any bidder conditioned on her beliefs is a decreasing function of the position at which the bidder appears. Since, all expectations are evaluated conditioned on the bidder's beliefs, we omit this conditioning from the notation. Formally, for each $1 \leq i \leq n-1$, we prove that

$$E[(v - v(S_{i-1})) \cdot X_i] \geq E[(v - v(S_i)) \cdot X_{i+1}] \quad (1)$$

Observe that we have the following.

$$E[(v - v(S_{i-1})) \cdot X_i] = E[(v - v(S_{i-1})) \cdot X_i | v > v(S_{i-1})] \cdot Pr[v > v(S_{i-1})]$$
$$= E[(v - v(S_{i-1})) | v > v(S_{i-1})] \cdot Pr[X_i = 1 | v > v(S_{i-1})] \cdot Pr[v > v(S_{i-1})]$$

The second equality follows by the fact that given the event that $v > v(S_{i-1})$, i.e. the bidder has the highest valuation so far, the probability of allocating the item to bidder i is independent of $v - v(S_{i-1})$. This follows since the underlying mechanism $\mathcal{M}_\tau$, and thus A_τ, does not look at the actual values but only the relative ordering when deciding whether to give the item or not to a bidder.

Conditioned on the beliefs of the bidder, we have $Pr[v > v(S_{i-1})] = 1/i$ and that the set of valuations in S_{i-1} when ordered by position form a random permutation. Thus,

$$Pr[X_i = 1 | v > v(S_{i-1})] \cdot Pr[v > v(S_{i-1})] = p_i$$

where p_i is the probability of accepting the i^{th} candidate by $\mathcal{M}_\tau$. But for mechanism $\mathcal{M}_\tau$, p_i is a decreasing function of i. Thus, it suffices to show that $E[v(S_{i-1}) | v \geq v(S_{i-1})]$ is a non-decreasing function of i. Before we prove this, we prove the following technical claim which is crucial in comparing the expected profit if the bidder arrives in position i or $i+1$. Here v_i is the random valuation of the i^{th} bidder by arrival order excluding the bidder with valuation v.

Claim. For each i, we have

$$E[v(S_{i-1}) | v > v(S_{i-1}) \ \& \ v < v_i] \leq E[v(S_{i-1}) | v > v(S_i)]$$

Proof. Let $v_2(S_i)$ be a random variable for the second maximal value in S_i.

$$\begin{aligned}
&E[v(S_{i-1}) | v > v(S_{i-1}) \ \& \ v < v_i] \\
&= E[v(S_{i-1}) | v_i > \max\{v(S_{i-1}), v\} \ \& \ v > v(S_{i-1})] \\
&= E[v(S_{i-1}) | v > \max\{S_{i-1}, v_i\} \ \& \ v_i > v(S_{i-1})] && (2) \\
&= E[v_2(S_i) | v > v(S_i) \ \& \ v_i > v(S_{i-1})] = E[v_2(S_i) | v > v(S_i)] && (3) \\
&\leq E[v(S_{i-1}) | v > v(S_i)] && (4)
\end{aligned}$$

Where equality (2) follows by the symmetry arguments on v and v_i. This is done by pairing each permutation in which $v_i > v$ to a permutation in which $v > v_i$. Second equality in (3) follows since for any permutation on v_1 to v_i the second highest value is the same. Equality (4) follows since for any permutation on the values the second highest value among the first i values is at most the highest value among the first $i-1$ values.

Now we prove the following claim which shows that $E[v(S_{i-1}) | v > v(S_{i-1})]$ is a non-decreasing function of i. This will complete the proof. Observe that

$$\begin{aligned}
&E[v(S_{i-1}) | v > v(S_{i-1})] \\
&= E[v(S_{i-1}) | v > v(S_{i-1}), v > v_i] \cdot Pr[v > v_i | v > v(S_{i-1})] \\
&+ E[v(S_{i-1}) | v > v(S_{i-1}), v < v_i] \cdot Pr[v < v_i | v > v(S_{i-1})] \\
&\leq E[v(S_{i-1}) | v > v(S_{i-1}), v > v_i] \cdot Pr[v > v_i | v > v(S_{i-1})] \\
&+ E[v(S_{i-1}) | v > v(S_{i-1}), \mathbf{v} > \mathbf{v_i}] \cdot Pr[v < v_i | v > v(S_{i-1})] && (5) \\
&= E[v(S_{i-1}) | v \geq v(S_i)] \leq E[v(S_i) | v \geq v(S_i)] && (6)
\end{aligned}$$

Inequality (5) follows by Claim 3. Inequality (6) follows since in every term we maximize over more elements.

Now, we prove the main theorem which follows directly from Lemma 6.

Proof. of Theorem 1 Given that the mechanism is incentive compatible for time strategies (Lemma 6), we get that the dominant strategy of the bidders is not to delay their arrival time. Thus, the rank given to the underlying mechanism is a random permutation of the bidders. Thus, the performance of the mechanism follows directly by Lemma 4.

References

[AAM03] Baruch Awerbuch, Yossi Azar, and Adam Meyerson. Reducing Truth-Telling Online Mechanisms to Online Optimization. In *In Proc. ACM Symposium on Theory of Computing*, pages 503–510, 2003.

[BIK07] M. Babaioff, N. Immorlica, and R. Kleinberg. Matroids, Secretary Problems, and Online Mechanisms. In *Proceedings 18th ACM-SIAM Symposium on Discrete Algorithms*, 2007.

[BIKK07] M. Babaioff, N. Immorlica, D. Kempe, and R. Kleinberg. A Knapsack Secretary Problem with Applications. In *Proceedings of 10th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, 2007.

[BIKK08] M. Babaioff, N. Immorlica, D. Kempe, and R. Kleinberg. Online Auctions and Generalized Secretary Problems. *SIGecom Exchange*, 7:1–11, 2008.

[BJS10] Niv Buchbinder, Kamal Jain, and Mohit Singh. Secretary problems via linear programming. In *14th Conference on Integer Programming and Combinatorial Optimization*, 2010.

[Fer89] T. S. Ferguson. Who Solved the Secretary Problem? *Statist. Sci.*, 4:282–289, 1989.

[Gar60] M. Gardner. Mathematical Games. *Scientific American*, pages 150–153, 1960.

[HKP04] M. T. Hajiaghayi, R. Kleinberg, and D. C. Parkes. Adaptive Limited-Supply Online Auctions. In *Proceedings of the 5th ACM Conference on Electronic Commerce*, 2004.

[Kle05] R. Kleinberg. A Multiple-Choice Secretary Algorithm with Applications to Online Auctions. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete algorithms*, 2005.

[LN00] Ron Lavi and Noam Nisan. Competitive Analysis of Incentive Compatible On-line Auctions. In *In Proc. 2nd ACM Conf. on Electronic Commerce*, pages 233–241, 2000.

[Vic61] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.