Critical state stability in type-II superconductors and superconducting–normal-metal composites

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This review is devoted to the problem of critical state stability in hard superconductors and superconducting normal composites. An introduction is given to the properties of hard and composite superconductors, and to the qualitative nature of the physical processes that occur in these materials in the critical state. The dynamics of the development of instabilities of various kinds are treated in detail. Stability criteria are obtained and discussed, and theory is compared with experiment. The interaction between flux jumps and plastic strain jerks and the training phenomenon in superconductors are also covered.

CONTENTS

I. INTRODUCTION

Ever since Kamerlingh Onnes discovered superconductivity in 1911, continuous research has been going on aimed at producing superconducting materials featuring the maximum possible critical parameters, such as the critical temperature T_c, the superconducting current density j_c, and the upper critical magnetic field H_{c_2}. While high-temperature superconductivity research has not as yet been successful, the discovery by Kunzler *et al.* (1961a, 1961b) of superconductivity in Nb_3Sn started the ever growing list of superconducting alloys featuring high critical current densities (10^5 A cm^{-2} and higher) and upper critical fields of $H_{c_2} \gtrsim 100$ kOe. Subsequent studies have shown the high values of critical current in Nb_3Sn and in other, later discovered, alloys (Nb-Zr, Nb-Ti, V_3Ga, MoRe, etc.) to be associated with the pinning effect, i.e., the attachment of Abrikosov (1957) vortex lines to crystal lattice defects (for more detail, see Saint-James *et al.*, 1969; Campbell and Evetts, 1972). Type-II superconductors featuring a strong interaction between the vortex structure and crystal lattice are usually referred to as hard superconductors. The hard superconductors developed up to now are characterized by critical current densities of up to about 10^7 A cm^{-2} and H_{c_2} values of up to 1,000 kOe. The characteristic values of T_c are in the range from 10 K (Nb-Ti, Nb-Zr) to about 18 K (Nb_3Sn), 23.2 K (Nb_3AlGe).

The unusual physical properties of hard superconductors have aroused great interest. Studies of these properties, moreover, have been promoted by the considerable promise that they hold for practical applications (see, Brechna, 1973).

However, in spite of considerable critical current densities and high values for the upper critical field, the magnetic field difference (and, consequently, the value of transport current) in a hard superconductor sample cannot exceed a certain value that is usually much lower than H_{c_2}. This is due to instabilities, so-called magnetic flux jumps, occurring in hard superconductors.

The present review is devoted to a study of the problems associated with the emergence of such instabilities. It is arranged as follows.

Section II contains a brief summary of the physical properties of hard superconductors that are essential for further discussion. The concept of the critical state is formulated, on the basis of which a study is made of the macroscopic properties of hard superconductors, including magnetic flux jumps. Superconducting composites are described, i.e., materials containing a combination of normal and superconducting materials. Section III describes the qualitative theory of magnetic-flux jumps and the closely associated oscillation effects, and analyzes the interaction of thermomagnetic (flux jump) and thermomechanical (plastic strain jerk) instabilities. Section IV contains a mathematical formulation of the problem of superconducting state stability in hard superconductors and superconducting composites. On the assumption that the critical current density is independent of the local value of magnetic field, criteria for critical state stability are obtained and the dynamics of perturbation development are studied for some specific cases. In Sec. V, the above listed methods have been generalized for the case of superconductors with properties varying over the cross-sectional area, in particular, with regard to the dependence of critical current density upon the local value of magnetic field. Section VI examines the effect of time-dependent boundary conditions upon stability and contains a more detailed comparison of theory with experiment than the other chapters. Section VII is concerned with the study of electric field and temperature oscillations in hard superconductors, occurring near the threshold of instability. Section VIII is devoted to studying the influence of transverse thermomagnetic effects (Nernst and Ettingshausen effects) upon stability and the dynamics of perturbation development in hard superconductors. Section IX is concerned with the superconducting state stability in superconductors subjected to high mechanical stresses causing plastic yield of the material. The relationship between the thermomagnetomechanical instability, observed under such conditions and the training effect, i.e., the dependence of superconducting current density upon the number of on-off cycles of transport current is discussed. Section X contains a brief discussion of some possibilities for further studies.

II. PHYSICAL PROPERTIES OF HARD SUPERCONDUCTORS. THE CRITICAL STATE. SUPERCONDUCTING COMPOSITES

In the equilibrium state, the vortex lines in a Type-II superconductor form a lattice having a mean density of $n = B/\phi_0$, where B is the magnetic induction and $\phi_0 = \pi\hbar c/e \simeq 2\times 10^{-7}$ G cm^2 is the magnetic flux quantum (Abrikosov, 1957; DeGennes, 1966). If a transport current is passed through a Type-II superconductor, the interaction of the current with a vortex leads to the emergence of the so-called Lorentz force acting on each one of the vortices (see Campbell and Evetts, 1972):

$$\mathbf{F}_L = \frac{1}{c}(\mathbf{j}\times\boldsymbol{\phi}_0), \tag{2.1}$$

where $\mathbf{j}$ is the current density, and $\boldsymbol{\phi}_0 = \phi_0\mathbf{B}/B$. Under the effect of this force, the vortices begin to move, energy dissipation occurs, and the superconductor undergoes transition to the resistive state (Anderson, 1962; Gorter, 1962a, 1962b; Kim *et al.*, 1963a, 1963b, 1964, 1965; Huebener, 1974; Gorkov and Kopnin, 1975).

However, in the presence of structural defects in the superconductor, the vortices may be attached to such defects (pinning effect) and form a metastable configuration of the magnetic flux (Saint-James *et al.*, 1969; Campbell and Evetts, 1972). The force of the vortex-defect interaction (pinning force) is a function of temperature T inasmuch as the energy and configuration of vortex lines depend considerably on temperature. The mutual repulsion of the vortices (Abrikosov, 1957; DeGennes, 1966; Saint-James *et al.*, 1969; Campbell and Evetts, 1972) causes the dependence of F_p upon the density of vortices, i.e., upon magnetic induction B. Owing to pinning, the resistive state in a Type-II superconductor occurs only if $F_L > F_p(T,B)$, where $F_p(T,B)$ can be conveniently written in the form

$$\mathbf{F}_p = \frac{1}{c}(\mathbf{j}_c\times\boldsymbol{\phi}_0) \tag{2.2}$$

and where $j_c = j_c(T,B)$ is the critical current density. Under conditions of a strong bond between the magnetic flux (Abrikosov vortex lattice) and metal lattice, as is the case in hard superconductors, j_c can attain very high values. A series of typical dependencies of j_c upon T and B are shown in Fig. 1.

Therefore, if $j < j_c(T,B)$, persistent currents can exist in a hard superconductor. In the case $j > j_c(T,B)$, a viscous flux flow mode of vortex lines begins in the superconductor, in which

$$F_L = F_p + \eta v,$$

where ηv is the force of viscous friction, η is the viscosity, and v is the velocity of motion of the vortex structure. Equations (2.1) and (2.2) imply that

$$j = j_c + \eta\frac{vc}{\phi_0}. \tag{2.3}$$

The relationship between v and the electric field E arising from the magnetic flux motion can be easily

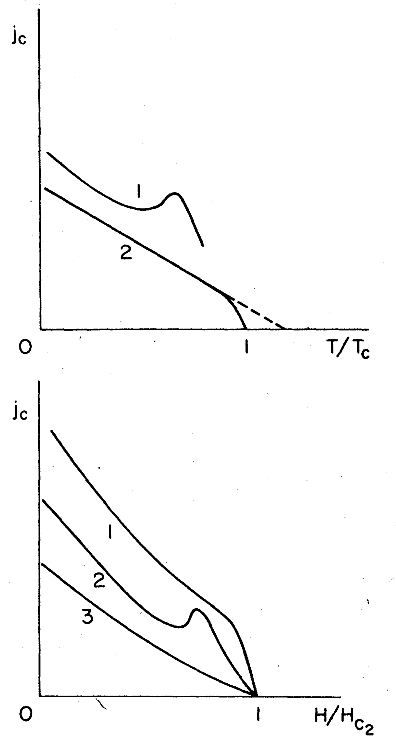

FIG. 1. Characteristic form of the dependence of j_c upon T (a) and B (b).

derived from the continuity equation for the vortex flux:

$$\partial n/\partial t = -\operatorname{div}(n\mathbf{v})$$

and the Maxwell equation

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,. \tag{2.4}$$

It follows from the latter two equations that

$$\mathbf{v} = \frac{c}{B^2}(\mathbf{E}\times\mathbf{B}).$$

Therefore,

$$j = j_c + \sigma_f E\,, \tag{2.5}$$

where $\sigma_f = \eta c^2/B\phi_0 \cong \sigma_n H_{c_2}/B$ (here, σ_n is the conductivity of the sample in the normal state). It should be stressed that Eq. (2.5) is only true if $E \neq 0$. Otherwise, j is an independent parameter.

The relationship $\sigma_f \propto B^{-1}$ is well supported experimentally and derived from the microscopic theory (Lynton, 1969; Campbell and Evetts, 1972; Huebener, 1974; Gorkov and Kopnin, 1975). For hard superconductors, $\sigma_f \sim 10^{16} H_{c_2}/B$ s^{-1}. Even in fields with $B \sim H_{c_1}$, this value is substantially smaller than the conductivity of pure metals. A typical current-voltage characteristic of a hard superconductor is shown in Fig. 2. The nonlinear portion of curve $j(E)$ at $E < E_0$ is caused by a number of factors such as inhomogeneity of the pinning centers, structural defects of the vortex lattice, thermal activation of vortices from the pinning centers, etc. The $E < E_0$ region is often referred to in the literature as the region of magnetic-flux creep. The value of E_0 apparently depends upon T and B. In hard superconductors, however, the size of the nonlinear portion of the current-voltage characteristic is usually small compared to the electric field value, so that $j_c \gg \sigma_f E_0$ and $dj/dE \gg \sigma_f$ at $E < E_0(T,B)$. In addition, it can be assumed for all actual values of the electric field that $j_c \gg \sigma(E)E$. It follows, then, that a current density close to the critical value is set up in a hard superconductor in response to any electric field. This concept of critical state was suggested (and developed in the course of further research) by Bean (1962, 1964), London (1963), and Kim *et al.* (1963a, 1963b). It has been repeatedly tested experimentally (see Campbell and Evetts, 1972; Grasmehr and Finzi, 1966; Bean *et al.*, 1966; Coffey, 1967) and describes well the phenomena occurring in hard superconductors. Note further that the relationship $j(E)$ can, for a number of purposes, be approximated by Eq. (2.5).

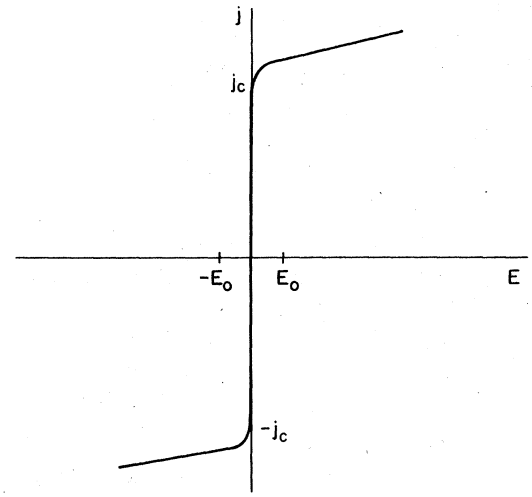

FIG. 2. Current-voltage characteristic of a hard superconductor.

The equation of the critical state $j_c = j_c(T,B)$ has been studied in a great number of experimental and theoretical papers, and it has been shown that the Kim-Anderson (1964) model provides a good approximation in numerous cases in the region of fields substantially less than H_{c_2}:

$$j_c = \frac{j_0(T)B_0(T)}{B + B_0(T)}\,,$$

where the value of $B_0(T)$ is usually on the order of several hundred Oe. In the case of fields comparable with H_{c_2}, the relationship $j_c(B)$ may have a more complex form and varies considerably depending on the type of superconducting alloy and the nature of its treatment. Observed in this region, in particular, is the so-called peak effect [see Saint-James *et al.*, 1969; Campbell and Evetts, 1972; and Fig. 1(b), curve 2]. For estimations in the present review the value of j_c will be approximated in the range of fields $1 - B/H_{c_2} \ll 1$ by the simplest dependence:

$$j_c = j_1(T)(1 - B/H_{c_2}).$$

Analogously, for estimation purposes one can assume that $j_0(T)$, $j_1(T) \propto 1 - T/T_c$.

If the changes in induction in the sample are small compared with the characteristic scale of variation of the function $j_c(B)$, the equation of the critical state can be written in the form $j_c = j_c(T, B_a)$, where B_a is the external field [Bean's model (Bean, 1962, 1964)]. Such conditions are quite often realized in the course of experiments.

In the presence of temperature gradients in the sample, temperature stresses apparently occur in the vortex lattice. In so doing, each line is affected by an additional force F_T. The expression for F_T has the form

$$\mathbf{F}_T = -S^* \nabla T, \tag{2.6}$$

where $S^* = S^*(T,B)$ is the transport entropy of a vortex line whose existence is associated with the presence of low-energy electron states localized in the core of the vortex (Caroli *et al.*, 1964; Mints and Rakhmanov, 1975). The function $S^*(T,B)$ has been studied in many papers (see Solomon and Otter, 1967; Lowell *et al.*, 1969; Maki, 1971; Kopnin, 1975). A good approximation for $S^*(T)$ is $S^* = S_0^*(T/T_c)(1 - T/T_c)$. The value of S^* can be estimated from the local density of states ν_e found by Caroli *et al.* (1964):

$$S^* \sim \nu_e K_B^2 T \sim \frac{m_e^{3/2} \varepsilon_F^{1/2} \xi^2}{\hbar^3} K_B^2 T,$$

where ε_F is the Fermi energy, K_B is Boltzmann's constant, and ξ is the coherence length. For characteristic parameter values at $T = 4$ K, $S^* \sim 10^{-7}$ erg cm^{-1} K^{-1}, which is in good agreement with experimental data.

Using Eq. (2.6), by analogy with (2.5), one can easily derive

$$\mathbf{j} = \mathbf{j}_c + \sigma_f \mathbf{E} + \frac{S}{B}(\mathbf{B} \times \nabla T), \tag{2.7}$$

where $S = cS^*/\phi_0$. Note that Eq. (2.7), like (2.5), is only true in the case $E \neq 0$.

The transfer of transport entropy S^* by a vortex line results in a corresponding contribution to the heat flux q. From Eq. (2.7) and the symmetry principle of kinetic coefficients (Landau and Lifshits, 1976), one can derive

$$\mathbf{q} = -\kappa \nabla T + \frac{ST}{B}(\mathbf{E} \times \mathbf{B}). \tag{2.8}$$

Here, κ is the heat conductivity of the superconductor (which, in the case of hard superconductors, is on the order of 10^3–10^4 erg cm^{-1} s^{-1} K^{-1}). Equation (2.8) is supported by the microscopic theory as well.

Note further that from the concept of a critical state there apparently follows the condition of "irreversibility," i.e., the current and electric field are always parallel and, consequently, $(\mathbf{j} \cdot \mathbf{E}) > 0$ at $E \neq 0$. Thus it can be readily seen that the current vector $\mathbf{j}$ is related to the vector $\mathbf{E}$ by $\mathbf{j} = j_c \mathbf{E}/|\mathbf{E}|$ if $E \neq 0$.

In many applications, hard superconductors are used in combination with normal metals (the so-called superconducting composites). Therefore, the present review will also include a discussion of such heterogeneous media.

The currently existing composites can be divided into three types, namely, (1) multifilament composites, consisting of a matrix of normal metal with a regular structure of superconducting wires embedded in it; (2) ribbon composites, consisting of layers of superconductor and normal metal in the form of wide ribbons; and (3) composites manufactured from normal and superconducting metal powders. The matrix of normal metal contains metals of good conductivity (Cu, Al, etc.) or alloys of lower conductivity (CuNi, etc.), or their combinations.

Of most interest in applications are composites consisting of a great number of normal and superconducting elements $N \gg 1$. In a number of cases, such materials can be regarded as an effective homogeneous anisotropic medium (Hart, 1969; Carr, 1974, 1975a, 1975b; Duchateau and Turk, 1975a, 1975b; Kremlev *et al.*, 1976a, 1977). The parameters of such a medium are found by averaging their local values over a region containing a rather high number of elements of the composite structure. For example, in the case of a multifilament composite, the heat capacity ν, the mean critical current density j_s, the electric conductivity $\sigma_\parallel$, and the heat conductivity $\kappa_\parallel$ longitudinal relative to the filaments, are calculated quite readily as

$$\begin{gathered} \nu = x_s \nu_s + x_n \nu_n, \quad j_s = x_s j_c, \\ \kappa_\parallel = \kappa_s x_s + \kappa_n x_n, \quad \sigma_\parallel = x_n \sigma_n + x_s \sigma_s. \end{gathered} \tag{2.9}$$

Here, x_n and x_s are relative concentrations of the normal and superconducting metals, respectively ($x_n + x_s = 1$); ν_s, κ_s, and σ_s are, respectively, heat capacity, heat conductivity, and electric conductivity in the resistive superconducting mode; ν_n, κ_n, and σ_n are the respective parameters of the normal metal.

Finding the mean values of transverse conductivities $\sigma_\perp$ and $\kappa_\perp$ presents a rather more complicated problem. However, if $\kappa_n \gg \kappa_s$ and $\sigma_n \gg \sigma_s$, while $x_n \sim x_s$, one can probably assume, for estimation purposes

$$\kappa_\perp = \kappa_n(1 - x_s^{1/2}), \quad \sigma_\perp = \sigma_n(1 - x_s^{1/2}). \tag{2.10}$$

It is worthy of note that, as a result, the electric and heat conductivities of composites for typical values of x_s turn out to be on the order of their respective values in the normal metal, i.e., two to four orders of magnitude higher than in hard superconductors.

In this review, we shall be concerned with the case of sufficiently high magnetic fields $B \gg H_{c_1}$. As is known (DeGennes, 1966), the induction B in this case can be assumed equal to the magnetic field intensity H.

III. QUALITATIVE THEORY OF FLUX JUMPS

By reference to the concept of the critical state, one can readily understand the physical nature of the flux jump. Let us consider the simplest example. Imagine a plane, semi-infinite plate in an external field parallel to its surface (Fig. 3). At the initial moment of time, the magnetic field is uniform and equal to H_0, after which the external field rises to some value H_a. With an increase of the external field, the magnetic flux penetrates the sample. The flux motion causes an electric field which generates persistent currents near the

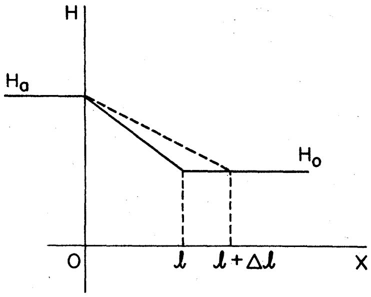

FIG. 3. Magnetic field distribution in a plane, semi-infinite sample.

superconductor surface, having, in accordance with the concept of the critical state, a density j_c. As a result, a magnetic field difference $\Delta H = H_a - H_0$ exists in the plate in the region $0 < x < l$, while $\partial H/\partial x \propto j_c$. Now, if the temperature in the plate increases as a result of some perturbation, j_c decreases. With a decrease of j_c, the magnetic flux penetrates the sample more deeply, more heat is released by the flux motion, and so on. Under certain conditions, such a perturbation may increase in an avalanche manner and bring about a collapse of the superconducting state.

Obviously, no instability may occur if $\partial j_c/\partial T > 0$ (peak effect). Livingston (1966) was the first to point this out. The validity of this assertion is reliably supported by a whole series of experiments (Livingston, 1966; Hart and Livingston, 1968; Wipf, 1968; Kroeger, 1969; Scanlan and Livingston, 1972; Onishi and Miura, 1973; Bethoux and Schumacher, 1973).

As can be readily seen, the normal current $j_N = \sigma E \propto \partial H/\partial t$ occurring with the movement of the magnetic flux compensates for the drop of $j_c(T)$ and thereby impedes the magnetic-flux movement inside the sample (an analog of viscous friction). As a result, the critical state stability increases with superconductor conductivity σ.

Therefore, a flux jump presents temperature and electromagnetic field perturbations increasing in a correlated manner. Each one of these processes is characterized by its respective diffusion coefficient, namely, the thermal diffusion coefficient $D_t = \kappa/\nu$ (ν is usually of the order of 10^4–10^5 erg cm^{-3} K^{-1}) and the magnetic diffusion coefficient $D_m = c^2/4\pi\sigma$ associated with normal currents in the resistive state. Characteristic values of D_t and D_m in hard superconductors are as follows: $D_t = 1$–10 cm^2 s^{-1}, $D_m = 10^2$–10^4 cm^2 s^{-1} ($T = 4$ K, $H \sim 10^3$–10^4 Oe, $\sigma = \sigma_f$).

Let us introduce the parameter τ:

$$\tau = D_t/D_m = \frac{4\pi\sigma\kappa}{c^2\nu}\ . \tag{3.1}$$

For hard superconductors, $\tau \ll 1$ (usually even in fields $H \sim H_{c_1}$). This means that the magnetic flux diffusion is considerably faster than that of the heat flux and that the heating of hard superconductors with a rapid variation of magnetic flux is adiabatic.

The inverse limiting case $\tau \gg 1$ can be realized in superconducting composites with the characteristic values of $D_m = 10^{-1}$–10^{-2} cm^2 s^{-1} and $D_t = 10^3$–10^4 cm^1 s^{-1}. Accordingly, fast heating of a composite occurs under conditions of frozen-in magnetic flux.

A. Hard superconductors

Let us now consider qualitatively the development of a small perturbation in a superconductor where $\tau \ll 1$. Assume that a fluctuation in some superconductor region causes the temperature to rise by the value ΔT_0. This means that a "priming" heat $Q_0 = \nu\Delta T_0$ has been applied to this spot. With such heating, the value j_c decreases and the magnetic flux starts to travel within the sample. As a result, additional heat Q_1 is released, which is equal to

$$Q_1 = \int j_c E\, dt\,.$$

Here we have taken into account that $\sigma_f E \ll j_c$ (note that hard superconductors provide an example of a system wherein heat release depends linearly upon E over a wide range of parameters). Inasmuch as the heating is adiabatic ($\tau \ll 1$), the new equilibrium temperature value can be found using the law of conservation of energy in the form

$$\nu\Delta T = Q_0 + Q_1 = \nu\Delta T_0 + Q_1\ . \tag{3.2}$$

To estimate the value of Q_1, we use the Maxwell equation

$$\nabla^2\mathbf{E} = \frac{4\pi\partial\mathbf{j}}{c^2\partial t} \tag{3.3}$$

and Bean's equation of the critical state, $j_c = j_c(T)$. Then $\partial j_c/\partial t = (dj_c/dT)\dot{T}$. The quantity $|\nabla^2 E|$ is of the order of E/b^2, where b is some characteristic dimension (for instance, in the case shown in Fig. 3, $b = l$). Then we derive from Eq. (3.3)

$$E \sim \frac{4\pi b^2}{c^2}\left|\frac{dj_c}{dT}\right|\dot{T}$$

and, accordingly,

$$Q_1 = \frac{1}{\gamma}\ \frac{4\pi b^2 j_c}{c^2}\left|\frac{dj_c}{dT}\right|\Delta T = \frac{\beta}{\gamma}\,\nu\Delta T\,,$$

where γ is a number of the order of unity which depends on the geometry of the problem, while

$$\beta = \frac{4\pi b^2 j_c}{c^2\nu}\left|\frac{dj_c}{dT}\right|\,. \tag{3.4}$$

The quantity β characterizes the spontaneous heating of superconductor caused by a small external perturbation. By substituting the expression for Q_1 into Eq. (3.2), we find that

$$T = \frac{\Delta T_0}{1 - \beta/\gamma}\ .$$

One can see from the latter relationship that, at $\beta \to \gamma$, the temperature becomes unstable and ΔT increases indefinitely for any value of the "priming" fluctuation ΔT_0. Consequently the critical state is only stable if

$$\beta < \gamma\,. \tag{3.5}$$

Note that the criterion expressed by Eq. (3.5) is often called the "adiabatic stability criterion." It was first derived for a plane, semi-infinite sample (Fig. 3) by Hancox (1965) on the basis of similar qualitative arguments. In this case, from the Maxwell equation

$$\mathrm{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \tag{3.6}$$

assuming that $j = j_c$, we derive

$$H(x) = \frac{4\pi j_c}{c}(l - x) + H_0 .$$

Hence, $l = c(H_a - H_0)/4\pi j_c$. On substituting this expression into the criterion (3.5), we find

$$\Delta H = H_a - H_0 < \left(\gamma \frac{4\pi\nu j_c}{|dj_c/dT|}\right)^{1/2}. \tag{3.7}$$

Using the criteria (3.5) and (3.7), one can easily estimate that the maximum sample thickness is on the order of 10^{-3}–10^{-2} cm, while the maximum stable magnetic field difference in the sample $\Delta H = H_j$ is on the order of 1–3 kOe. Both these estimates are in agreement with experimental data (Neuringer and Shapira, 1966; Shiiki and Kudo, 1974).

Let us analyze the dynamics of the perturbation. Assume that the perturbations of T and E of interest to us vary over a characteristic time t_j:

$$E, \Delta T \propto \exp(t/t_j) .$$

Let us write t_j in the form $t_j = t_\kappa/\lambda$ where $t_\kappa = b^2/D_t$ is the characteristic time of heat diffusion, and λ is the eigenvalue of the problem to be found. The nearly adiabatic nature of heating in hard superconductors ($\tau \ll 1$) implies that $|\lambda| \gg 1$. On the other hand, as will be shown below, $t_j \gg t_m = b^2/D_m$ or $|\lambda|\tau \ll 1$.

We now use the equation of heat conductivity:

$$\nu\dot{T} = \kappa\nabla^2 T + j_c E . \tag{3.8}$$

Following its time integration and after estimating $\nabla^2 T$ as $-\Delta T/b^2$, one can easily derive

$$\nu(\Delta T - \Delta T_0) = \int j_c E\,dt - \gamma\frac{\nu\Delta T}{\lambda} . \tag{3.9}$$

From the Maxwell equation (3.3) for E, with due regard for the normal current $j_N = \sigma_f E$, we find

$$E \sim \frac{4\pi b^2}{c^2}\left|\frac{dj_c}{dT}\right|\dot{T}(1 - \lambda\tau/\gamma),$$

whence

$$\int j_c E\,dt = (\beta - \lambda\tau)\frac{\nu\Delta T}{\gamma}$$

and, from (3.9), we obtain

$$\Delta T = \frac{\Delta T_0}{1 + \gamma/\lambda + \lambda\tau/\gamma - \beta/\gamma} .$$

One can see from this that for $\lambda = \lambda(\beta, \tau)$ satisfying the dispersion equation

$$\beta/\gamma = 1 + \gamma/\lambda + \lambda\tau/\gamma, \tag{3.10}$$

the final temperature deviation ΔT may turn out to be considerable for a small initial perturbation ΔT_0. Thus Eq. (3.10) describes the spectrum of eigenvalues of perturbation "frequencies" in the critical state for $|\lambda| \gg 1$. In so doing, $\mathrm{Im}\lambda$ coincides with the oscillation frequency and $\mathrm{Re}\lambda$ with the growth increment (damping decrement) of the eigensolutions.

The dependences of $\lambda_1 = \mathrm{Im}\lambda$ and $\lambda_2 = \mathrm{Re}\lambda$ upon β are shown in Fig. 4. The value of β_c, at which the eigenvalue of λ first appears so that $\lambda_1 = 0$ and $\lambda_2 > 0$, is found from the condition $\partial\lambda/\partial\beta = \infty$, whence

$$\beta_c = \gamma(1 + 2\tau^{1/2}), \qquad \lambda_2 = \lambda_c = \gamma/\tau^{1/2} \tag{3.11}$$

at $\beta = \beta_0 = \gamma$; it follows from (3.10) that $\lambda_2 = 0$ and $\lambda_1 = \lambda_c$. As it is seen from Eq. (3.11), the conditions $|\lambda| \sim \lambda_c \gg 1$ and $|\lambda|\tau \sim \lambda_c\tau \ll 1$ are satisfied for the fluctuations of interest as assumed.

In the derivation of Eq. (3.10), the characteristic space scale of variation of E and ΔT perturbations is independent of λ. As can be demonstrated, this is only true if the sample is thermally insulated. Accordingly, Eqs. (3.11) relate to the case of adiabatic thermal boundary conditions. In the case when the sample surface is cooled intensely (isothermal boundary conditions), it should be taken into consideration that no perturbation arises in a layer having a thickness of about $b/|\lambda|^{1/2}$. Then, by substituting $b(1 - 1/\lambda^{1/2})$ for b, we find, by analogy with (3.10),

$$\beta/\gamma = 1 + 2/\lambda^{1/2} + \lambda\tau/\gamma .$$

The eigenvalue spectrum $\lambda(\beta) = \lambda_2(\beta) + i\lambda_1(\beta)$ has an appearance similar to that shown in Fig. 4. For β_c and λ_c we derive

$$\beta_c = \gamma(1 + 3(\tau/\gamma)^{1/3}), \qquad \lambda_c = (\gamma/\tau)^{2/3} . \tag{3.12}$$

At

$$\beta = \beta_0 = \gamma\left(1 + \left(\frac{2\tau}{\gamma}\right)^{1/3}\right),$$

$\lambda = i\lambda_1$, where $\lambda_1 = 2^{1/3}\lambda_c$.

As would be expected, at $\tau \ll 1$ the heat transfer conditions have little effect upon the stability criterion. A corresponding estimate for the variation of H_j at $T = 4$ K yields no more than 5–10% difference, which is in agreement with experimental data (see Irie *et al.*, 1977). At the same time, the dynamical evolution of the perturbations varies appreciably. The correction to the adiabatic stability criterion (3.5) associated with the nonzero value of τ may be appreciable, reaching tens of percent, even if $\tau \ll 1$, due to the one-third

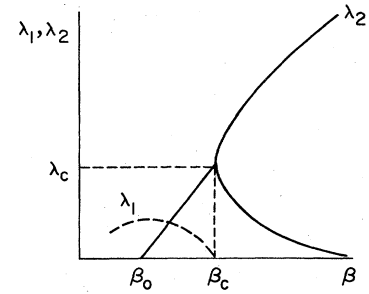

FIG. 4. Qualitative appearance of the eigenvalue spectrum of $\lambda(\beta)$.

power. Note that near-isothermal conditions are characteristic of liquid-helium-cooled hard superconductors.

B. Composite superconductors

In the case of superconducting composites, one has $\tau \gg 1$, and one can assume that $\partial j/\partial t = 0$ upon rapid heating. Then, using for simplicity Bean's equation of the critical state, we obtain

$$\partial j/\partial t = \sigma\dot{E} + \frac{dj_c}{dT}\dot{T} = 0\,,$$

whence

$$E = \sigma^{-1}\left|\frac{dj_c}{dT}\right|\Delta T\,. \tag{3.13}$$

The power released per unit volume is

$$\dot{Q} = j_c E = \frac{j_c \Delta T}{\sigma}\left|\frac{dj_c}{dT}\right|\,.$$

The critical state is stable if $\dot{Q}$ does not exceed heat removal due to heat conductivity, q:

$$q = \kappa\nabla^2 T > \frac{j_c \Delta T}{\sigma}\left|\frac{dj_c}{dT}\right|\,.$$

Inasmuch as $|\nabla^2 T| \sim \Delta T/b^2$,

$$\frac{b^2 j_c}{\kappa\sigma}\left|\frac{dj_c}{dT}\right| = \frac{\beta}{\tau} < \gamma_1 \tag{3.14}$$

or, in terms of magnetic field difference

$$\Delta H < \frac{4\pi}{c}\left(\gamma_1 \frac{\kappa\sigma j_c}{|dj_c/dT|}\right)^{1/2}\,.$$

Here γ_1 is a number of the order of unity, which depends upon details of temperature distribution.

Equation (3.14) was derived on the assumption of ideal cooling of the sample boundaries. If the external cooling is weak, the released power should not be allowed to exceed that transferred to the outside, i.e.,

$$\gamma_2 W_0\,\Delta T > \frac{j_c b \Delta T}{\sigma}\left|\frac{dj_c}{dT}\right|\,,$$

where $W_0\Delta T$ is the heat flux from the sample surface, while $\gamma_2 \sim 1$ depends on the ratio of the sample volume to the surface being cooled and details of temperature distribution. Thus the stability criterion has the form

$$\beta/\tau < \gamma_2 \frac{W_0 b}{\kappa}\,. \tag{3.15}$$

Or, in terms of ΔH,

$$\Delta H < \frac{4\pi\gamma_2}{c}\frac{W_0\sigma}{|dj_c/dT|}\,.$$

Note that, as will be shown below, the condition of applicability of the criterion (3.15) has the form $W_0 b/\kappa \ll 1$, which is usually the case in liquid-helium-cooled composites ($W_0 \lesssim 10^7$ erg cm^{-2} s^{-1} K^{-1}, $\kappa \lesssim 10^7$ erg cm^{-1} s^{-1} K^{-1}, $b \sim 10^{-1}$ cm). Using Eq. (3.15), one can evaluate the maximum field difference in the sample and thickness b: $\Delta H \lesssim 10$ kOe, $b \sim 10^{-1}$ cm. The stability criteria (3.14) and (3.15) were first obtained by Hart (1968, 1969) and are often referred to as the "dynamic stability criteria."

From the foregoing discussion, one can readily understand the influence of the normal current j_N excited in the conductor upon the development of instability. A temperature increase causes j_c to drop; however, this drop is compensated for by an increase of j_N, whereby the magnetic flux is damped (an analog of viscous friction) and the heat release decreases accordingly.

Let us now consider the dynamics of the perturbation. Assume, as in Sec. III.A, that $t_j = t_\kappa/\lambda$. Then the condition $\partial j/\partial t = 0$ means that $t_j \ll t_m$ and $|\lambda|\tau \gg 1$. On the other hand, instability develops slowly as compared with the diffusion of heat, i.e., $t_j \gg t_\kappa$ and $|\lambda| \ll 1$ (all of these assumptions are confirmed by the results of thorough calculation in Sec. IV).

We write the energy balance equation

$$\nu\dot{T} = \frac{\lambda\kappa}{b^2}\Delta T = j_c E + \kappa\nabla^2 T\,, \tag{3.16}$$

whence

$$\beta/\tau - \lambda = \gamma_1\,. \tag{3.17}$$

Let us assume ideal heat removal, for the sake of simplicity. By analogy with the case $\tau \ll 1$, at $t_j \gg t_\kappa$, no perturbation is present in the layer of thickness $b/(|\lambda|\tau)^{1/2} \ll b$ and, in Eq. (3.17), $b[1 - 1/(\lambda\tau)^{1/2}]$ should be substituted for b, thus leading to

$$\beta/\tau - \lambda - \frac{2\gamma_1}{(\lambda\tau)^{1/2}} = \gamma_1\,.$$

The dependences of λ_1 and λ_2 upon β have the form shown in Fig. 4. Now, however, $|\lambda| \ll 1$. For β_c, λ_c, β_0, and $\lambda(\beta_0)$, we derive

$$\beta_c = \gamma_1\tau(1 + 3(\gamma_1\tau)^{-1/3})\,,$$

$$\lambda_c = \frac{\gamma_1^{2/3}}{\tau^{1/3}}\,,$$

$$\beta_0 = \gamma_1\tau(1 + (2/\gamma_1\tau)^{1/3})\,,$$

$$\lambda(\beta_0) = i2^{1/3}\lambda_c\,.$$

Note that the correction to the stability criterion for the nonzero value of τ^{-1} may be substantial because of the $\frac{1}{3}$ power, even in the case of relatively high τ.

C. Nonlinear portion of the current-voltage characteristic and instability delay

It follows from the foregoing discussion that, upon violation of the stability criteria (3.5) ($\tau \ll 1$) or (3.14), (3.15) ($\tau \gg 1$), instability can only occur under conditions in which the perturbation covers a considerable portion of the bulk of the superconductor (in the case shown in Fig. 3, this bulk should have a volume of about $l \times L_y \times L_z$, where $L_y, L_z \gg l$). Moreover, the "priming" perturbation should be rather strong.

Indeed, the current-voltage characteristic of hard superconductors at $E \to 0$ is nonlinear (Fig. 2). Accordingly, if $E < E_0$, then $\sigma(E) > \sigma_f$. Hence the parameter

$$\tau(E) = 4\pi\sigma(E)\kappa/c^2\nu > \tau = \frac{4\pi\sigma_f\kappa}{c^2\nu}\,.$$

It follows from the results presented in Secs. III.A and III.B that stability increases with an increase of σ.

Thus, if the electric field is less than E_0, instability does not occur even if the stability criterion (3.5) has been violated. A flux jump will develop with greater field difference in the sample depending on the value of $\sigma(E)$. Inasmuch as, at $E \ll E_0$, $\sigma(E) \gg \sigma_f$, such a flux jump delay may turn out to be quite substantial. This circumstance helps one to understand, for example, the scatter of the experimental values of H_j when the instability was not intentionally initiated in the course of experiment. Another effect associated with the delay in the flux jump is the dependence of the stability threshold upon the rate of external field variation, which has been observed in numerous experiments (see Sec. VI). The variable external field $H_a(t)$ induces in the sample an electric field $E_a \propto \dot{H}_a$. An increase of E_a causes a decrease of $\sigma(E)$ and, consequently, an increase of $\dot{H}_a$ is accompanied by a decrease of the magnetic field difference in the sample upon which a flux jump occurs (Mints and Rakhmanov, 1979b).

An analogous effect is likely to take place in superconducting composites as well. In the case of low electric fields, the current-voltage characteristic of the composite is nonlinear (Kaido, *et al.*, 1976; Dorofeev *et al.*, 1980). Accordingly, the effective conductivity of the composite for $E \to 0$ may increase considerably over the electrical conductivity of the normal matrix.

D. Oscillation effects

It follows from the results presented in Secs. III.A and III.B above that temperature and electric field oscillations may arise in the critical state for a certain range of parameters (Mints, 1978). Assume, for example, that we increase the external magnetic field (Fig. 3). At some value of H_a, the parameter β reaches the value β_0. Upon a further field increase, the parameter β enters the region $\beta_0 < \beta < \beta_c$ where $\mathrm{Im}\lambda \neq 0$ and $\mathrm{Re}\lambda \geq 0$ and oscillations may be observed in the sample. Then, after H_a reaches the value of $H_0 + H_j$ ($\beta = \beta_c$), a flux jump occurs. It is just such a pattern that has been observed in a great number of experiments (as we shall discuss in more detail below).

Note that the magnitude of the electric field in the sample has a considerable effect. Indeed, let $\beta_0 < \beta < \beta_c$ at $E > E_0$, i.e., at $\sigma = \sigma_f$. Then, if the field in the sample is not supported by an external source,

$$E \propto \exp(\lambda_2 t/t_\kappa)\cos(\lambda_1 t/t_\kappa)$$

and the electric field will decrease during a time $t \sim t_\kappa/\lambda_1$ such that the condition $E < E_0$ will certainly be met and the perturbation attenuated inasmuch as this is accompanied by an increase in conductivity σ and, consequently, in the values $\beta_0(\tau)$ and $\beta_c(\tau)$. Thus the initial fluctuation rise with an increment λ_2/t_κ will bring about an increase of magnetic flux in the sample by a finite value. Accordingly, there can be experimentally observed limited flux jumps with an amplitude proportional to the value of initial perturbation (however, at an adequately high initial fluctuation, such instability may also terminate in the transition of the sample to normal state in the region of fluctuation growth).

If a "background" electric field E_a induced by an external source (for instance, by a variable external magnetic field) exists in the sample, electric field (and temperature) oscillations with an amplitude less than E_a may be observed in the superconductor. In particular, if there are oscillations in the flux flow, their amplitude should be less than the difference $E_a - E_0$. Inasmuch as the rate of variation of the external field $\dot{H}_a$ is, as a rule, much less than the rate of magnetic field variation in the sample upon flux jump, the number N of oscillations preceding the instability can be readily estimated. Indeed, $\beta \propto (H_a - H_0)^2$ (see Fig. 3). Oscillations may be observed in the range $\Delta\beta = \beta_c - \beta_0 \ll \beta_c$. The corresponding magnetic field range ΔH_a is equal to $\Delta H_a = \Delta\beta(\partial\beta/\partial H_a)^{-1} = (\Delta\beta/2\beta_c)H_j$. The time during which H_a is within this range equals $\Delta t \sim \Delta H_a/\dot{H}_a$. Then,

$$N \sim \Delta t \lambda_1/t_\kappa \sim \Delta H_a \lambda_1/\dot{H}_a t_\kappa$$

and $N \gtrsim 1$ if

$$\dot{H}_a < \Delta H_a \lambda_1/t_\kappa \sim \Delta\beta H_j \lambda_1/\beta_c t_\kappa\,.$$

If $\beta_c < \beta$, the critical state is certainly unstable at least in the linear approximation. Therefore, we shall denote as the stability boundary a line in the parameter space of the problem, on which there first appears a positive real value of the increment of instability rise. Note further that, if $\lambda_c \neq 0$ at $\beta = \beta_c$, oscillations may be observed prior to flux jump. In this case, indeed, the derivative $\partial\lambda/\partial\beta$ in the vicinity of $\beta = \beta_c$ becomes infinite and λ may be represented as

$$\lambda = \lambda_c\left[1 + a\left(\frac{\beta - \beta_c}{\beta_c}\right)^{1/2}\right],$$

where a is some real number. Thus, for $\beta < \beta_c$, $\mathrm{Im}\lambda \neq 0$.

E. Flux jumps and training of superconductors

As is known (Saint-James *et al.*, 1969; Brechna, 1973), a training effect occurs in hard superconductors and superconducting composites, i.e., the critical current depends on the sample history. For example, if the current is increased up to the disappearance of superconductivity and then disconnected, with subsequent repetition of the process, the following transition to the resistive state will occur at a higher current value. By repeating this process several times, one finally attains the maximum transport current density. Note that other methods of training are possible (Saint-James *et al.*, 1969).

Training occurs both in coils and in short samples of superconductors in the presence of stresses which cause plastic deformation of the material (Anashkin *et al.*, 1975, 1977, 1979; Schmidt, 1976; Pasztor and Schmidt, 1978, 1979). From qualitative considerations, we shall discuss here the critical state stability in short superconducting samples in the presence of plastic yield of the material. From the obtained criterion of stability relative to thermomagnetomechanical instability it will be shown that plastic yield may lead to training in short samples.

We shall first discuss the stability of plastic yield in normal metals in which plastic strain (stress) jerks are known to exist, causing the so-called discontinuous flow mode (Basinski, 1957). Appropriate criteria for plastic yield stability have been obtained both from studying the dynamics of dislocation motion (Malyghin, 1975) and from the macroscopic approach (Petukhov and Estrin, 1975; Petukhov, 1977; Mints and Petukhov, 1980). We shall now derive, from qualitative considerations, the criterion obtained by Petukhov and Estrin (1975).

The additional release of heat due to the work associated with plastic strain amounts to $\hat{\sigma}\dot{u}$, where $\hat{\sigma}$ is the applied stress, $\dot{u}=\dot{u}(T,\hat{\sigma},u)$ is the rate of plastic strain, and u is the value of plastic strain. In the case when only part of the work is released as heat, appropriate changes can be introduced with the aid of a suitable factor.

Assume that a temperature fluctuation ΔT occurs in the sample, causing heat to be released due to plastic strain:

$$Q_1=\hat{\sigma}\frac{\partial\dot{u}}{\partial T}\Delta T.$$

The plastic yield is stable if Q_1 does not exceed the heat transferred to the outside $q=W_0\Delta T$. ($W_0 b/\kappa$ is assumed to be small.) Then, the stability criterion of interest to us has the appearance

$$\hat{\sigma}\frac{\partial\dot{u}}{\partial T}\frac{b}{W_0}<\gamma_3, \tag{3.18}$$

where $\gamma_3\sim 1$. A strain (stress) jerk occurs slowly as compared with heat diffusion. Indeed, from the equation of heat conductivity one can readily derive

$$\lambda=\frac{\partial\dot{u}}{\partial T}\frac{b^2\hat{\sigma}}{\kappa}-\gamma_3\frac{W_0 b}{\kappa},$$

where λ, as usual, denotes the characteristic time of instability $t_j=t_\kappa/\lambda$. Hence the stability criterion has the form of Eq. (3.18) and $\lambda=0$ at the stability threshold.

Consider now the case when a mechanical stress causing plastic deformation is applied to a superconductor in the critical state. If the characteristic times for development of the flux jump and the plastic strain jerk are of the same order of magnitude, both instabilities appear to be interacting strongly and initiating each other. Consequently, a significant variation of the stability criterion is to be expected if the flux jump occurs slowly, i.e., $|\lambda|\ll 1$. It is precisely this case that is observed in superconducting composites and, as will be shown below, in hard superconductors as well, under conditions of very low surface cooling, or if the value of transport current I in the sample is close to $I_c=j_c S$, where S is the cross-sectional area of the conductor.

Let us now derive the stability criterion from qualitative considerations. The summary power of heat release per unit volume is now equal to

$$Q_1=\hat{\sigma}\frac{\partial\dot{u}}{\partial T}\Delta T \quad \text{in the regions where } j=0,$$

$$Q_2=Q_1+\frac{j_c}{\sigma}\left|\frac{dj_c}{dT}\right|\Delta T \quad \text{in the regions where } j=j_c.$$

The critical state is stable if the released heat has had time to be removed from the sample, In the case of, say, weak external cooling, one can readily derive from these considerations the stability criterion in the form

$$\hat{\sigma}\frac{\partial\dot{u}}{\partial T}\frac{b}{W_0}+\frac{I}{I_c}\frac{j_c b}{W_0\sigma}\left|\frac{dj_c}{dT}\right|<\gamma_3. \tag{3.19}$$

The ratio I/I_c included in (3.19) is indicative of the fact that heat release associated with the redistribution of magnetic flux occurs in only part of the sample.

The criterion (3.19) leads one to understand the training effect as successive strain hardening of the superconductor stimulated by thermomagnetomechanical instability. Indeed, if condition (3.19) is violated, a magnetic flux jump occurs accompanied by a plastic strain jerk and heating of the sample. As is known, a strain jerk leads to strain hardening of the material, i.e., the value of $\partial\dot{u}/\partial T$ for given $\hat{\sigma}$ decreases. And, as follows from Eq. (3.19), the limiting value of I grows upon subsequent introduction of current into the sample. If the applied stress $\hat{\sigma}$ is not too high, training may help attain the limiting value of superconducting current corresponding to the given sample in the absence of mechanical stresses.

Note that strain hardening is associated with the separation of weakly pinned dislocations. Following a strain jerk, the structure of dislocations in the sample takes a more stable configuration. In view of this, the inhomogeneity of the physical properties of the sample, in particular, of its mechanical properties, may be of essential importance in the development of thermomagnetomechanical instabilities. In this case, instabilities will develop in the "weak" links of the superconductor and training will depend upon the strain hardening of such links.

The value of $\partial\dot{u}/\partial T$ characteristic of metals is approximately equal to $3\times10^{-2}\ \mathrm{cm^{-1}\,K^{-1}}$. From (3.18) we determine that the $\hat{\sigma}$ value at which serration yield occurs is of the order of 10 to 30 $\mathrm{kg\,mm^{-2}}$. Inasmuch as in this region the characteristic value of the ratio

$$\hat{\sigma}\left(\frac{\partial\dot{u}}{\partial T}\right)\Big/\left(\frac{j_c}{\sigma}\left|\frac{dj_c}{dT}\right|\right)$$

is approximately equal to unity, the presence of current in the sample may result in a considerable reduction of the stress at which a thermomagnetomechanical instability occurs as compared with the stress at which serration yield occurs in the absence of a current.

IV. THEORY OF CRITICAL STATE STABILITY

A. Basic equations

The study of the critical state stability and the dynamics of perturbation development is carried out by means of linear analysis of the stability of the solutions of the Maxwell and heat equations (Hart, 1968; Kremlev, 1973, 1974; Duchateau and Turk, 1975a, 1975b; Mints and Rakhmanov, 1975b). We shall briefly describe this method below.

Obviously, for the present purposes, it is sufficient to consider only those of the initial perturbations which constitute the biggest threat to stability. It is clear from the qualitative treatment contained in Sec. III that these are perturbations covering the maximum (possible under given geometric conditions) volume. In so doing, heat transfer to the unperturbed region of the sample volume is minimal and, consequently, the stability is minimal. In this chapter, we shall use Bean's model $[j_c = j_c(T)]$ as the equation of the critical state.

The development of temperature and electric field perturbations in a superconductor is described by a system of electromagnetic (Maxwell) and heat equations. In a linear approximation, this system has the form

$$\nu \frac{\partial}{\partial t}(\Delta T) = \kappa \nabla^2 (\Delta T) + j_c E ,$$
$$\text{curl}\,\text{curl}\mathbf{E} = -\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t} , \tag{4.1}$$
$$j = j_c + \sigma E ,$$

where ν, κ, j_c, σ represent either the values of respective quantities in a hard superconductor or the averaged values of these parameters in a composite. We now write ΔT and E as

$$T = T_c\,\theta(x/b)\exp(\lambda t/t_\kappa) ,$$
$$E = \frac{\kappa T_c}{j_c b^2}\,\varepsilon(x/b)\exp(\lambda t/t_\kappa) , \tag{4.2}$$

where λ is the eigenvalue of the problem to be found. One can easily derive from the original equations

$$\lambda\theta = \nabla^2\theta + \varepsilon ,$$
$$\text{curl}\,\text{curl}\boldsymbol{\varepsilon} = \lambda\beta\mathbf{e}\,\theta - \lambda\tau\boldsymbol{\varepsilon} , \tag{4.3}$$

where $\mathbf{e}$ is a unit vector in the direction of $\boldsymbol{\varepsilon}$.

Boundary conditions should be imposed on Eqs. (4.3). The thermal boundary conditions in a linear approximation have the form

$$W_0\theta - \frac{\kappa}{b}(\mathbf{n}\cdot\nabla\theta) = 0 , \tag{4.4}$$

where $\mathbf{n}$ is normal to the sample surface.

To determine the electrodynamic boundary conditions, one should, generally speaking, solve Maxwell equations outside the superconductor:

$$\text{curl}\mathbf{H} = 0 ,$$
$$\text{div}\mathbf{H} = 0 ,$$

with the boundary condition of $H_a = H_a(t)$ as $|\mathbf{x}| \to \infty$. Then the solutions are joined along the continuity of $\mathbf{E}$ and $\mathbf{H}$ on the sample surface. Note further than, in the linear problem under consideration, the spectrum of eigenvalues $\lambda = \lambda(\beta, \tau, W_0, \dots)$ cannot depend explicitly on $H_a(t)$. Therefore it can be assumed that $\dot{H}_a = 0$. The dependence of the stability criterion upon $\dot{H}_a(t)$ occurs only if the variable external field affects considerably the unperturbed state of the system, for example if it affects the conductivity value $\sigma(E)$ or causes inhomogeneous heating.

Therefore, after the equations and boundary conditions have been written down, the finding of the spectrum of eigenvalues of λ is straightforward. The solution of Eqs. (4.3) depends upon a set of arbitrary constants C_j. By substituting these solutions in the boundary conditions, we shall derive a homogeneous linear system of equations for C_j:

$$\sum_j A_{ij}C_j = 0 ,$$

where $A_{ij} = A_{ij}(\lambda, \tau, \beta, W_0, \dots)$. It is the condition of existence of a nontrivial set of C_j that defines the spectrum of eigenvalues of λ. Thus the dispersion equation for finding λ has the form:

$$\det\|A_{ij}\| = 0 . \tag{4.5}$$

B. Plane plate in a parallel external magnetic field ($\tau \ll 1$)

Consider now by way of example the critical state stability in a plane plate of a hard superconductor (Fig. 5). In the case of plane geometry (Fig. 5), one should apparently consider perturbations covering large spaces along the axes $0Y, 0Z$ (the limit being provided by the entire sample). Then, by virtue of the homogeneity of the plate, $\partial\theta/\partial x \gg \partial\theta/\partial y, \partial\theta/\partial z$, and Eqs. (4.3) with their boundary conditions can be written as

$$\theta'''' - \lambda(1+\tau)\theta'' - \lambda(\beta - \lambda\tau)\theta = 0 , \quad \varepsilon = \lambda\theta - \theta'' \tag{4.6}$$

$$w\theta(\pm 1) \pm \theta'(\pm 1) = 0 ,$$
$$\lambda\theta'(\pm 1) - \theta'''(\pm 1) = 0 , \quad w = W_0 b/\kappa . \tag{4.7}$$

It can be shown that taking into consideration the size of the perturbations in the directions $0Y(L_Y)$ and $0Z(L_Z)$ leads to corrections in the stability criterion on the order of $b^2/(L_Y^2 + L_Z^2) \ll 1$.

As can be seen from Fig. 5, two different cases of magnetic field distribution are possible in the sample. In the first case, the critical state occurs in the entire volume of the sample. The magnetic field difference ΔH is maximal at the given j_c and b:

$$\Delta H = H_p = \frac{4\pi b j_c}{c} .$$

This case will be further referred to as "complete penetration of the magnetic field." Referred to as "incomplete penetration of the magnetic field" is the case when $\Delta H < H_p$ and the critical state occurs only in the region $|x| > l$. In the region $|x| < l$ we apparently have for ε and θ:

$$\varepsilon = 0 , \quad \lambda\theta - \theta'' = 0 . \tag{4.8}$$

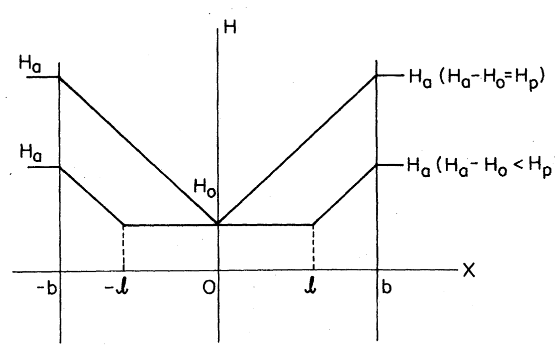

FIG. 5. Magnetic field distribution in a plane plate.

If $\Delta H = H_p$, the vector $\mathbf{j}$ changes its direction abruptly at $x=0$, i.e., there is a singularity in the coefficients of the original equations. Therefore Eqs. (4.6) should be solved independently to the left and right of the plane $x=0$ and, at $x=0$, natural joining conditions should be imposed:

$$\varepsilon(0)=0, \qquad \theta(+0)=\theta(-0),\ \theta'(+0)=\theta'(-0). \tag{4.9}$$

Analogously, if $\Delta H < H_p$, the solutions of Eqs. (4.6) and (4.8) should be joined at $|x| = l$ by the continuity of θ, θ', ε.

Let us first consider the simpler case when the magnetic field fully penetrates the sample. The solution of Eqs. (4.6) has the form

$$\begin{aligned}
&\theta = C_1 e^{k_1\tilde{x}} + C_2 e^{-k_1\tilde{x}} + C_3 e^{ik_2\tilde{x}} + C_4 e^{-ik_2\tilde{x}}, \quad x>0\\
&\theta = C_5 e^{k_1\tilde{x}} + C_6 e^{-k_1\tilde{x}} + C_7 e^{ik_2\tilde{x}} + C_8 e^{-ik_2\tilde{x}}, \quad x<0\\
&k_{1,2} = \left[\left(\frac{\lambda^2(1-\tau)^2}{4} + \lambda\beta\right)^{1/2} \pm \frac{\lambda(1+\tau)}{2}\right]^{1/2},\\
&\tilde{x} = x/b.
\end{aligned} \tag{4.10}$$

Then, with the aid of the boundary conditions, we derive the dispersion equation having the form

$$\begin{aligned}
&\Delta = \Delta_a + w\Delta_i\\
&\Delta_a = -16ik_1k_2(k_1^2+k_2^2)[k_1(\lambda+k_2^2)\sinh k_1 \cos k_2 + k_2(\lambda-k_1^2)\cosh k_1 \sin k_2],\\
&\Delta_i = 16i\{2k_1k_2(\lambda+k_2^2)(\lambda-k_1^2) - k_1k_2[(\lambda+k_2^2)^2+(\lambda-k_1^2)^2]\cosh k_1\cos k_2 + (k_1^2-k_2^2)(\lambda-k_1^2)(\lambda+k_2^2)\sinh k_1 \sin k_2\}.
\end{aligned} \tag{4.11}$$

One can readily solve Eq. (4.11) numerically and make certain that the eigenvalue spectrum has the appearance shown in Fig. 4. Presented in Fig. 6 is the numerically derived dependence $\beta_c(\tau)$ at $\tau<1$ and different w.

We shall now derive an analytical solution to the problem for the limiting case of $\tau \ll 1$ (hard superconductor). As already noted, at sufficiently low τ, $\lambda_c \gg 1$. Then, upon series expansion of Eq. (4.11) at $|\lambda| \gg 1$, we obtain

$$\lambda^2 + w\lambda^{3/2} + \frac{\pi}{\tau}\left(\frac{\pi}{2} - \beta^{1/2}\right)\lambda + \frac{w\pi}{\tau}\left(\frac{\pi}{2} - \beta^{1/2}\right)\lambda^{1/2} + \frac{\pi^2}{2\tau}\left(w + \frac{\pi^2}{8}\right) = 0. \tag{4.12}$$

For $w \gg (|\lambda|)^{1/2}$, Eq. (4.12) coincides with Eq. (3.12) if $\gamma = \pi^2/4$ (isothermal limit), and at $w \ll 1$—with Eq. (3.10) (adiabatic limit). An analytical solution to Eq. (4.12) can be obtained in two cases, namely, $w \gg (|\lambda|)^{1/2}$ and $w \ll (|\lambda|)^{1/2}$. In the former case, the values of β_c, λ_c, β_0, $\lambda(\beta_0)$ are found from the expressions (3.12), assuming that $\gamma = \pi^2/4$. In the latter case,

$$\begin{aligned}
&\beta_c = \frac{\pi^2}{4}\left[1 + 2\left(1 + \frac{8w}{\pi^2}\right)^{1/2}\tau^{1/2}\right],\\
&\lambda_c = \frac{\pi^2}{4\tau^{1/2}}\left(1 + \frac{8w}{\pi^2}\right)^{1/2},\\
&\beta_0 = \pi^2/4,\\
&\lambda(\beta_0) = i\lambda_c,
\end{aligned} \tag{4.13}$$

which, at $w \to 0$, coincides with the expressions (3.11) at $\gamma = \pi^2/4$.

Curves of $\lambda = \lambda(\beta)$ for $\beta \geq \beta_c$ are shown qualitatively in Fig. 7 at different τ and w. With an increase of τ, the

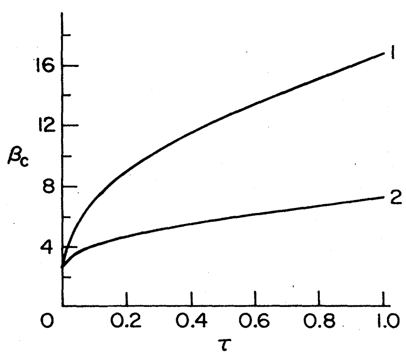

FIG. 6. Value of β_c as a function of τ at $\tau<1$: Curve 1, $w \gg 1$; curve 2, $w=1$.

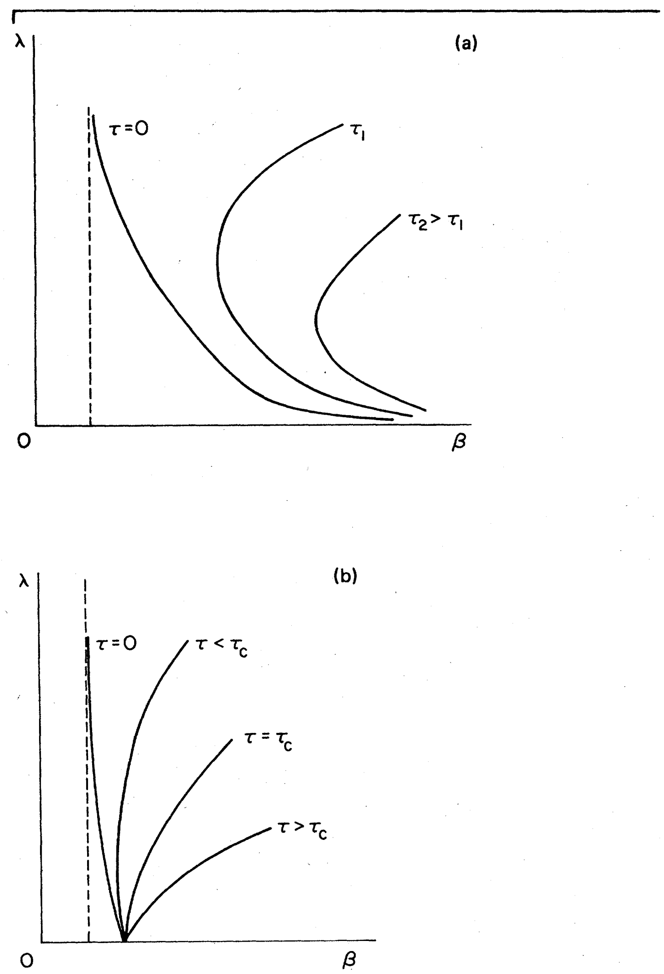

FIG. 7. Evolution of the eigenvalue spectrum of $\lambda(\beta)$ in the region $\beta \geq \beta_c$ with an increase of τ: (a) for $w>0$; (b) for $w=0$.

curves for $\lambda(\beta)$ shift to the right and are deformed so that λ_c decreases. One can readily understand such behavior by remembering that the increase of τ is accompanied by the growth of the damping function of the normal current $j_N = \sigma E \propto \lambda\tau$ and that perturbations with high λ are damped more strongly with an increase of τ. As to perturbations with low λ, these are affected more strongly by heat transfer. And, at $w=0$, there exists a point $[\lambda=0, \beta(0)]$ whose position is not affected by the value of τ. If $\tau > \tau_c = \frac{1}{21}$, then, at $w=0$, the parameter β_c equals $\beta(0)$ and ceases to depend on τ (Kremlev, 1973). For the case of a plane geometry, $\beta(0)=3$.

At $\lambda \to 0$, we derive from Eq. (4.11)

$$1.2(\tau-\tau_c)\lambda^2 + (\beta-3)\lambda + 3w = 0$$

from which it follows readily (Maksimov and Mints, 1979) that

$$\beta_c = 3 + 3.8 w^{1/2}(\tau-\tau_c)^{1/2}, \quad \lambda_c = 1.6\left(\frac{w}{\tau-\tau_c}\right)^{1/2},$$

$$\beta_0 = 3, \quad \lambda(\beta_0) = i\lambda_c.$$

If $w \ll \tau - \tau_c$, then $\lambda_c \ll 1$.

Analogous results can be obtained in the case of incomplete penetration of the magnetic field into the sample (Maksimov and Mints, 1980). The essential distinction between the cases $\Delta H = H_p$ and $\Delta H < H_p$ becomes apparent at $w \ll 1$. Thus, at $w=0$, the curve of $\lambda(\beta)$ for $\beta > \beta_c$ has the same appearance as that shown in Fig. 7(b); however, the value of τ_c now depends on the ratio $b/(b-l)$:

$$\tau_c = \frac{5}{6}\left(\frac{b}{b-l}\right)^2 - \frac{9}{7}\frac{b}{b-l} + \frac{1}{2}.$$

C. Stability at low τ

1. Simplified theory

The fact that at low τ the loss of stability is caused by "rapid" ($\lambda_c \gg 1$) perturbations makes for a marked simplification of the problem of determining the stability criterion (Wipf, 1967; Swartz and Bean, 1968; Mints and Rakhmanov, 1975b). As follows from the foregoing results, the nonzero value of the ratio between the thermal and magnetic diffusion coefficients does not have an excessively strong effect upon the stability criterion, and it can be assumed in the main approximation that $\tau=0$. However, the dynamics of the process depend upon the value of τ. In particular, the coupled nature of the rise of temperature and electromagnetic field perturbation leads to the inequality $t_m \ll t_j \ll t_\kappa$. It follows from the condition $t_j \ll t_\kappa$ that the magnetic flux jump develops nearly adiabatically and, in the main approximation, the heat conduction term can be omitted from Eqs. (4.1). Upon eliminating ΔT from the system of differential equations (4.1), we obtain

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \beta\frac{\mathbf{E}}{b^2} - \frac{4\pi\sigma}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

Inasmuch as $t_j \gg t_m$, the second term on the right-hand side is of the order of $t_m E/t_j b^2 \ll E/b^2$ and can be discarded. Then, in dimensionless coordinates, we have

$$\operatorname{curl}\operatorname{curl}\mathbf{E}(\mathbf{x}/b) = \beta\mathbf{E}(\mathbf{x}/b), \qquad \frac{\partial}{\partial t}(\Delta T) = \frac{j_c E}{\nu}. \tag{4.14}$$

Since we have neglected the redistribution of heat while deriving Eq. (4.14), only the electrodynamic boundary conditions should be imposed on the equation. Stability is lost if there exists a nontrivial solution to Eq. (4.14). Note further that Eq. (4.14) can be derived from Eqs. (4.3) by passage to the limit $|\lambda| \to \infty$, $\tau \to 0$, $|\lambda|\tau \to 0$.

Inasmuch as in the approximation under consideration $\tau=0$, the values of β_c and β_0 coincide. No oscillation frequencies or perturbation rise increments can be found within the bounds of the simplified theory, and one can only find the critical state stability criteria.

Consider now the plane plate case already discussed above (Fig. 5). Here, Eq. (4.14) and its boundary conditions have the form

$$\varepsilon'' + \beta\varepsilon = 0, \qquad \varepsilon'(\pm 1) = 0. \tag{4.15}$$

The conditions at the boundaries where the current changes direction or turns to zero are identical and have the form of $\varepsilon=0$. By solving Eq. (4.15), we obtain in each continuity region of j

$$\varepsilon = C_1 \cos\sqrt{\beta}\,\tilde{x} + C_2 \sin\sqrt{\beta}\,\tilde{x}.$$

In the case of complete penetration of the magnetic field into the sample, we have $\varepsilon(0)=0$. Then a nontrivial set of constants C_i exists if $\cos\sqrt{\beta}=0$. Hence, $\beta_c = \pi^2/4$.

Inasmuch as $\varepsilon=0$ at the boundaries of regions with differing directions of superconducting current, the instability develops independently in each one of those regions. The interaction between them only occurs owing to heat transfer, i.e., upon taking into account the nonzero value of τ.

2. The effect of transport current upon stability

Consider now a plane sample carrying a transport current I. Depending on the values of I, ΔH, and on the order of sequence in which the current and external fields are varied, different magnetic-flux distributions can be set up in the plate. Examples of such distributions are shown in Fig. 8. Any one of such distributions consists of a number of regions differing from each other by the direction or value (j_c or 0) of the current. In the approximation $\tau=0$, the stability in each layer is disturbed independently. For the two external regions, the boundary conditions on Eq. (4.15) have the form $\varepsilon'(\pm 1)=0$ and $\varepsilon=0$ at the internal boundaries. Accordingly, $\beta < (\pi b/2l_i)^2$, where l_i is the size of the ith region, or

$$\Delta H_i < H_j = \left(\pi^3 \nu j_c \left|\frac{dj_c}{dT}\right|^{-1}\right)^{1/2}, \tag{4.16}$$

where ΔH_i is the magnetic field difference in the ith region. In the internal regions, $E=0$ at both boundaries. In this case,

$$\Delta H_i < 2H_j.$$

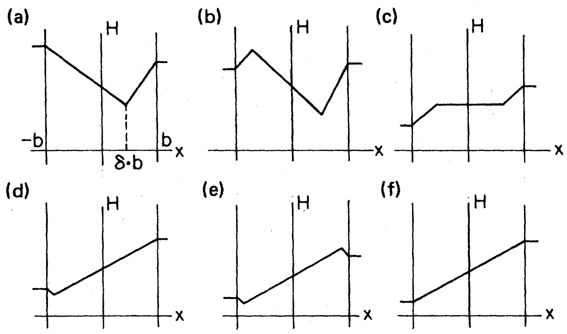

FIG. 8. Examples of different magnetic field distributions in a plane plate.

The overall critical state stability in the sample is determined by the least stable region.

It is worthy of note that the stability of the magnetic-flux distributions shown in Fig. 8(e) and 8(d) differs considerably. Thus, in the former case, the critical thickness of the sample is greater by a factor of ~2; consequently the maximum transport current in the sample is higher by a factor of ~2.

Let us now address the simplest illustrative example presented in Fig. 8(a) (complete penetration, the magnetic flux distributed asymmetrically relative to the $0X$ axis). Using the criterion (4.16), one can readily derive (Mints and Rakhmanov, 1975b)

$$\beta < \beta_c = \frac{\pi^2}{4(1+|I|/I_c)}, \tag{4.17}$$

where $I_c = 2bj_c$ is the critical current per unit length of the sample, and $I = 2\delta b j_c$ is the transport current per unit length of the plate.

If $I \to I_c$, the planes where $E=0$ and $E'=0$ converge. Therefore the case of $I=I_c$ calls for special consideration, and it would be wrong to assume $I=I_c$ in the criterion (4.17). At $I=I_c$, the currents throughout the sample flow in the same direction. The boundary conditions of Eq. (4.15) have the form $E'(\pm 1)=0$, and a nontrivial solution exists in the case $\sin\sqrt{\beta}=0$. Hence $\beta_c=0$. One can readily understand what causes such a sharp drop in stability. If no flux jump occurs upon the emergence of a perturbation, a new state of equilibrium with a temperature $T_0+\Delta T$ is set up in the sample. Inasmuch as $\delta = I/I_c(T)$ the boundary of regions with different directions of current shifts.

If $\delta=1$, no new equilibrium position is set up since the system has lost a degree of freedom essential for stability. In the $\tau=0$ approximation, stability drops to zero at $I=I_c$. If one takes into consideration that $\tau\neq 0$ the degree of freedom essential for stability is not lost because the drop of j_c is compensated for by the normal current. However, at $\tau \ll 1$, this fact leads to only a slight increase of stability $\beta_c \sim \tau \ll 1$. At $I=I_c$, the increment of perturbation rise decreases sharply. It can be shown that $\beta_c = 3w\tau/(3+w)$ and $\lambda_c=0$ for any w and $\tau \ll 1$. In this case, $\beta_c \leqslant \beta_0$ and the oscillations preceding the instability are absent. Nonzero values of λ_c and, consequently, oscillations, only occur at $\tau \gtrsim 1$.

3. The contact of a hard superconductor with a normal metal

As already noted, in technical materials a hard superconductor, as a rule, maintains contact (thermal and electric) with a normal metal. Therefore the effect of a normal metal cladding upon the critical state stability in a hard superconductor has been subjected to extensive experimental and theoretical study. We shall now examine this problem using the methods described above.

In the case under consideration, the equations for field and temperature in the normal state should be added to Eqs. (4.3) describing the development of a small perturbation in the superconductor, and these should then be joined with solutions for perturbations in the continuity of E, H, T and heat fluxes at the normal metal-superconductor interface. Selecting E and ΔT to be the same as before [Eq. (4.2)], we use the Maxwell and heat equations in a linear approximation to derive, for a normal metal,

$$\lambda\theta = \frac{\kappa_n}{\kappa_s}\nabla^2\theta$$

$$\mathrm{curl\,curl}\,\varepsilon = -\lambda\tau_1\varepsilon \tag{4.18}$$

where $\tau_1 = \tau\sigma_n/\sigma_s$. The subscripts n and s denote parameters in the normal metal and in the superconductor, respectively. The subsequent procedure for determining the stability criteria and spectrum of eigenvalues of λ remains unchanged.

We shall assume below that $\kappa_n \gg \kappa_s$, $\sigma_n \gg \sigma_s$, which corresponds to coatings of metals with good conductivity (Cu, Al). This case is of particular interest since this is where the effect of cladding upon the stability and dynamics of the critical state is most essential. In addition, $\nu_n \ll \nu_s$ as a rule. The case of $\kappa_n \lesssim \kappa_s$, $\sigma_n \lesssim \sigma_s$ can be investigated using the same methods.

Let us first look at the problem qualitatively (Kremlev, 1973; Onishi, 1974; Mints and Rakhmanov, 1976a, 1977b; Maksimov and Mints, 1979). For $\tau \ll 1$, instability in an uncoated superconductor develops "rapidly": $\lambda_c \gg 1$. On the other hand, a variable external field only penetrates a normal metal to a depth on the order of the skin depth: $\delta_{sk} = c/(2\pi\sigma_n\omega)^{1/2}$. By substituting $\omega \sim t_j^{-1}$ for estimation, we obtain $\delta_{sk}/b \sim (\lambda_c\tau_1)^{-1/2}$, where $\tau_1 \gg \tau$. If $\lambda_c\tau_1 \gg 1$ the magnetic flux cannot enter the sample at a rate corresponding to that of flux jump propagation in an uncoated sample as early as at a normal cladding thickness of $d \ll b$. Thus screening currents occurring in the coating damp "rapid" perturbations. In the region of low values of λ ("heat" branch of the spectrum), the effect of normal cladding is less pronounced. Indeed, the heat propagates rapidly over the coating ($\kappa_n \gg \kappa_s$), the heat capacity of the latter at $d \lesssim b$ is low, and the heat boundary conditions actually realized on the superconductor surface correspond to cooling by an external coolant (Kremlev, 1973; Mints and Rakhmanov, 1977b).

Presented by way of illustration in Fig. 9 are curves $\lambda(\beta)$ for $\beta \geqslant \beta_c$ at different d and w. Thus, under conditions of adiabatic insulation of the sample, an increase of the cladding thickness affects only the sta-

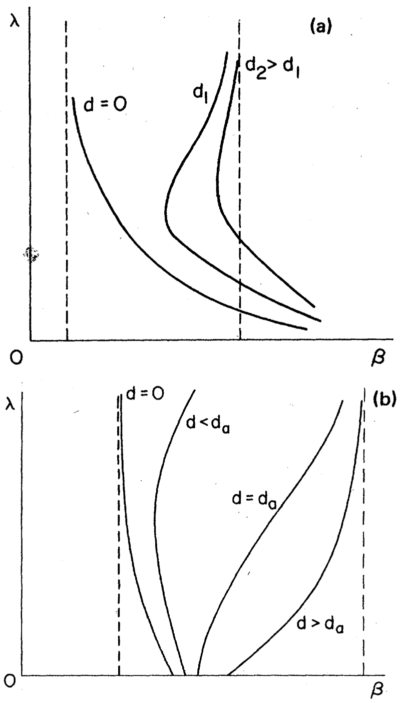

FIG. 9. Evolution of the eigenvalue spectrum of $\lambda(\beta)$ in the region $\beta \geq \beta_c$ with an increase of the cladding thickness d: (a) for $w>0$; (b) for $w=0$.

bility while $d<d_a$ (Fig. 9b). For $d>d_a$, $\lambda_c=0$, and the effect of coating upon stability is low in proportion to $\nu_n d \ll \nu_s b$ (Kremlev, 1973; Maksimov and Mints, 1979). For $w \gg 1$, the stability increase is more significant. However, in this case it is likewise limited [Fig. 9(a)]. The point is that the flux jump may occur owing to redistribution of the magnetic flux already in the material without the arrival of magnetic flux from the outside. When this happens, the instability develops in two stages: first, a rapid redistribution of magnetic field in the superconductor under conditions of constant magnetic flux in the bulk, and then a slow entry of the magnetic flux into the sample over a time depending on the time of magnetic diffusion in the cladding. This pattern of instability development was observed experimentally by Onishi (1974), who monitored the evolution of magnetic field distribution in the sample upon flux jump.

Evidently, an increase in the coating thickness can affect the critical state stability only so long as d is less than d_c—the skin depth corresponding to the characteristic time for a flux jump to develop under conditions of fixed magnetic flux.

For $d>d_c$, one can make use of simplified theory, i.e., Eqs. (4.14), for determining the stability criterion mainly in the $\tau \ll 1$ approximation. The boundary conditions on the superconductor surface now have the form $E(\pm 1)=0$, which is evidently equivalent to the condition of preservation of frozen-in flux. For example, it can be readily found for a plane sample at $I=0$, $\Delta H = H_p$ that

$$\varepsilon'' + \beta\varepsilon = 0\,, \quad \varepsilon(\pm 1) = \varepsilon(0) = 0\,.$$

Hence $\beta_c = \pi^2$, i.e., at $d>d_c$ the parameter β_c has increased by a factor of 4 (accordingly, the maximum values of ΔH or of the superconductor thickness have doubled).

It can be readily demonstrated that no sharp drop in stability occurs at $d>d_c$ if $I \to I_c$. Normal currents excited in the cladding compensate for the drop in j_c due to heating, and the degree of freedom essential to stability is retained.

In order to find the values of d_a and d_c, one should know the spectrum of eigenvalues of λ, to which end the problem is to be solved with due regard for the redistribution of heat throughout the sample. Thus, in the simplest case of plane geometry and in the case $I=0$, $\Delta H = H_p$, the dispersion equation has the form (Mints and Maksimov, 1979)

$$\begin{aligned} &W_d Z_d (k_1^2 + k_2^2)(k_1^3 \tan k_2 + k_2^3 \tanh k_1) + Z_d[(k_1^6 - k_2^6)\tanh k_1 \tan k_2 + 2k_1^3 k_2^3 (1 - 1/\cosh k_1 \cos k_2)] \\ &\quad + k_1 k_2 (k_1^2 + k_2^2)(k_1^3 \tanh k_1 - k_2^3 \tan k_2) + W_d\left(\frac{2k_1^3 k_2^3}{\cosh k_1 \cos k_2} + k_1 k_2 (k_1^4 + k_2^4) + k_1^2 k_2^2 (k_1^2 - k_2^2)\tanh k_1 \tan k_2\right) = 0\,, \end{aligned} \tag{4.19}$$

$$W_d = \left(\lambda \frac{\nu_n \kappa_n}{\nu_s \kappa_s}\right)^{1/2} \frac{w + \left(\lambda \frac{\nu_n \kappa_n}{\nu_s \kappa_s}\right)^{1/2} \tanh\left[\left(\lambda \frac{\nu_n \kappa_s}{\nu_s \kappa_n}\right)^{1/2} \frac{d}{b}\right]}{\left(\lambda \frac{\nu_n \kappa_n}{\nu_s \kappa_s}\right)^{1/2} + w \tanh\left[\left(\lambda \frac{\nu_n \kappa_s}{\nu_s \kappa_n}\right)^{1/2} \frac{d}{b}\right]}\,, \quad Z_d = (\lambda\tau_1)^{1/2} \tanh\left[(\lambda\tau_1)^{1/2} \frac{d}{b}\right].$$

The solution to this equation can be found numerically, and, in the limiting cases of $|\lambda| \ll 1$, $\gg 1$, analytically as well. An analysis of the dependence of λ on β and d/b made with the aid of Eq. (4.19) confirms the results derived from the qualitative theory (Fig. 9). Accordingly, the value of d_a at which λ_c turns to zero is equal to

$$d_a = \frac{8b}{3\left(11 \frac{\nu_n}{\nu_s} + 105\tau_1\right)}\,.$$

If $d - d_a \ll d_a$, then $\lambda_c \ll 1$. In this case, $\lambda_c \propto 1 - d/d_a$. If $d \ll d_a$, the instability and oscillations occur at $|\lambda| \gg 1$ ($\tau < \tau_c$). Then we derive from Eq. (4.19), within the desired accuracy,

$$\lambda^2\tau\left(1+\frac{2\sigma_n d}{\sigma_s b}\right)-\lambda\left(\beta-\frac{\pi^2}{4}\right)+\frac{\pi^4}{16}=0,$$

whence

$$\beta_c=\frac{\pi^2}{4}\left\{1+2\left[\tau\left(1+2\frac{\sigma_n d}{\sigma_s b}\right)\right]^{1/2}\right\},\quad \lambda_c=\frac{\pi^2}{4}\frac{1}{\left[\tau\left(1+2\frac{\sigma_n d}{\sigma_s b}\right)\right]^{1/2}}$$

$$\beta_0=\frac{\pi^2}{4},\quad \lambda(\beta_0)=i\lambda_c.$$

In the case of $w\gg1$, a spontaneous perturbation growth at $|\lambda|\ll1$ is only possible for $\beta\gg1$. Therefore, with the aid of cladding, the parameter β_c can be increased up to π^2. If $\tau\ll1$, $\pi^2\tau_1\gg1$ then the problem of determining the value of d_c can be solved analytically. Thus, in the $\tau=0$, $\pi^2\tau_1\gg1$, $d>d_c$ approximation in the region of $|\lambda|\gg1$, the dispersion equation has the form

$$\lambda(\pi^2-\beta)-2\pi^2\left(\frac{\lambda}{\tau_1}\right)^{1/2}+\pi^4=0,$$

whence

$$\beta_c=\pi^2(1-1/\pi^2\tau_1),\quad \lambda_c=\pi^4\tau_1$$
$$\beta_0=\pi^2(1-2/\pi^2\tau_1),\quad \lambda(\beta_0)=\tfrac{1}{2}i\lambda_c. \tag{4.20}$$

By defining d_c as $\delta_{sk}(\lambda_c)$ one can easily derive

$$d_c=\frac{b}{\pi^2\tau_1}.$$

Assuming that $D_t=1\ \mathrm{cm^2\,s^{-1}}$, $\sigma_n=10^{20}\ \mathrm{s^{-1}}$, $b=10^{-2}$ cm, one can find $d_c\sim10^{-3}$ cm, which is in agreement with experimental estimates (see Lazarev and Goridov, 1972).

4. The effect of geometry upon critical state stability (cylindrical samples)

Using an example of cylindrical samples (Fig. 10), we shall now demonstrate the effect of geometry upon the critical state stability in hard superconductors. In order to isolate qualitative phenomena associated with this effect, it suffices to restrict oneself to an "adiabatic" approximation, i.e., to assume that $\tau=0$.

a. Wire in a transverse field

In numerous cases, stability experiments are conducted using a wire placed in an external magnetic field transverse to its axis [Fig. 10(a)]. We shall now find the maximum wire radius R_m at which the critical state in the wire is stable for any external field H_a. For determining R_m, it is quite sufficient to consider the case of complete penetration of the magnetic field into the wire, for it is in this particular case that the field difference in the wire is maximum and, consequently, the stability is minimum. In this case,

$$\Delta H=H_a-H_0=H_p=8j_cR/c,$$

where H_a is the field at $r\to\infty$ and H_0 is the field on the wire axis.

It follows from symmetry that, for purposes of determining the stability criterion, it is enough to consider perturbations of an electric field which only has a component along the wire axis. Inasmuch as at $y=0$, $\mathcal{E}=0$,

$$\mathcal{E}=\mathcal{E}_0(r)\sin\varphi,$$

$$\mathcal{E}_0''+\frac{1}{r}\mathcal{E}_0'+(\beta-1/r^2)\mathcal{E}_0=0,$$

$$\beta=\left(\frac{R}{R_0}\right)^2=\frac{4\pi R^2 j_c}{c^2\nu}\left|\frac{dj_c}{dT}\right|,$$

where r,φ are polar coordinates [see Fig. 10(a)], and where r is normalized for R. A nonsingular solution of this equation is $\mathcal{E}_0=CJ_1(\sqrt{\beta}r)$. At $\dot{H}_a=0$, the solution of the Maxwell equations outside of the wire has the form

$$\dot{H}_r=-\frac{B}{r^2}\cos\varphi,\quad \dot{H}_\varphi=\frac{B}{r^2}\sin\varphi.$$

Here H_r and H_φ are the r and φ components of the magnetic field. By using the continuity condition on E and H at $r=1$ and the existence of nonzero values of B and C, we derive the equation for β_c: $J_0(\sqrt{\beta_c})=0$, whence $\sqrt{\beta_c}\simeq2.4$, and for R_m we have

$$R_m=2.4R_0,$$

where R_0 lies in the range of from 10^{-2} cm ($j_c\sim10^5\ \mathrm{A\,cm^{-2}}$) to 10^{-3} cm ($j_c\sim10^6\ \mathrm{A\,cm^{-2}}$). To within an order of magnitude, R_m agrees with the experimental data available from the literature (see Shiiki and Kudo, 1974; Irie *et al.*, 1977).

If the wire is clad with normal metal to a thickness $d>d_c$, the boundary condition to the equation for $\mathcal{E}_0$ has the form $\mathcal{E}_0(1)=0$. Accordingly, for β_c we derive $J_1(\sqrt{\beta_c})=0$, whence

$$R_{m1}=3.8R_0.$$

Thus a normal cladding leads to a radius increase by a factor of 1.6, as distinct from the case of plane geometry where the maximum thickness doubled. Note further that the ratio R_m/R_0 is independent of temperature.

b. Quenching current of a superconducting wire

Let us now find the maximum quenching transport current I_m of a superconducting wire having a radius R (Mints and Rakhmanov, 1976a) [Fig. 10(b)]. In so doing, we shall assume that the external transverse field (if any) is included in the problem as a j_c-defining parameter, but it should not affect the current distribution in the sample.

The equation for the perturbations of interest has

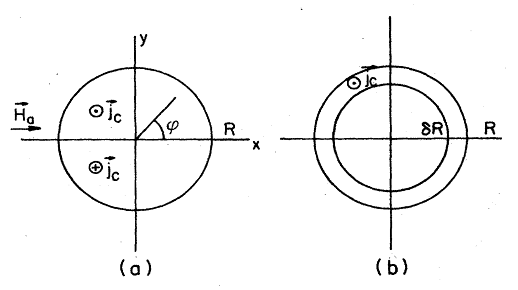

FIG. 10. Cylindrical samples: (a) wire in a transverse external magnetic field $H_a>H_p$, $I=0$; (b) wire with preset transport current.

the form

$$\varepsilon'' + \frac{1}{r}\varepsilon' + \beta\varepsilon = 0\,,$$

where $\beta = (R/R_0)^2$ (ε has only a z component and is independent of φ). The boundary condition for ε is $\varepsilon'(1) = \varepsilon(\delta) = 0$, where $\delta = (1 - I/I_c)^{1/2}$, $I_c = \pi R^2 j_c$. Then, in order to determine β_c, we obtain

$$N_1(\sqrt{\beta_c})J_0(\delta\sqrt{\beta_c}) - N_0(\delta\sqrt{\beta_c})J_1(\sqrt{\beta_c}) = 0\,, \qquad (4.21)$$

where J_k and N_k are the Bessel and Neumann functions, respectively. The critical value of the transport current I_m is found from the condition $(R/R_0)^2 = \beta_c(I_m)$. The ratio I_m/I_c found with the aid of Eq. (4.21) is presented in Fig. 11 (curve I) as a function of the ratio R/R_0. At $I_m \ll I_c$, the stability criterion for the wire coincides naturally with the results obtained for the case of a plane geometry.

It follows from Fig. 11 that in the $\tau = 0$ approximation I_m is always smaller than I_c. The reason for this is the same as in the case of a plane geometry, i.e., at $I = I_c$ the system loses the degree of freedom essential to stability. With due regard for the nonzero value of τ, one can derive that $I_m = I_c$ if

$$R < R_c = \left(\frac{8w\tau}{4+w}\right)^{1/2} R_0 \ll R_0\,.$$

If the wire is clad with a normal metal layer having a thickness $d > d_c$, then at $r = 1$ the boundary condition $\varepsilon'(1) = 0$ is replaced with $\varepsilon(1) = 0$. Accordingly, we have instead of Eq. (4.21)

$$N_0(\sqrt{\beta_c})J_0(\delta\sqrt{\beta_c}) - N_0(\delta\sqrt{\beta_c})J_0(\sqrt{\beta_c}) = 0\,.$$

The dependence of I_m/I_c upon R/R_0 is shown in Fig. 11 (curve 2). Plotted in the same figure are the experimental points obtained by Lange and Verges (1974); no fitting parameters were used. As distinct from the case of uncoated wire, at $I_m = I_c$, the value of R_c/R_0 stays on the order of unity: $I_m = I_c$ if $R < R_c = 2.4R_0$. Thus for $d > d_c$ the maximum radius of a wire with current I_c is 1.6 times less than in the case shown in Fig. 10(a) ($I = 0$).

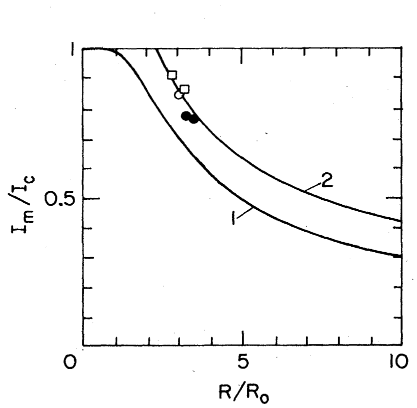

FIG. 11. Maximum transport current in the wire as a function of the ratio R/R_0: Curve 1, uncoated wire; curve 2, wire clad with normal metal to a thickness of $d > d_c$. The points show experimental data from Lange and Verges (1974).

D. Stability at high τ

1. Plane plate in an external parallel magnetic field

We shall now continue to study the simplest example, that of a plane plate with $I = 0$ (see Fig. 5). The dispersion equation for the case $\Delta H = H_p$ has already been derived above [Eq. (4.11)]. As in the case $\tau \ll 1$, it can be solved numerically. Shown in Fig. 12 is the dependence of β_c upon τ at different w, found with the aid of numerical solution of this equation (curves A_1, B_1) (Kremlev *et al.*, 1977). For $\tau \gg 1$ the solution to the problem can be obtained analytically (Kremlev *et al.*, 1977; Maksimov and Mints, 1979).

As already noted in Sec. III, for $\tau \gg 1$ the values of λ at which instabilities (flux jumps, oscillations) develop are such that $|\lambda| \ll 1$. Moreover, if the external heat removal is not too low ($w\tau \gg 1$), then $|\lambda|\tau \gg 1$. This conclusion is supported by an analysis of Eq. (4.11). In the case $w\tau \ll 1$ (adiabatic limit), $|\lambda|\tau \ll 1$. Accordingly, as follows from the results presented in Sec. IV B, if $\tau > \tau_c \ll 1$ then $\beta_c \to 3$, $\lambda_c \to 0$.

For $w\tau \gg 1$ an analytical solution to the problem can be obtained in two cases:

(1) $w \gg 1$ (ideal cooling limit); the dispersion equation for λ has the form

$$\lambda^{3/2} - \lambda^{1/2}\pi[(\beta/\tau)^{1/2} - \pi/2] + \pi^2/2\tau^{1/2} = 0\,,$$

whence

$$\beta_c = \tfrac{1}{4}\pi^2\tau(1 + 2.2\tau^{-1/3}),\quad \lambda_c = \left(\frac{\pi}{2}\right)^{4/3}\tau^{-1/3},$$
$$\beta_0 = \tfrac{1}{4}\pi^2\tau(1 + 0.9\tau^{-1/3}),\quad \lambda(\beta_c) = 1.26i\lambda_c\,. \qquad (4.22)$$

(2) $\tau^{-1} \ll w \ll 1$; the dispersion equation has the form

$$\lambda^{3/2}\tau - (\beta - w\tau)\lambda^{1/2} + w\tau^{1/2} = 0\,,$$

whence

$$\beta_c = w\tau[1 + 1.9(w\tau)^{-1/3}],\quad \lambda_c = \left(\frac{w^2}{4\tau}\right)^{1/3},$$
$$\beta_0 = w\tau[1 + 0.8(w\tau)^{-1/3}],\quad \lambda(\beta_0) = i1.26\lambda_c\,. \qquad (4.23)$$

Note that it is this particular case that is characteristic

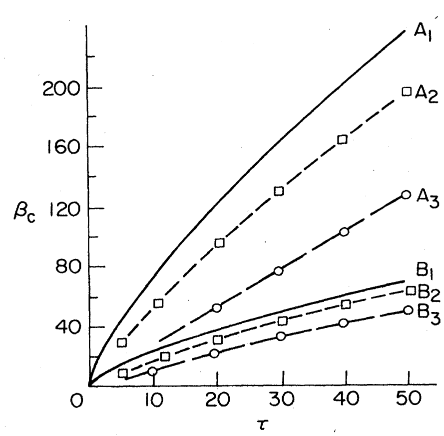

FIG. 12. β_c as a function of τ at high τ: Curves A, $w \gg 1$; curves B, $w = 1$; A_1, B_1, numerical calculation; A_2, B_2, calculated from Eqs. (4.22) and (4.23), respectively; A_3, B_3, "dynamic" approximation (Hart, 1968).

of liquid-helium-cooled superconducting composites.

The dependences β_c for (4.22) and (4.23) are shown in Fig. 12 (curves A_2 and B_2). As seen from the figure, their accuracy is within 20% as early as $\tau = 10$. Also plotted in the figure are the values of β_c corresponding to the "dynamic criterion" approximation [see Eqs. (3.14) and (3.15)], i.e., $\beta_c = (\pi^2/4)\tau$ ($w \gg 1$), $\beta_c = w\tau$ ($w \ll 1$). In this approximation, the value of β_c differs from the exact value by a factor of about two, even at $\tau = 50$. One can analogously investigate the case of incomplete penetration of the magnetic flux into the sample (Maksimov and Mints, 1980).

2. Simplified theory

Inasmuch as for $\tau \gg 1$ and $w\tau \gg 1$ the values of λ_c are such that $\lambda_c \ll 1$ and $\lambda_c \tau \gg 1$, the characteristic time of the flux jump lies in the range of $t_\kappa \ll t_j \ll t_m$. This fact helps considerably to simplify the procedure for determining the stability criterion. The condition of $t_j \ll t_m$ implies that the instability rises under conditions of frozen-in magnetic flux (see Sec. III.B). In this case, the local current density does not vary: $\partial j/\partial t = 0$; hence the relationship between field and temperature perturbations has the form of Eq. (3.13). By substituting into the heat equation and using dimensionless variables, one can readily derive

$$\nabla^2\theta + (\beta/\tau - \lambda)\theta = 0. \tag{4.24}$$

This equation was derived from qualitative considerations by Hart (1968). Thermal boundary conditions should apparently be imposed on Eq. (4.24). The existence of eigensolutions to such a problem with $\lambda > 0$ would mean a loss of stability. Equation (4.24) can also be obtained directly from (4.3) by passage to the limit $\lambda \to 0$, $|\lambda|\tau$, $\tau \to \infty$.

Consider now the case of a plane sample with transport current I (Fig. 8). When this case is used as an example, it reveals a number of common properties of a developing instability at $\tau \gg 1$. We shall first study the case of complete penetration [Figs. 8(a), 8(b), 8(d)–8(f)]. Then, making use of the solution of Eq. (4.24) and arbitrary thermal boundary conditions, one can readily derive for β_c:

$$\tan\left(\frac{\beta_c}{\tau}\right)^{1/2} = w\left(\frac{\tau}{\beta_c}\right)^{1/2}. \tag{4.25}$$

Note that, inasmuch as θ and θ' are continuous on the surfaces of discontinuity of j while $\beta = \text{const}$, there is no need here to solve Eq. (4.24) independently in each continuity region. In Fig. 13, $\sqrt{\beta_c/\tau}$ is plotted versus w. Thus, using the given approximation, we find the stability criterion in the main approximation in $\tau \gg 1$ (dynamic criterion). Accordingly, $\beta_c = \beta_0$, and the perturbation spectrum cannot be studied in this approximation.

It follows from Eq. (4.25) that the parameter β_c is independent of transport current. In the case $\tau \ll 1$, the instability develops independently in each of the superconducting regions differing in the direction of the current. For $\tau \gg 1$ the instability develops instantly throughout the entire sample inasmuch as $t_j \gg t_\kappa$ and heat contact is established between the different regions of the superconductor. Therefore, in the main approximation with respect to $\tau \gg 1$, the stability depends only upon the full size of the regions where $j \neq 0$ and, consequently, heat release takes place upon the emergence of perturbation (Kremlev *et al.*, 1976a, 1977).

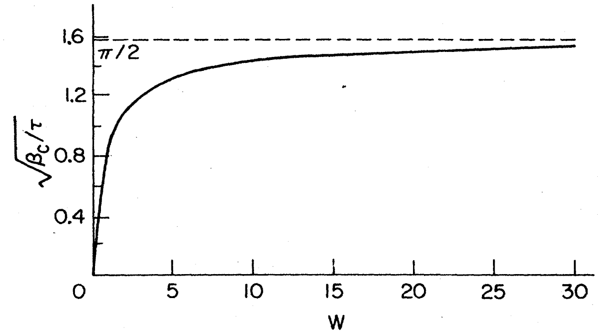

FIG. 13. Dependence of the ratio $\sqrt{\beta_c/\tau}$ upon w.

The dependence of the stability criterion on transport current becomes clear when one takes into account the terms of the next order with respect to τ. Thus, in the case of a plane plate, one can readily derive with the aid of the fourth-order Eq. (4.6) (at $I = I_c$ and $\tau^{-1} \ll w \ll 1$)

$$\beta_c = w\tau(1 + 2/\sqrt{\tau}), \quad \lambda_c = w/\sqrt{\tau},$$

$$\beta_0 = w\tau, \quad \lambda(\beta_0) = i\lambda_c.$$

On comparing the latter expression with Eq. (4.23), one can easily see that the ratio of the β_c values in these two cases equals $\sim 1 + 1.9/(w\tau)^{1/3}$ (since $w \ll 1$). Let $w = 0.1$ and $\tau = 10^2$; then the β_c values differ from each other by about 100%. On the other hand, in the case under consideration, the critical thickness of the sample or its maximum field difference is proportional to $\beta_c/w\tau$. Then the difference between these values may reach approximately 100%. This difference becomes less pronounced only at $\tau > 10^3$. Therefore, for liquid-helium-cooled composites, the criterion of "dynamic stabilization" may only be a good approximation if $\tau > 10^3$.

Within the bounds of the simplified theory, the stability criterion for the case of incomplete penetration has the form:

$$\beta < \beta_c,$$

where

$$\tan\left[\left(\frac{\beta_c}{\tau}\right)^{1/2}\left(\frac{I}{I_c}\right)\right] = w\left(\frac{\tau}{\beta_c}\right)^{1/2}.$$

Equation (4.25) reduces to the above if the half-thickness of the sample, b, in Eq. (4.25) is replaced by the size of the region with current ($b \to b \cdot I/I_c$).

3. Adiabatically heat-insulated samples

The case of $w = 0$ deserves to be discussed separately because, in this limit, $w\tau = 0$ and the $|\lambda|\tau \gg 1$ approximation is inapplicable. Consider a plane sample in an external field. If $\Delta H = H_p$, the problem has already been solved above: for $\tau > \frac{1}{21}$, $\lambda_c = 0$, $\beta_c = 3$, i.e., in the $\tau \gg 1$

limit, $\beta_c/\tau = 3/\tau \to 0$ which, in the main approximation, corresponds to the "dynamic criterion" at $w=0$.

If $\Delta H < H_p$, heat transfer inside the sample becomes substantial. In the approximations of the simplified theory this heat transfer is equal to zero. Therefore, in order to get the correct answer, one should consider the dynamics of the perturbations. Appropriate results have been obtained by Maksimov and Mints (1980). If $\tau > \tau_c = \frac{5}{6}(I_c/I)^2 - \frac{9}{7}(I_c/I) + \frac{1}{2}$, the $\lambda_c = 0$ and $\beta_c = 3(I_c/I)^3$. If the sample is sufficiently thick such that $\tau \ll \tau_c$, the appropriate calculation yields:

$$\beta_c/\tau = 3.8\tau^{-1/2}(I_c/I)^2, \quad \lambda_c = 2.5 I_c^2/I^2\tau .$$

Let us now evaluate $(\beta_c/\tau)^{1/2}$ numerically. Let $I_c/I = 10$ and $\tau = 100 < \tau_c$. Then $(\beta_c/\tau)^{1/2} \simeq 6$ at $w=0$. If the same sample is cooled with liquid helium ($w=0.1$) then, using the simplified theory, we obtain the estimate $(\beta_c/\tau)^{1/2} = 10$, i.e., a value of the same order of magnitude as at $w=0$.

4. Cylindrical samples

Using Eq. (4.24), one can study the stability of cylindrical samples. In the case of a wire with preset current [Fig. 10(b)], Eq. (4.24) may be written ($r > \delta$)

$$\theta'' + \frac{1}{r}\theta' + \frac{\beta}{\tau}\theta = 0 .$$

Its solution,

$$\theta = c_1 J_0[r(\beta/\tau)^{1/2}] + c_2 N_0[r(\beta/\tau)^{1/2}] ,$$

should be joined with the solution for θ in the region of $r < \delta$, using the continuity of θ and θ'

$$\theta' + \frac{1}{r}\theta = 0 ,$$

and substituted in the thermal boundary conditions. On so determining the parameter β_c, one can find, for example, the maximum (quenching) transport current in the wire, I_m. The equation for I_m has the form (Kremlev *et al.*, 1977)

$$N_1(R\delta_m/R_0)[wJ_0(R/R_0) - (R/R_0)J_1(R/R_0)] = J_1(R\delta_m/R_0)[wN_0(R/R_0) - (R/R_0)N_1(R/R_0)]$$

$$\delta_m = \left(1 - \frac{I_m}{I_c}\right)^{1/2}$$

where $I_c = \pi R^2 j_c$, $(R/R_0)^2 = \beta/\tau$. The R/R_0 dependence of I_m/I_c for $w \gg 1$ is shown in Fig. 14.

At $R < R_c$ (see Fig. 14), the stability is not disturbed even if $I = I_c$. At $I \to I_c$, $\delta \to 0$. Then, by passage to the limit $\delta_m \to 0$ from Eq. (4.26) we obtain (Kremlev *et al.*, 1977)

$$wJ_0(R_c/R_0) - (R_c/R_0)J_1(R_c/R_0) = 0 . \qquad (4.27)$$

The w dependence of R_c/R_0 is shown in Fig. 15. $R_c/R_0 \to 0$ at $w \to 0$ and $R_c/R_0 \simeq 2.4$ at $w \gg 1$. The accuracy of these results is the same as in the case of a plane geometry, $\sim(w\tau)^{-1/3}$.

5. Superconducting composites

As has already been noted, the limit $\tau \gg 1$ is realized when describing superconducting composites using the

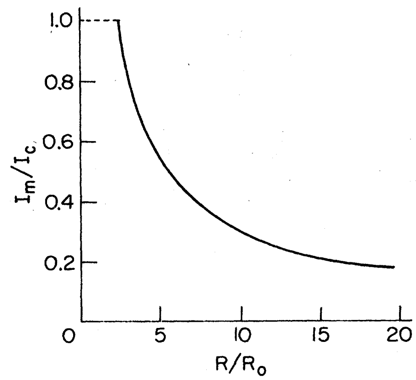

FIG. 14. Ratio I_m/I_c as a function of R/R_0 for $w \gg 1$.

effective medium model (see Secs. II and III). We shall now discuss the limits of applicability of such an approach to stability studies, as well as some specific problems associated with the superconducting state stability in composites.

For the effective medium model to describe adequately the properties of the composite, the following standard conditions must be met:

(1) The total number of individual superconducting elements, N, should be large, i.e., $N \gg 1$;

(2) The instability rise time t_j should be much greater than the corresponding relaxation times in a single element of the structure;

(3) The variables (θ, ε, etc.) should not vary strongly on a scale comparable with the dimensions of a unit element of the structure.

Let us now write conditions (2) and (3) in algebraic form, for example, in application to multifilament composites (Kremlev *et al.*, 1976a, 1977). The smallest scale on which, for $\tau \gg 1$, the solutions to Eqs. (4.3) vary, is about equal to $L/(|\lambda|\tau)^{1/2}$, where L is some characteristic dimension of the sample (for example, its thickness, radius, or the like). Inasmuch as in the region of parameters of interest to us $|\lambda| \sim \lambda_c$, while the characteristic dimension of a unit structural element is $\sim L/(N)^{1/2}$, the inequality $L/(N)^{1/2} \gg L/(\lambda_c\tau)^{1/2}$ should be satisfied, or

$$N \gg \lambda_c \tau . \qquad (4.28)$$

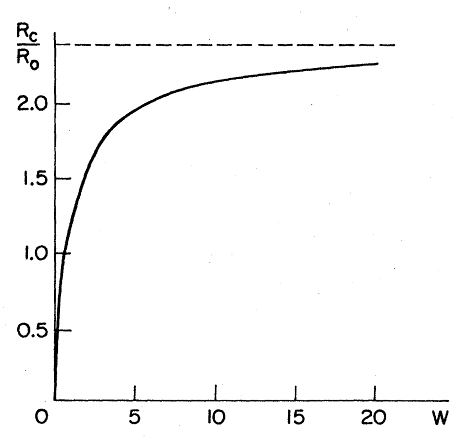

FIG. 15. Ratio R_c/R_0 as a function of w.

In a single element of the structure, there are two different magnetic diffusion times and two thermal relaxation times, namely, t_{mn} and t_{ms}—the characteristic times for magnetic diffusion in a normal metal and in a superconductor, respectively, on the scale of $L/(N)^{1/2}$, and $t_{\kappa n}$ and $t_{\kappa s}$—the characteristic times for thermal diffusion on the same scale. Accordingly $t_{mn} \gg t_{ms}$ and $t_{\kappa n} \ll t_{\kappa s}$. As can be readily seen, the value of $L/(\lambda_c \tau)^{1/2}$ is of the order of the skin depth at a frequency $\sim t_j^{-1}$. Therefore, the condition (4.28) is equivalent to the requirement that $t_j \gg t_{mn}$. In order to fully satisfy the third condition, it is necessary that $t_j \gg t_{\kappa s} \sim \nu_s x_s b^2/\kappa_s N$, or

$$N \gg \lambda_c \frac{x_s \nu_s \kappa}{\nu \kappa_s} . \tag{4.29}$$

Therefore the stability of a superconducting composite can be studied with the aid of averaged equations if $N \gg 1$ and the conditions (4.28) and (4.29) are met. In particular, for the case of a plane plate discussed above, these conditions take the form [see Eqs. (4.22) and (4.23)]

$$N \gg \tau^{3/2}, \quad N \gg \frac{\kappa}{\kappa_s} \frac{\nu_s x_s}{\nu} \tau^{-1/3} \quad (w \gg 1)$$
$$N \gg (w\tau)^{2/3}, \quad N \gg \left(\frac{w^2}{\tau}\right)^{1/3} \frac{\kappa}{\kappa_s} \frac{\nu_s x_s}{\nu} \quad (w \ll 1, \quad w\tau \gg 1) . \tag{4.30}$$

Here N is the number of filaments in a $b \times b$ square. At $w\tau \to 0$ and $N \gg 1$, the effective medium model is always applicable because $\lambda_c \to 0$.

Let us now estimate the right-hand sides of inequalities (4.30) for the case of a liquid-helium-cooled composite. In this case, one can assume that $w = 0.1$, $\tau = 10^2$, $\nu_s x_s/\nu \sim 1$, $\kappa_n/\kappa_s = 10^3$. Then it follows from (4.30) that $N \gg 10$.

The material from which the winding of a modern superconducting system is manufactured has a rather complex structure. A detailed description of such materials can be found, for example, in the book by Brechna (1973) and in the specialized literature. The primary element of a composite material is a superconducting wire including, as a rule, a normal-metal matrix and several hundred or thousand superconducting filaments embedded in it and having a diameter of from several microns to several dozens of microns. The wire diameter d is, as a rule, on the order of 1 mm. The wire is twisted about its axis at some twisting pitch L_p which, for technological reasons, is always greater than the wire perimeter πd. Several wires make up a structural unit, and so on. In a number of cases, for increasing thermal stability, channels are provided in the composite material via which liquid helium is supplied to the sample volume. Depending on the purpose of the magnet system, composite material of one or another structure is selected for its manufacture.

Consider first the stability of a single multifilament wire. In a nontwisted composite the magnetic-flux distribution is the same as in a solid hard superconductor (provided we neglect inessential details of flux distribution in a single filament). All filaments act collectively to screen the magnetic field. An instability occurs as a collective effect throughout the entire system of filaments even if each one of the filaments taken separately is stable. It is to this case that the results obtained above correspond. Consider now a twisted wire in an external variable magnetic field transverse to the wire axis [see the geometry shown in Fig. 10(a)]. Owing to symmetry, there are two regions in the wire, differing from each other by the direction of current, namely, $y > 0$ and $y < 0$. As the current moves along the axis of the wire, a filament goes alternately from one region to the other. As a result, the current cannot flow in the superconductor above but flows from filament to filament through the normal matrix. As can be readily understood, the amount of screening in a twisted conductor at a low rate of external field variation is much smaller than in a nontwisted one. Accordingly, the magnetic flux is distributed over the cross-sectional area almost uniformly, i.e., $H = \text{const} = H_a(t)$, while the magnetic field difference occurs only in single filaments:

$$\Delta H \sim r_0 j_c / c ,$$

where r_0 is the filament radius. No collective instability occurs. In order to ensure stability, it is sufficient to use adequately small filaments (Wilson *et al.*, 1970).

However, if a current $I \sim I_c$ flows in the sample, or the external field varies rapidly (Wilson *et al.*, 1970),

$$\dot{H}_a > \dot{H}_0 = \frac{c^2(H_a - H_0)}{2\pi L_p^2 \sigma}$$

twisting does not lead to the elimination of collective screening of the external field and, consequently, of a collective instability. Since, in actual conditions, a composite is always loaded with the current $I \sim I_c$ while $2\pi R/L_p \ll 1$, the foregoing results for $\tau \gg 1$ are applicable to the study of stability in twisted multifilament composites.

The stability criteria obtained in Secs. IV.D.1 and IV.D.2 of this section help us to find restrictions on the current density in the composite associated with its dimensions, as well as the critical number of filaments, N_c, at which the composite is unstable as a whole. For example, using Eq. (4.23), we find for N_c (Kremlev *et al.*, 1977):

$$N_c \simeq 0.2\tau w \frac{\nu}{\nu_s x_s} \left(\frac{r_c}{r_0}\right)^2 [1 + 1.9(w\tau)^{-1/3}], \quad \tau^{-1} \ll w \ll 1 .$$

Here,

$$r_c = 2.4c[\nu_s / 4\pi j_c \,|dj_c(T)/dT|\,]^{1/2}$$

is the critical radius of a single filament. Let us take for the purpose of estimation $\tau = 10^3$, $w = 0.1$, $\nu_s x_s/\nu = 1$; then we find for N_c: $N_c \sim 10^2 (r_c/r_0)^2$. Note that, inasmuch as $I_s \propto N r_0^2$, a reduction in the filament radius r_0 does not result in an increased current density (however, the use of very fine filaments may turn out to be useful for reducing losses in the composite).

From the condition $\beta/\tau < w$, we write the restriction on the current density:

$$j_s \lesssim \left(\frac{W_0 \sigma_{\parallel} T_1}{b}\right)^{1/2}, \quad T_1 = j_c \left|\frac{dj_c}{dT}\right|^{-1} . \tag{4.31}$$

For characteristic values of the parameters, $j_s \lesssim 10^5$

A cm^{-2}.

After the transition to the next structural level, one can again make use of stability criteria such as those found in Secs. IV.D.1 and IV.D.2. The only requirement is that one replace the appropriate averaged parameters. If a superconducting material is provided with channels for internal helium cooling, the stability criterion assumes, in the main approximation $\tau \gg 1$, the form (Mints and Rakhmanov, 1980)

$$\beta/\tau < w + w_i , \tag{4.32}$$

where $w_i = W_0(2x_H b^2/\kappa_\perp a)$, a is the radius of the cooling channels, and x_H is the fraction of the cross-sectional area taken by these channels ($x_n + x_s + x_H = 1$). In deriving the criterion (4.32), the rate of helium movement in the channels was assumed to be adequately high: $V \gg V_c = W_0 L_z/\nu_H a$, where ν_H is the heat capacity of helium and L_z is the longitudinal (along the velocity) dimension of perturbed region. If $V \ll V_c$, the internal cooling channels fail to ensure an increase in stability relative to collective perturbations. It is easy to obtain (Mints and Rakhmanov, 1980) an optimum value of $x_H \sim 0.5$. Then, inasmuch as $b/a \gg 1$, we have the following restriction for j_s:

$$j_s \lesssim \left(\frac{W_0 \sigma_\parallel T_1}{a}\right)^{1/2}. \tag{4.33}$$

Since usually $\sigma_\parallel \sim \sigma_n$, $a \sim 10^{-1}$ cm, j_s does not exceed 10^4–10^5 A cm^{-2}.

However, the foregoing rigid restrictions upon j_s are not obligatory for successful operation of a superconducting system. Indeed, as already pointed out in Sec. III, for the instability under consideration to occur, an initial perturbation is required covering the entire cross section of the sample and having a longitudinal dimension bigger than the transverse one. On the other hand, all of the estimates have been made for the region of the linear current-voltage characteristic of the composite where $\sigma_\parallel \sim \sigma_n$. In the region of low fields E, the effective conductivity $\sigma_\parallel$ may turn out to be considerably higher than σ_n. If we have in mind a scale on the order of the dimensions of a single filament, such perturbations are hard to eliminate, while sufficiently fine filaments should be used for stabilization without loss in current density. There are considerably fewer reasons for perturbations to occur on the scale of an entire winding or considerable portion thereof, and such perturbations are technically easier to eliminate. Nonetheless stabilization of the entire winding appears impractical because, in order to reach even the limit (4.33), helium must be pumped through at a comparatively fast rate. Indeed, since $\nu_H \simeq 10^7$ erg cm^{-3} K^{-1}, $V_c \sim L_z s^{-1}$, while L_z is the length of cable in the winding, i.e., dozens of meters. In the case $V < V_c$, Eq. (4.33) will be replaced with the restriction (4.31), where b is the characteristic dimension of the winding. If $b \sim 10$ cm, j_s decreases by an order of magnitude. For these reasons, in practice one must totally stabilize the superconducting structural element, say, of the radius R and preclude the possibility of strong perturbations of the scale larger than R.

V. THE MAGNETIC FIELD DEPENDENCE OF THE CRITICAL CURRENT DENSITY AND STABILITY CRITERION

The magnetic field dependence of the critical current density may affect the stability criterion in two ways. First, the dependence of j_c upon the local value of the magnetic field may turn out to be substantial. This case cannot be reduced to those considered above and must be discussed separately. However, as will be shown below, this possibility is only realized in a limited range of magnetic fields. Second, even if the dependence of j_c upon the local value of H is insubstantial, the external magnetic field value H_a is incorporated as a parameter in the stability criteria obtained above. This case lends itself to a rather simple analysis.

When j_c is a function of the local field H, the coefficients of linear equations for ΔT and E perturbations become functions of coordinates. One cannot find their exact solutions in the general case even if use is made of the $\tau = 0$ or $\tau = \infty$ approximations. There are two possible ways of carrying out a stability study: (1) by presetting a specific model dependence $j_c(H)$ and solving, by one method or another, the equations for small perturbations (see Wipf, 1967; Yamafuji et al., 1969; Takeo, 1971; Morton and Darby, 1973; Kremlev *et al.*, 1976b) or, (2) by trying, under some general assumptions, to find an approximate solution in the case when the function $j_c(H)$ has arbitrary form (Mints and Rakhmanov, 1976b, 1977a, b). We shall take the latter approach.

We derive the equation for the perturbation of the electric field E in the $\tau = 0$ approximation for the case of plane geometry. Taking into account the dependence of j_c upon H, the derivative $\partial j_c/\partial t$ is written as

$$\frac{\partial j_c}{\partial t} = \frac{\partial j_c}{\partial T}\dot{T} + \frac{\partial j_c}{\partial H}\dot{H}.$$

Using the Maxwell equation curl $\mathbf{E} = -\dot{\mathbf{H}}/c$ analogously with Eq. (4.14), we derive for $\varepsilon(\tilde{x})$:

$$\varepsilon'' + \alpha(\tilde{x})\varepsilon' + \beta(\tilde{x})\varepsilon = 0 ,$$
$$\alpha(\tilde{x}) = -\frac{4\pi b}{c}\frac{\partial j_c}{\partial H}, \quad \beta(\tilde{x}) = \frac{4\pi b^2 j_c}{c^2 \nu}\left|\frac{\partial j_c}{\partial T}\right| , \tag{5.1}$$

with the boundary conditions of Eq. (5.1) being obviously $\varepsilon'(\pm 1) = 0$ and $\varepsilon = 0$ at points where $j = 0$. The stability criterion is found as usual.

Now we transform Eq. (5.1) to a more convenient form. The dependence $H(\tilde{x})$ is found from the equation

$$\frac{dH}{d\tilde{x}} = \frac{4\pi b}{c} j_c(H) , \tag{5.2}$$

whence

$$d\tilde{x} = \frac{c\,dH}{4\pi b j_c(H)} .$$

Let us introduce a new variable,

$$y = \frac{H_a - H(\tilde{x})}{\Delta H} , \tag{5.3}$$

where ΔH is the magnetic field difference in the given region. Using Eqs. (5.2) and (5.3), one can readily derive from Eq. (5.1)

$$d^2\varepsilon/dy^2 + \tilde{\beta}\varepsilon = 0\,, \qquad (5.4)$$

where

$$\tilde{\beta} = \frac{(\Delta H)^2}{4\pi\nu T_1(H)}, \quad T_1(H) = \frac{j_c}{|\partial j_c/\partial T|}\ .$$

The boundary conditions for $\varepsilon(y)$ are now written as

$$\varepsilon(\pm 1) = 0, \quad \left.\frac{d\varepsilon}{dy}\right|_{y=0} = 0\,. \qquad (5.5)$$

Note here that if $j_c(T,H)$ has the form $j_c = j_0(T)\cdot f(H)$, T_1 does not depend on H and, consequently, does not depend on y. Equation (5.4) may then be solved exactly and the stability criterion has, as before, the form

$$\Delta H < H_j = (\pi^3 \nu T_1)^{1/2}\,.$$

In many cases, the dependence $j_c(H)$ may be represented as $j_c = \alpha(T)/[B_0(T) + H]^{\gamma}$ or $j_c = j_1(T)(1 - H/H_{c_2})^{\delta}$. If $H \gg B_0$ or $H \ll H_{c_2}$, then j_c is written as the product $j_0(T)\cdot f(H)$.

In order to solve Eq. (5.4) in the more general case use can be made of the WKBJ method. Using the standard procedure (see Heading, 1962), one can readily derive the stability criterion in the form

$$\int_0^1 \tilde{\beta}^{1/2} dy = (4\pi\nu)^{-1/2} \int_{H_0}^{H_a} T_1^{-1/2}(H)\,dH$$

$$= \int_{l/b}^{1} [\beta(\tilde{x})]^{1/2} d\tilde{x} < \frac{\pi}{2} \qquad (5.6)$$

(it is assumed here that the current flows in a single direction at $l/b < \tilde{x} < 1$).

The accuracy of the derived criterion can be estimated as

$$\varepsilon \sim \frac{1}{\pi^2}\,\Delta H^2 \left(\frac{1}{T_1}\frac{\partial T_1}{\partial H}\right)^2 .$$

For Eq. (5.6) to be applicable ε must satisfy the condition $\varepsilon \ll 1$. This condition is observed throughout the entire range of magnetic fields, with the exception of the immediate vicinity of the upper critical field H_{c_2}. Indeed, for $H \ll H_{c_2}$, $T_1^{-1}\,\partial T_1/\partial H \sim H_{c_2}^{-1}$. If $H \simeq H_{c_2}$, then j_c can be represented as

$$j_c = j_1(T)\left(1 - \frac{H}{H_{c_2}}\right). \qquad (5.7)$$

Then $T_1 \sim T_c(1 - H/H_{c_2})$ and $\varepsilon \sim (1/\pi^2)[H/(H_{c_2} - H)]^2$. Thus $\varepsilon \sim 1$ only if $H_{c_2} - H_a \sim H/\pi \sim 1$ kOe. For the dependence $j_c(H)$ having the form of Eq. (5.7), the problem has an exact solution (Kremlev *et al.*, 1976b).

Therefore, with the exception of a limited vicinity near H_{c_2}, the stability criterion can be obtained either in Bean's model or by the WKBJ method. A simple qualitative analysis of these criteria shows that, in the case of near-planar geometry in the range of fields not too close to H_{c_2}, the dependence of H_j on j_c and, consequently, on H_a is small. The paper by Hancox (1965) contains a convincing experimental proof of this assertion. It has been shown that upon a threefold decrease of j_c (H_a varying from 5 to 30 kOe) the value of H_j only varied by 5%. The same result was obtained by a number of authors in experiments involving other superconducting materials (Leblanc and Vernon, 1964; Borovik *et al.*, 1965; Lange, 1965; Neuringer and Shapira, 1966) and confirmed repeatedly in more recent papers.

If the sample size is less than some critical size

$$b < b_c = \frac{c}{4}\left(\frac{\pi\nu}{j_c\,|\partial j_c(T)/\partial T|}\right)^{1/2},$$

no flux jump occurs in the sample, i.e., for $b < b_c$ the maximum field difference H_p is less than H_j. Obviously, $b_c \propto (j_c\,|\partial j_c(T)/\partial T|)^{1/2}$ may depend quite strongly on H_a. For example, in one and the same sample flux jumps may occur in the low field region and disappear in high fields. This fact has been observed repeatedly in experiments (see Zebouni *et al.*, 1964). It is worthy of note that flux jumps may again occur at $H \sim H_{c_2}$ because of the growth of $|\partial j_c/\partial T| \propto (1 - H/H_{c_2})^{-1}$.

Consider now the case $\tau \gg 1$ in the "dynamic" approximation. Here, the equation for temperature perturbation θ remains unchanged, $\lambda_c = 0$, but the parameter β is a function of the coordinates. In the plane geometry case under consideration we derive

$$\theta'' + \frac{\beta(\tilde{x})}{\tau}\,\theta = 0\,.$$

When solving this equation by the WKBJ method, one finds that the stability criterion has the form [$\Delta H = H_p$; see Eq. (4.25)]:

$$\tan\left[\int_0^1 \left(\frac{\beta(\tilde{x})}{\tau}\right)^{1/2} d\tilde{x}\right] < \left.\frac{w}{[\beta(\tilde{x})/\tau]^{1/2}}\right|_{\tilde{x}=1}.$$

The accuracy of this approximation

$$\varepsilon \sim \frac{\Delta H^2}{\pi^2}\left(\frac{1}{T_1}\frac{\partial T_1}{\partial H}\right)^2$$

is the same as in the case $\tau \ll 1$.

A comparison of the stability criteria obtained in Bean's model and by the WKBJ method shows that, at least for a qualitative analysis of the situation, use can be made of the approximation $j_c = j_c(H_a)$, or it can be assumed that $j_c = j_0(T)\cdot f(H)$.

It should also be noted that one can solve the problem of critical state stability analogously when the nonuniformity of the superconductor properties is due to other reasons (nonuniformity of the sample material, nonuniform temperature background, etc.).

VI. COMPARISON OF THEORY AND EXPERIMENT

A. General comments

Quite a few papers have been devoted to the experimental study of flux jumps. The time dependence of the magnetization, voltage, temperature, ultrasonic attenuation, and other characteristics of the sample were determined in the course of the experiments. In some papers, several quantities were measured simultaneously (Claiborne and Einspruch, 1966; Neuringer and Shapira, 1966). Various measuring techniques were used. However, we shall not dwell in the present review on experimental details.

An experiment involving the study of flux jumps is usually staged in the following manner. A superconductor is placed in an external magnetic field whose value rises (decreases) or oscillates at some amplitude about a preset value of H_0. When some external field value is reached, instability occurs in the superconductor. Accordingly, the temperature and electric field rise in an avalanche manner, the magnetization jumps as well as ultrasonic attenuation, etc. Shown in Fig. 16 are characteristic time dependences for the aforementioned quantities, taken from the literature [Claiborne and Einspruch, 1966, Fig. 16(a); Neuringer and Shapira, 1966, Fig. 16(b); Chikaba, 1970, Fig. 16(c)]. In Fig. 16(b) and 16(c), one can see clearly the electric field and temperature oscillations preceding a magnetic-flux jump. Shown in Fig. 17 is the dependence of hysteresis losses q_h in Nb-Ti-Zr wires of various diameters d on the amplitude of the external magnetic field $H(t)=H_m\cos(\omega t)$. If $H_m<H_j$, $q_h \propto H^n$, where $n\sim 3$–4. At $H_m>H_j$, flux jumps occur in the sample and q_h increases sharply. If the wire diameter is sufficiently small that $H_p<H_j$, no instability occurs, as well as no sharp increase of losses (see bottom curve in Fig. 17). As can be seen from the foregoing data, the presence of instability is reliably registered in the course of an

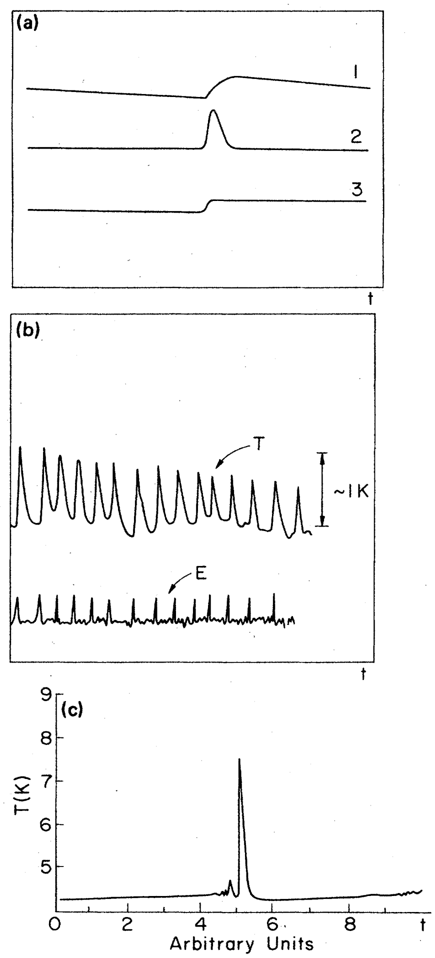

FIG. 16. Time dependences of physical quantities during a flux jump. (a): 1, sound absorption coefficient; 2, temperature; 3, magnetic flux exclusion (Claiborne and Einspruch, 1966). (b) temperature and electric field (Neuringer and Shapira, 1966). (c) temperature (Chikaba, 1970).

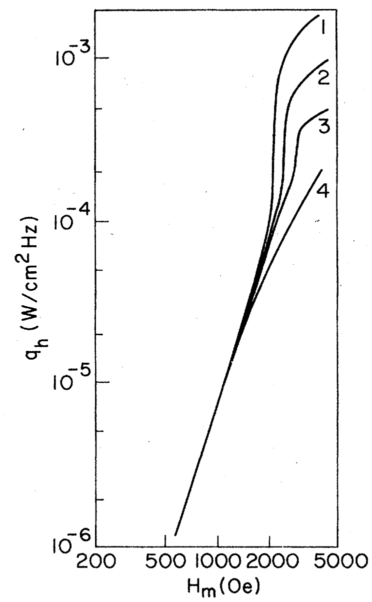

FIG. 17. Hysteresis losses in Nb-Zr-Ti wires of different diameters as a function of the amplitude of the variable magnetic field at a frequency of 35 Hz (Shiiki and Kudo, 1974): Curve 1, wire diameter of 508 μm; curve 2, wire diameter of 250 μm; curve 3, wire diameter of 101 μm; curve 4, wire diameter of 45 μm.

experiment by sharp variations in a whole series of macroscopic parameters of the superconductor.

A number of authors (Goodman and Wertheimer, 1965; Keyston and Wertheimer, 1966; Wertheimer and Gilchrist, 1967; Harrison *et al.*, 1973, 1974, 1975; Harrison and Wright, 1974) have studied flux jumps by way of "visual" observation using the Faraday effect and high-speed filming.

However, in spite of the availability of numerous experimental papers, most of these papers do not lend themselves to quantitative comparison with theory or with each other. In our opinion, there are two reasons for this. First, there is a delay in the emergence of instability, due to the nonlinear portion of the current-voltage characteristic (this effect has already been discussed above in Sec. III. C and will be studied in detail below). In order to find the value of H_j (or β_c) corresponding to $\sigma=\sigma_f$, one should initiate instability in the course of the experiment. When the value of the initiating influence is insufficient to cause transition to the flux flow mode, the occurrence of a flux jump will depend upon the value of this influence. Most authors fail to take this into account so that an instability in their experiments is caused either by uncontrolled perturbations or by variable external field, current, temperature, or the like. However, even in cases when flux jumps are purposely initiated [for example, by mechanical vibrations (Evetts *et al.*, 1964) or by rapid heating (Sutton, 1973)], we are often unable to say with certainty that the measured value of H_j corresponds to $\sigma=\sigma_f$ since we do not know whether the external influence upon the critical state was sufficiently strong. Thus, according to Sutton (1973), flux jumps were excited in a Nb-50% Ti sample by heating at a rate of $\dot{T}_0 = 0.2-0.3\ \mathrm{K\,s^{-1}}$. According to our estimates (see, Sec. VI. B) based on the data of Wipf and Lubell (1965) for a Nb-25% Zr alloy (quite close to Nb-Ti from the viewpoint of stability), this heating rate is close to the order of magnitude required for the transition of a superconductor to the flux flow mode $\dot{T}_f \sim 1\ \mathrm{K\,s^{-1}}$. In order to find the value of $\dot{T}_f$ more accurately, one should know the current-voltage characteristic of the superconductor in the region of low fields or the dependence $H_j(\dot{T}_0)$. The flux flow mode is attained in a region where H_j ceases to depend on $\dot{T}_0$ (assuming uniform heating of the sample).

Another reason for the difficulties resides in the absence of an adequate set of data on the properties of the samples being investigated. In order to determine the stability criterion and time characteristics of flux jumps and oscillations in the flux flow mode, one should know the following parameters: $j_c(T,H)$, σ_f, ν, κ, W_0, b, as well as the corresponding characteristics of normal cladding, if any. As far as we know, no one has ever conducted such comprehensive studies of sample parameters concurrently with a study of the desired level of excitation of flux jumps. As a rule, one finds in experimental papers only data on the geometric characteristics of the samples and their chemical composition; critical current density values are sometimes provided for a single temperature value. The use of data obtained in other samples for the purposes of comparing theory with experiment cannot guarantee a high accuracy, inasmuch as characteristics of a material depend, some to a greater (j_c, σ_f, E_0) and some to a lesser (ν) extent, on the pretreatment and history of the material. Further, the dynamics of instability development (and, to a lesser degree, the stability criterion for hard superconductors) depend on the value of the surface heat removal coefficient W_0. It is practically impossible, however, to evaluate this parameter from the data provided in most experimental papers. On analyzing and comparing the available literature on the parameters of hard superconductors, one concludes that the maximum accuracy to which characteristics such as H_j, maximum sample thickness, or transport current can be calculated does not exceed several tens of percent. Dynamic characteristics can only be estimated within an order of magnitude. Therefore, when comparing theory and experiment, we shall only refer to the terms of the main approximation in the appropriate expressions.

We cite but one example for the sake of illustration. The papers by Neuringer and Shapira (1966) and Shiiki and Kudo (1974) contain measured values of H_j for Nb-Ti, Nb-Zr, and Nb-Ti-Zr samples. These values are in the 1.5–2.5 kOe range. At the same time, according to Sutton (1973) (single-layer samples), measured values of H_j are in the 5–8 kOe range. Based on the data provided in these papers, one cannot state with assurance the reasons for such a marked difference in the results.

We shall discuss experiments related to oscillation effects in the critical state, as well as to the effect of mechanical processes on stability, in full in the appropriate sections (Secs. VII and IX).

B. Hard superconductors

1. The effect of variable external conditions

As noted above, flux jumps are often studied in a variable external magnetic field. In so doing, numerous authors have observed a regular dependence of the field difference before the flux jump, ΔH_1, upon the rate of external field variation, $\dot{H}_a$ (Rothward *et al.*, 1968; Corsan, 1964; Wipf and Lubell, 1965; Benaroga and Mogenson, 1966; Gandolfo *et al.*, 1966, 1969; Watson, 1966, 1967; Wertheimer and Gilchrist, 1967; Chikaba *et al.*, 1968; McIntruff, 1968; Kwasnitza, 1973; Shimamoto, 1974; Shiiki and Kudo, 1974; Shiiki and Aihara, 1974; Subramangam and Chopra, 1975; Irie *et al.*, 1977). At low $\dot{H}_a$, ΔH_1 drops to assume a constant value at $\dot{H}_a \sim 10^3-10^4\ \mathrm{Oe\,s^{-1}}$ (see Figs. 18 and 19 below). "Disruptions" characterized by a regular dependence of ΔH_1 upon $\dot{H}_a$ are often observed: sometimes a flux jump occurs at a lower value of H_a (see Fig. 18). Moreover, some authors (Neuringer and Shapira, 1966; Harrison *et al.*, 1975) noted the absence of a regular dependence of ΔH_1 on $\dot{H}_a$. Shimamoto (1974) also observed this effect in a number of cases. Marked quantitative deviations have been observed in measured dependences $\Delta H_1(\dot{H}_a)$ in seemingly similar samples. Since the condition $\dot{H}_a \cdot t_j \ll \Delta H_1$ is certain to be met under experimental conditions, the above mentioned deviations cannot be attributed to a difference in the moment of registering the emergence of the flux jump.

All these facts, which appear to be conflicting, find a natural interpretation within the assumption that a variable external magnetic field serves as an instability-initiating perturbation. Indeed, stability grows with an increase of the sample conductivity σ. The conductivity of a hard superconductor decreases with the growth of E and reaches the minimum σ_f (see Fig. 2). A variable external field induces in the sample a background electric field $E_a \propto \dot{H}_a$ and, thereby, the sample conductivity σ becomes a function of $\dot{H}_a$. For this reason, the field difference before the flux jump, ΔH_1, decreases with increasing $\dot{H}_a$ (in this discussion, H_j will denote the field difference at $\sigma = \sigma_f$). If the sample is affected by strong irregular perturbations in the course of experiment, "disruptions" may appear on smooth curves of $\Delta H_1(\dot{H}_a)$ and, at a higher repetition rate of such perturbations, the regular dependence $\Delta H_1(\dot{H}_a)$ may disappear.

Based on the foregoing considerations, one can develop a simple semiquantitative theory for finding the dependence of ΔH_1 on $\dot{H}_a$ (Mints and Rakhmanov, 1979b). Consider a plane hard superconductor sample in an external magnetic field (Fig. 3). Let us find the distribution in this sample of the electric field E_a caused by a variable external magnetic field $\dot{H}_a$. Inasmuch as we are interested in the field region $H_{c_1} \ll H \ll H_{c_2}$, we shall take j_c in the form $j_c = \alpha(T)/H$ (Kim-Anderson model, 1964). Then, using the appropriate Maxwell equation, we derive for $H(x)$ ($j = j_c$):

$$H^2(x) = H_a^2 - \frac{8\pi\alpha(T)x}{c} = H_a^2(1 - x/l)\,. \qquad (6.1)$$

Hence, using the equation $\partial E/\partial x = (1/c)\dot{H}(x)$, one can readily find

$$E_a = -\frac{2\dot{H}_a H(x) l}{c H_a} . \tag{6.2}$$

Expressions (6.1) and (6.2) have been derived under the assumption that heating of the sample due to the variable magnetic field is low: $T - T_0 \ll T_c - T_0$. As can be readily seen, the following conditions must be met for this purpose:

$$\begin{aligned} \dot{H}_a &\ll (8\pi)^3 \kappa (T_c - T_0)(\alpha/c)^2 H_a^{-5} , \\ \dot{H}_a &\ll (8\pi)^2 W_0 (T_c - T_0)(\alpha/c) H_a^{-3} . \end{aligned} \tag{6.3}$$

Using the characteristic values at $T = 4$ K: $T_c - T_0 = 10$ K, $\kappa = 10^3$ erg cm^{-1} K^{-1}, $W_0 = 10^6$ erg cm^{-2} s^{-1} K^{-1}, $H_a = 3 \times 10^3$ Oe, $j_c = 10^5$ A cm^{-2}, we obtain $\dot{H}_a \ll 10^4$ Oe s^{-1}.

The portion of the superconductor contained in the volume $0 < x < x_f$ is in the flux flow mode ($E > E_0$). The value of x_f is found from the condition $E_a(x_f) = E_0$:

$$x_f = l \left(1 - \frac{c^2 E_0^2}{4 \dot{H}_a^2 l^2}\right).$$

By introducing the notation $H(x_f) = H_f$, we find that the magnetic field difference in the region where $E > E_0$ is equal to

$$H_a - H_f = H_a \left(1 - \frac{c E_0}{2 \dot{H}_a l}\right). \tag{6.4}$$

Let us introduce, in the sample region where $E < E_0$, some averaged value of τ equal to $\tau_a > \tau(\sigma_f) = \tau_f$. Assume that $\tau_a \gg \tau_f$ and $\tau_a \gg 1$. At $\tau \ll 1$, a flux jump develops rapidly, $t_j \ll t_\kappa$. However, at $\tau \gg 1$ such rapid perturbations are suppressed by the normal current j_N. Then, in the first approximation in $\tau_a \gg 1$ and $\tau_f \ll 1$, the presence in the superconductor of a region where $E < E_0$ leads to an effective reduction of the volume in which "rapid" perturbations may develop. Consequently, in the absence of "priming" effects other than the variable external field, instability develops in the region $0 < x < x_f$. A portion of the magnetic field difference in the sample, $H_a - H_f$, must be stabilized rather than the overall difference, $H_a - H_0$. Accordingly, the stability criterion has the form

$$H_a - H_f < H_j . \tag{6.5}$$

In the paper by Mints and Rakhmanov (1979b), this statement has been rigorously proven for a plane sample at $w \gg 1$, $\Delta H = H_p$, $\tau_f \ll 1$, and $\tau_a \gg 1$.

The flux flow mode is attained at a current density $j_f = j_1 + \sigma_f E_0$, where $j_1 \sim j_c \sim j_f$. Then E_0 can be conveniently written as

$$E_0 = \frac{j_f - j_1}{\sigma_f} = K(T, H) \frac{j_c}{\sigma_f} ,$$

where $K(T, H) \ll 1$. Using the dependence $j_c(H) = \alpha(T)/H$ and $\sigma_f = \sigma_n H_{c_2}/H$, we finally derive

$$E_0 = K(T, H) \frac{\alpha(T)}{\sigma_n H_{c_2}(T)} . \tag{6.6}$$

The coefficient $K(T, H)$ characterizes the initial stage of the magnetic flux motion and inhomogeneity of pinning centers. It is natural to assume that, in the field region $H_{c_1} \ll H \ll H_{c_2}$, the magnetic field dependence of K is rather weak. Then, by using Eqs. (6.5) and (6.6), one can readily derive that the stability criterion has the following form, depending on the rate of external field variation:

$$\Delta H < \Delta H_1 = \frac{H_j}{2} + \left(\frac{H_j^2}{4} + \frac{A(T)}{\dot{H}_a}\right)^{1/2} , \tag{6.7}$$

where

$$A(T) = \frac{4\pi K(T) \alpha^2(T)}{\sigma_n H_{c_2}(T)} .$$

ΔH_1 reaches the lower limit of H_j at $\dot{H}_a \gg A(T)/H_j^2$.

As follows from Eq. (6.7), at $\dot{H}_a \to 0$, $\Delta H_1 \to \infty$. Actually, the latter value is known to be restricted and reaches some value H_m at $\dot{H}_a = 0$. The value of H_m cannot be found from general considerations since it depends on (1) the portion of current-voltage characteristic at $E \to 0$, which has not yet been sufficiently studied, and (2) uncontrolled perturbations that are always present in the course of experiment. Equation (6.7) could be modified in order to take into account the arrival of ΔH_1 at the limit H_m at $\dot{H}_a \to 0$. However, we shall not do this because it would mean introducing and additional fitting parameter into the theory. The function $\Delta H_1(\dot{H}_a)$ depends naturally on the specific form of the dependence $j_c(H)$; however, its qualitative appearance remains unchanged.

By analogous reasoning, one can find the stability criterion under other types of external influence, for example, external heating of the sample. Thus, in the foregoing case, the dependence of the field difference ΔH_1 before the flux jump on the heating rate $\dot{T}_0$ is found from the equation (Mints and Rakhmanov, 1979b)

$$\frac{2A(T)\alpha(T)}{|\alpha'(T)|} \dot{T}_0^{-1} = (\Delta H_1 - H_j)\Delta H_1^2 - \frac{(\Delta H_1 - H_j)^3}{3} . \tag{6.8}$$

Qualitatively, the form of the dependence of ΔH_1 on $\dot{T}_0$ is the same as the form of its $\dot{H}_a$ dependence. For $\dot{T}_0 \gg A(T)\alpha(T)/|\alpha'(T)| H_j^3$, ΔH_1 reaches the value $\sim H_j$.

In deriving Eq. (6.8), it was assumed that the heating was sufficiently slow for a homogeneous temperature to become established in the sample. It can be readily shown that for this to be the case it is necessary that $\dot{T}_0 \ll \kappa T_0/\nu l^2$ or, for characteristic values of the parameters, $\dot{T}_0 \ll 10^3$ K s^{-1}.

Shown in Fig. 18 are experimental values of ΔH_1 depending on T_0 at different $\dot{H}_a$ obtained by Watson (1967) in porous glass samples with indium pressed into the pores (sample with $T_c = 4.04$ K). Figure 19 contains data from the paper by Wipf and Lubell (1965) (Nb-Zr sample at $T_0 = 2.5$ and 4.2 K). Calculated curves obtained with the aid of Eq. (6.7) are shown in the figures. The value of $A(T)$ in the case of each T_0 was selected such that the calculated curves should not lie below the experimental points. The value of H_j was estimated by Watson from the known experimental T_c data and from the heat capacity of his samples (see solid curve in Fig. 18). As can be seen from Figs. 18 and 19, the agreement of theory with experiment deteriorates at low $\dot{H}_a$ ($\dot{H}_a < 2$ kOe min in Fig. 18 and $\dot{H}_a < 50$ Oe s^{-1} in Fig. 19), where ΔH_1 is probably close to the maximum. In the region of high $\dot{H}_a$, the agreement of theory with experiment found by Watson somewhat deteriorates as

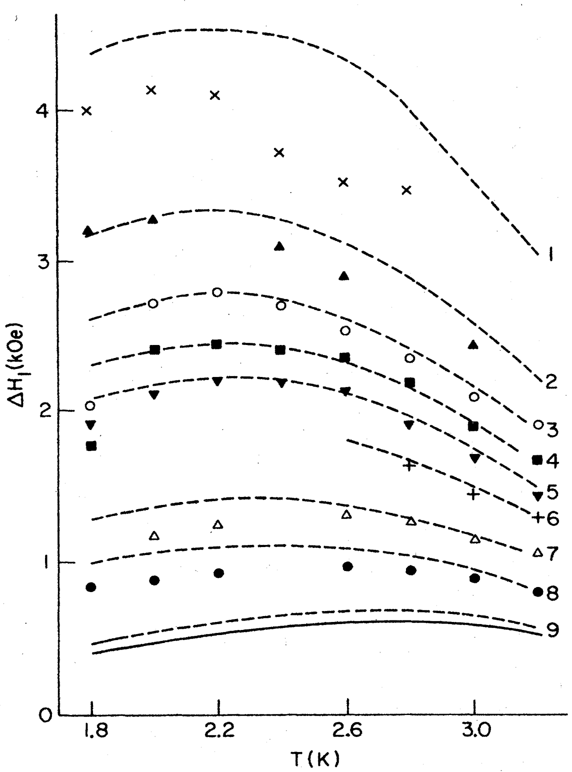

FIG. 18. ΔH_1 as a function of T_0. Experimental data from Watson, 1967 (symbols) and theory (dashed curves). The solid line corresponds to the value of $H_j(T)$ found by Watson (1967): Curve 1 and data marked with ×, $\dot{H}_a = 2$ kOe min^{-1}; curve 2 and ▲, 3 kOe min^{-1}; curve 3 and ○, 4 kOe min^{-1}; curve 4 and ■, 6 kOe min^{-1}; curve 5 and ▼, 10 kOe min^{-1}; curve 6 and +, 15 kOe min^{-1}; curve 7 and △, 30 kOe min^{-1}; curve 8 and ●, 58.5 kOe min^{-1}; curve 9, 10^3 kOe min^{-1}.

well, because the accuracy of the value of H_j known to us is still inadequate. Plotted in Fig. 20 is the dependence $A(T)$ obtained with the aid of Eq. (6.7) and the experimental points of Fig. 18. In the vicinity of T_c, the upper critical field $H_{c_2} \propto (1 - T/T_c)$ (see DeGennes,

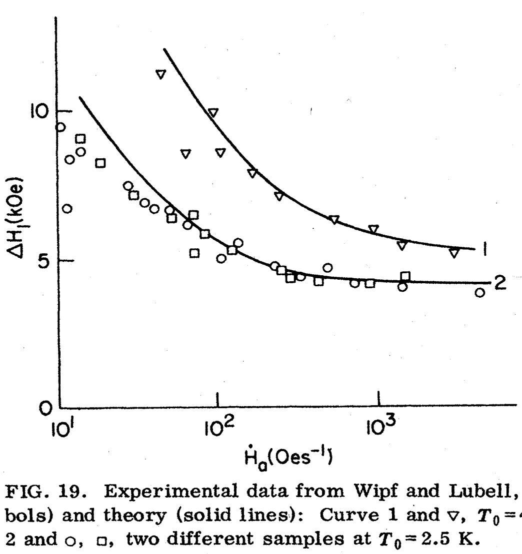

FIG. 19. Experimental data from Wipf and Lubell, 1965 (symbols) and theory (solid lines): Curve 1 and ▽, $T_0 = 4.2$ K; curve 2 and ○, □, two different samples at $T_0 = 2.5$ K.

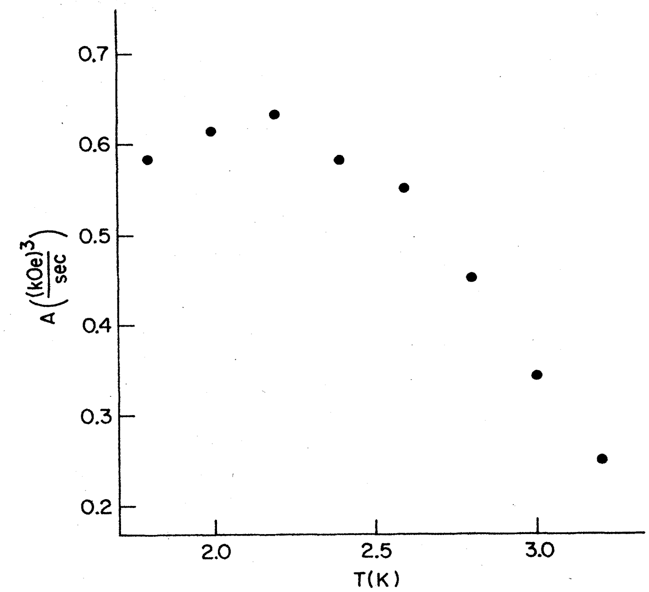

FIG. 20. A as a function of T for the curves shown in Fig. 18.

1966). In the same region, $\alpha \propto (1 - T/T_c)$. If $K(T)$ has no singularity at $T = T_c$, $A(T) \propto (1 - T/T_c) \to 0$, at $T \to T_c$. Indeed, as can be seen from Fig. 20, $A(T)$ drops sharply at $T_0 > 2.5$ K.

In plotting the theoretical curves shown in Fig. 19, use was made of two fitting parameters, A and H_j, since the value of H_j had not been determined in the paper (Wipf and Lubell, 1965). Accordingly, the values of A and H_j used in the plotting of theoretical curves in Fig. 19 were 4×10^9 Oe^3 s^{-1} and 5 kOe at $T_0 = 4.2$ K, and 0.85×10^9 Oe^3 s^{-1} and 4 kOe at $T_0 = 2.5$ K. Expressions (6.6) and (6.7) enable one to estimate the value of E_0. Let $A = 4 \times 10^9$ Oe^3 s^{-1}, $\alpha(T) = \alpha_0(1 - T/T_c)$, $\alpha_0 = 3 \times 10^8$ A cm^{-2}Oe, $T = 4$ K, and $T_c = 10$ K. Then $E_0 \sim 10^{-7}$ V cm^{-1}. Using the characteristic values of H_j and A from Eq. (6.8), it is readily seen that for samples of the Nb-Zr, Nb-Ti type one should expect a marked effect of $\dot{T}_0$ on ΔH_1 at $\dot{T}_0 = 10^{-2}$–10^{-1} K s^{-1}. At $\dot{T}_0 \sim 1$ K s^{-1}, $\Delta H_1 \simeq H_j$.

Some authors have observed an increase in $\Delta H_1(\dot{H}_a)$ in the region of high values for $\dot{H}_a$ ($\dot{H}_a > 10^4$ Oe s^{-1} at $T_0 = 4$ K for liquid-helium-cooled samples) (Rothwarf *et al.*, 1968; Chikaba *et al.*, 1968). These authors attribute such an increase in stability to strong heating of the sample by a variable external magnetic field (on the T dependence of H_j, see Sec. VI.B.2 below). As noted by Rothwarf *et al.* (1968), the start of the increase in ΔH_1 coincides with the transition from the nucleate boiling mode of liquid helium on the sample surface to the film boiling mode. Accordingly, the value of W_0 drops by an order of magnitude. Estimates show, however, that a uniform heating of the sample may prove insufficient (by a factor of two or more) for quantitative interpretation of the observed growth of ΔH_1. It can be shown that nonuniform heating may lead to a somewhat more rapid rise in stability than uniform heating. No quantitative description of this effect has been provided, and the causes of the growth in ΔH_1 at high $\dot{H}_a$ remain to be explained.

We have been discussing throughout this section the magnetic field difference in a superconductor before the flux jump. In many experiments, the authors observe series of flux jumps in an increasing (decreasing) external magnetic field [see Fig. 16(b)]. For these cases, H_j is assumed to be the difference between external fields ΔH_a upon successive jumps. This is only correct in the case of complete flux jumps, i.e., with a uniform field distribution in the sample as a result of instability (Neuringer and Shapira, 1966). If this is not the case, one can readily understand that $\Delta H_a < H_j$. The value of H_a may affect considerably the "degree of completeness" of the instability and, consequently, the difference ΔH_a. Such nonlinear effects have not been studied as yet.

2. Field differences in the sample before the flux jump—temperature dependence

The parameter measured most frequently in the course of experimental studies is the magnetic field difference in the sample before the flux jump. For $\tau \ll 1$ we have, for the case of a plane geometry

$$\Delta H < H_j = [\pi^3 \nu(T) j_c(T)/|(\partial j_c/\partial T)|]^{1/2}$$

(sample without normal cladding)

$$\Delta H < 2H_j$$

(sample with a $d > d_c$ cladding).

It is natural to assume that the ratio $j_c/j_c'(T)$ depends only slightly on the value of j_c (and, consequently, on pretreatment). This statement has been supported experimentally (Hancox, 1965).

Experiments (Goldsmid and Corsan, 1964; Swartz *et al.*, 1964; Corsan, 1964; Hancox, 1965; Carden, 1965; Lange, 1965) carried out in porous Nb_3Sn samples have demonstrated the effect of heat capacity upon critical state stability. Upon penetration of liquid helium into the pores, the effective heat capacity of the sample increases owing to heat transfer between the helium and the superconductor. The experiments serve to confirm qualitatively the dependence $H_j \propto \nu^{1/2}$.

In a fairly wide temperature range, the values of $\nu(T)$ and $j_c(T)$ can be approximated to a good accuracy as $\nu(T) = aT^3$ and $j_c(T) = j_0(1 - T/T_c)$, respectively. Then we derive for H_j (Swartz and Bean, 1968)

$$H_j = \pi^{3/2}(T/T_c)^{3/2}\nu^{1/2}(T_c)(1 - T/T_c)^{1/2}. \tag{6.9}$$

The $H_j(T)$ curve is shown in Fig. 21. Plotted in the same figure are experimental points taken from the paper by Irie *et al.* (1977). Theory and experiment are in qualitative agreement. A more accurate comparison appears difficult in view of the fact that the experiment was carried out in wires in an external transverse field at $\Delta H < H_p$. Thus the analytical solution to this problem tends to be complicated. The current distribution in the case under consideration was calculated numerically by Ashkin (1979) and Zenkevitch *et al.* (1980). The current and magnetic field distribution and, consequently, the value of $H_j(T)$ (the maximum field difference), depend on the ratio $H_a/H_p(T)$, where $H_p = 8j_c(T)R/c$. Instead of Eq. (6.9), we derive for $H_{j\perp}$: $H_{j\perp}(T) = A(R,T)H_j(T)$, where $A(R,T) \sim 1$ is some function to be determined.

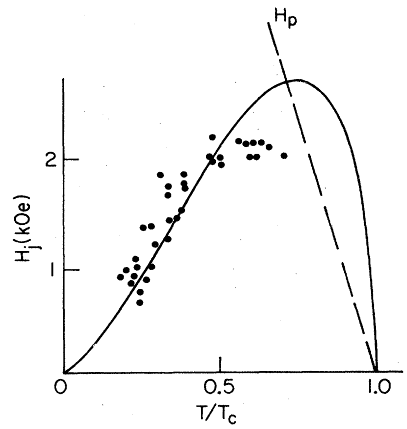

FIG. 21. H_j as a function of T/T_c, plotted from Eq. (6.9). Experimental data from Irie *et al.* (1977).

If $H_j > H_p$, as already noted, no flux jumps will occur (at a given value of H_a or some other "priming" influence). This statement has been experimentally supported by Shiiki and Kudo (1974) and Irie *et al.* (1977) (see Figs. 17 and 22). As can be seen from Fig. 17, in a wire with $H_p < H_j$, 2.5–3 kOe the region where losses increase sharply due to flux jumps disappears. Figure 22 shows the dependence $H_p(T)$ and $H_j(T)$ as measured by Irie *et al.* (1977). When the equality $H_p = H_j$ is reached, the instabilities disappear.

If $H_{j\perp} \simeq H_p$, the value of $H_{j\perp}$ is readily found analitically (see Sec. IV. C.4.a). Proceeding from the value of β_c found in that section for a normal metal-clad wire, one can easily derive

$$\frac{4\pi}{c^2} b^2 j_c \left| \frac{dj_c}{dT} \right| < 3.8\nu(T) \propto T^3 .$$

A corresponding dependence was found experimentally by Lange and Verges (1974) for Nb-Ti samples (see Fig. 23). The theoretical curve is plotted as a solid line. Corresponding to this curve is the heat capacity $\nu(T) = 4.6 \times 10^2 T^3$ erg cm^{-3} K^{-4}.

Apart from the papers mentioned above, the value of H_j has been measured in numerous other experiments which are not listed here for lack of space. The agreement of theory with experiment is within 30–40%, at least when one can assume that instability has been initiated with adequate intensity and that Eq. (6.9) is applicable.

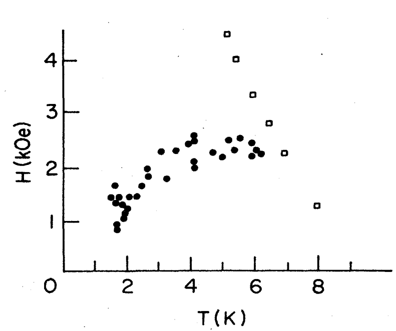

FIG. 22. Experimental values of H_j (•) and H_p (□) depending on temperature, for Nb-Ti-Zr samples (Irie *et al.*, 1977).

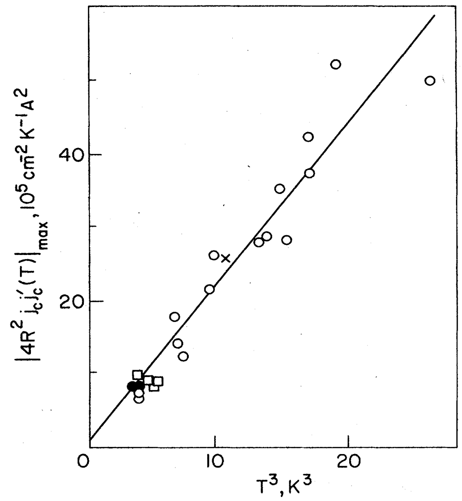

FIG. 23. The value of $(4R^2 j_c|\partial j_c/\partial T|)_{max}$ as a function of T^3 (Lange and Verges, 1974). Theory and experimental data for different samples.

3. Samples with normal-metal cladding

As shown in Sec. IV, a normal-metal cladding affects considerably both the dynamics of instability development and the stability criterion. We have already discussed in Sec. IV. C.3 the experiments (Onishi, 1974) designed to study the time variation of the magnetic flux during a flux jump in a normal metal-clad superconductor.

Lange and Verges (1974) have measured the temperature dependence of the maximum transport current density in a copper-clad Nb-Ti wire. From their results, one can find the value of R_c/R_0 at which flux jumps start to occur. As derived in Sec. IV, $R_c/R_0 \simeq 2.4$. Then, on determining the value of R_0 from experiment, one can find the dependence of I_m/I_c on R/R_0 without resorting to fitting parameters. The corresponding points are plotted in Fig. 11. They are in good quantitative agreement with theory.

Lazarev and Goridov (1972) have determined the effect of copper cladding on the maximum transport current density in a superconducting wire. The results of this experiment are shown in Fig. 24. As can be seen in the figure, starting at $d > d_c = 10^{-3}$ cm, the transport current in the wire does not depend on the cladding thickness d. The value of d_c is actually independent of the external magnetic field. In Sec. IV, d_c was estimated at

$$d_c \sim \frac{c^2 \nu_s R}{4\pi^3 \kappa_s \sigma_n} ,$$

$\partial d_c/\partial H \simeq 0$, in virtue of smallness of the derivative $\partial(\nu_s/\kappa_s\sigma_n)/\partial H$, which is in agreement with the experimental result. It follows from Fig. 24 that the ratio of transport currents in clad and unclad samples, I_{m1} and I_m, respectively, decreases with an increase in the external field. This is owing to the fact that the ratio R/R_0 drops with an increase of H_a (see Fig. 11). For example, if $j_c = j_0(T)(1 - H/H_{c_2})$, $R/R_0 \sim 1 - H/H_{c_2}$. At $R_0 < R < R_c = 2.4 R_0$, the ratio I_{m1}/I_m is not too high (about 1.2 or less), which coincides with the experimental results.

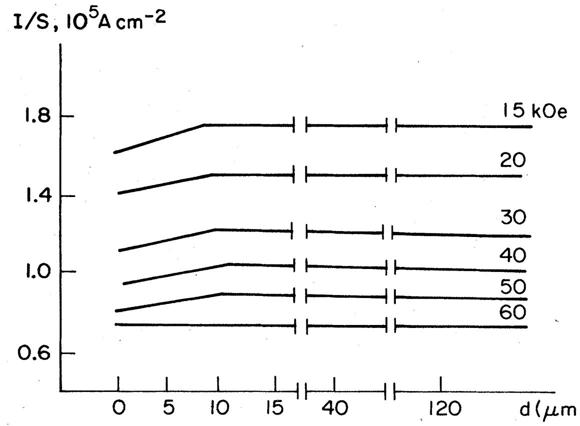

FIG. 24. Dependence of transport current in the wire upon thickness of normal cladding (Lazarev and Goridov, 1972).

The growth of stability in normal-metal-clad samples has been studied by Shiiki and Kudo (1974) and Irie *et al.* (1977), as well as in some earlier papers (see the review by Wilson *et al.*, 1970).

4. Brief review of some other experimental papers

We shall briefly discuss here three more groups of experimental papers of interest from the standpoint of processes accompanying the development of an instability.

In the papers by Shiiki and Kudo (1974), Shimamoto (1974), Harrison *et al.* (1974) and Irie *et al.* (1977), as well as in some other papers, the effect of external cooling upon the critical state stability is studied. In accordance with theory, in the case of a hard superconductor ($\tau \ll 1$), the value of H_j increases slightly (within about 10%) at $T = 4$ K with an increase in surface cooling. The importance of surface cooling grows with τ. Inasmuch as $\tau \propto \kappa/\nu \propto T^{-p}$ where $p > 0$, the effect of cooling grows at low temperatures, reaching about 20% at $T = 2$ K (Irie *et al.*, 1977).

The dependence of the stability criterion on the presence of transport current in the sample was demonstrated by McIntruff (1968). In the samples used by McIntruff, flux jumps disappeared at $H_j > H_p$ only to reappear upon the passage of a sufficiently high transport current through the sample.

The time of instability development has been measured in the course of numerous experiments (Swartz and Rosner, 1962; Lubell *et al.*, 1964; Goedemoed *et al.*, 1965; Borovik *et al.*, 1965; Neuringer and Shapira, 1966; Harrison *et al.*, 1973, 1974, 1975; Onishi, 1974). Depending on sample properties, cladding, and environmental conditions, the measured value lies in the range of from 10^{-6} to 10^{-4} s; this order of magnitude is in agreement with the theoretical estimate of t_j. It should be stressed, however, that t_j^{-1} is the increment of the initial increase in instability. Accordingly, the time t_j cannot coincide with the time of full perturbation increase, whose value is subject to nonlinear effects.

C. Composite superconductors

All of the difficulties involved in the comparison of theory with experiment in hard superconductors have as much (if not more) bearing on the case of superconducting composites. In addition, other difficulties arise in the course of interpreting experimental data for such materials. For example, most experiments involve coil samples, in which it is difficult to separate the effects due to the interaction (thermal, electrical, mechanical) of the turns with each other from phenomena due to the properties of the material as such. If it is known that at some current value I the coil passes to a resistive mode, a special study is required to find out whether this transition is due to thermomagnetic instability or to some other reason (for example, to movement of the winding turns). On the other hand, as noted in Sec. III.E, flux jumps in superconducting composites may interact strongly with mechanical instabilities such as plastic strain jerk. For this reason, we are unable to provide any reliable comparison of theory and experiment in this particular case.

VII. OSCILLATIONS IN THE CRITICAL STATE. LIMITED FLUX JUMPS

We shall now discuss in more detail the electric field and temperature oscillations occurring in a superconductor in the critical state near the stability threshold. As has already been noted in Sec. III, the existence of a background electric field E_a is essential for observing such oscillations. The field E_a should be induced by an external source. Depending on the source, oscillation effects can be divided into two groups, namely, oscillations which occur under nonstationary external conditions and those which occur under steady conditions. Note that only oscillations of the former type have been observed experimentally, with the field E_a being induced by a variable external field $H_a(t)$.

A. Oscillations and limited flux jumps occurring under nonstationary external conditions

Consider a plane hard superconductor sample in a variable external magnetic field $H_a(t)$. We assume that $I=0$, ΔH, $H_j<H_p$, while the sample surface is either well cooled or the value of τ is sufficiently small—$\tau<\tau_c$ (see Sec. IV.B). Under such conditions, as shown above, $\lambda_c\neq 0$ and the emergence of temperature and electric field oscillations is possible near the threshold of critical state stability.

Let the external field be absent at the initial moment of time. With an increase of $H_a(t)$, the magnetic field difference ΔH grows in the sample and, consequently, the parameter β is proportional to ΔH^2. At some value of $\Delta H=H_0$, $\beta=\beta_0$, while in the eigenvalue spectrum there appear values of $\lambda=i\lambda_1+\lambda_2$ with a nonzero imaginary part and $\lambda_2>0$ (see Fig. 4). If the amplitude of a variable electric field

$$E(t)=E_0\exp(\lambda_2 t/t_\kappa)\cos(\lambda_1 t/t_\kappa)$$

is less than E_a, electric field and temperature oscillations occur in the sample. The value of ΔH grows with H_a and, at $\Delta H=H_j$, a flux jump occurs in the superconductor. The formula for the number of oscillations,

$$N\sim\frac{\beta_c-\beta_0}{\beta_c}\frac{\omega H_j}{\dot{H}_a},$$

where $\omega=\lambda_1/t_\kappa$, was derived in Sec. III.D. Therefore, the number and frequency of oscillations depend on the sample properties, as well as on the external thermal and electrodynamic conditions.

If the amplitude of the electric field oscillations becomes higher than E_a, limited flux jumps will probably be observed in the sample.

A flux jump results in the heating of the sample and a consequent decrease in the field difference. Following the flux jump, the sample cools down to $T=T_0$ and, on subsequent increase of the external field, the effect is repeated. A similar pattern of oscillations preceding the instability has been observed experimentally by numerous authors (Zebouni *et al.*, 1964; Neuringer and Shapira, 1964, 1966; Goedemoed *et al.*, 1965; DelGastillo and Oswald, 1968; Chikaba, 1970; Shimamoto, 1974). DelGastillo and Oswald (1968) have demonstrated that in Nb_3Sn samples both oscillations and limited flux jumps can occur, depending on the conditions (Fig. 25). It is worthy of note that these authors observed a series of correlated limited jumps (see Fig. 25, curves B and C). The nature of such correlation has not been revealed.

With an increase in H_a, the value of the field at which one gets complete penetration of magnetic flux into the sample drops because $H_p\propto j_c(H)$. Therefore, at a certain value of H_a, it may turn out that $H_j>H_p$ and that oscillations and instabilities disappear in the region of high external fields. However, inasmuch as the value of j_c varies only slightly with the field variation by the value of H_j, the probability is high that, following the last flux jump, a magnetic field difference of $H_0<\Delta H<H_j$ sets up in the sample and a series of oscillations are observed which are not accompanied by a flux jump. The value of ΔH drops with an increase in H_a and, at $\Delta H=H_0$, oscillations start to decay and disappear. A similar effect was observed by Zebouni *et al.* (1964). It can be readily seen that the number N_1 of such oscillations may turn out to be much higher than the number N of oscillations before the flux jump estimated above.

As follows from theory, the oscillation frequency may vary, depending on the conditions, in the range of from 0 to $t_\kappa^{-1}\tau^{-2/3}$ (see Sec. IV). By using $\tau=10^{-3}$ and $t_\kappa=10^{-4}$

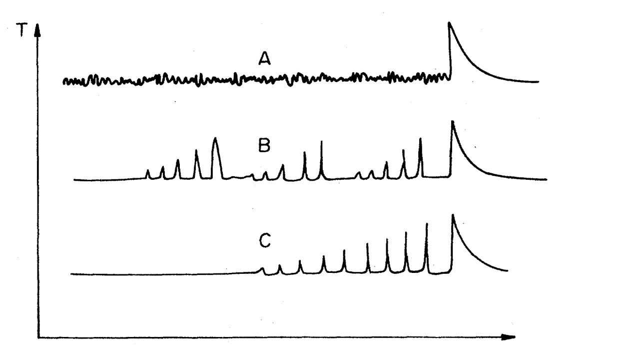

FIG. 25. Oscillations and limited flux jumps in Nb_3Sn strips (DelGastillo and Oswald, 1968).

s, we derive for ω, $0<\omega<10^6$ s^{-1}. Accordingly, one can readily find for the number of oscillations: $0<N<(H_j/\dot{H}_a)\ 10^5$ s.

The qualitative results obtained above are applicable to some degree of accuracy if the sample conductivity is independent of coordinates and time. This approximation is only true if the greater part of the sample bulk in the critical state has undergone a transition to the flux flow mode. For this to happen, it is necessary that $E_a>E_0$ or $\dot{H}_a>cE_0/l$. Assuming that $E_0=10^{-7}$ V cm^{-1} and $l=10^{-2}$ cm, we obtain $\dot{H}_a>10^3$ Oe s^{-1}. If this is not the case, the qualitative pattern of oscillations apparently stays unchanged. However, the numerical values of frequencies and rise increments may differ considerably from the formulas provided in the present review.

In order to calculate the oscillation amplitude, one should solve the equation for small perturbations in the critical state, taking into account the initial and inhomogeneous boundary conditions. In the case of $H_p>H_j$ (oscillations before successive flux jumps), the solution of such a problem encounters serious computational difficulties. These difficulties are due to the presence of the boundary of the critical state region where the superconducting current density abruptly turns to zero. The position of this boundary is a function of time. Not solution to the problem is available in this case for any situation of practical interest.

If $H_0<H_p<H_j$ (the last series of oscillations unaccompanied by a flux jump), the foregoing difficulty is removed. The corresponding linearized equations (4.1) are solved with the aid of a Laplace transform. We give below their solution for the case of a plane plate at $I=0$. Following the procedure of Mints and Rakhmanov (1979a), one can readily derive ($\omega t\gg 1$):

$$T=T_0+\frac{j_c b^3}{6c\kappa}\left(\mathrm{Res}\dot{H}_{a\lambda}(0)+a(\tilde{x})\,|\dot{H}_{a\lambda}(i\omega)|\,e^{\Gamma t}\cos(\omega t+\psi)\right),$$
$$E=\frac{b}{c}\left(\mathrm{Res}\dot{H}_{a\lambda}(0)+b(\tilde{x})\,|\dot{H}_{a\lambda}(i\omega)|\,e^{\Gamma t}\cos\omega t\right), \tag{7.1}$$

where $\dot{H}_{a\lambda}(z)$ is the Laplace transform from $\dot{H}_a$ at $\lambda=z$,

$$\dot{H}_{a\lambda}=\frac{1}{t_\kappa}\int_0^\infty \dot{H}_a(t)\exp(\lambda t/t_\kappa)dt\,,$$

$\mathrm{Res}\dot{H}_{a\lambda}(0)$ is the residue of this function at $z=0$, and $b\,\mathrm{Res}\dot{H}_{a\lambda}(0)/c\simeq E_a$. [$H_a(t)$ is assumed to vary monotonically.] The values of ω, Γ, $a(\tilde{x})$, $b(\tilde{x})$ depend on the value of τ and external cooling. Consider now a series of limiting cases describing roughly possible experimental situations.

Let $w\gg 1$ and $\tau\ll 1$, which corresponds to the case of a hard superconductor at $E_a>E_0$, cooled with liquid helium. For this case,

$$\omega=\frac{\pi^{4/3}}{2^{8/3}\tau^{2/3}}\,t_\kappa^{-1}\,,\quad \Gamma\sim\frac{\pi}{\tau}\left(\sqrt{\beta}-\frac{\pi}{2}\right)t_\kappa^{-1}-\omega\,,$$
$$a(\tilde{x})\simeq 2\cos\left(\frac{\pi\tilde{x}}{2}\right),\quad b(\tilde{x})\simeq\frac{0.6}{\tau}\cos\left(\frac{\pi\tilde{x}}{2}\right). \tag{7.2}$$

The case of a helium-gas-cooled sample can be described assuming $w\ll 1$. Then

$$a,\ b\cong\text{const},\quad \psi=0\,. \tag{7.3}$$

If $E_a\ll E_0$, the situation can be simulated by assuming $\tau\gg 1$. Then

$$\omega\sim\tau^{-1/3}t_\kappa^{-1}$$
$$a(x),b(x)\sim\cos\left(\frac{\pi x}{2b}\right). \tag{7.4}$$

In the latter two cases [cf. Eqs. (7.3) and (7.4)], "slow" oscillations are observed, i.e., $\omega t_\kappa\ll 1$. Note that in the case of fast oscillations the electric field amplitude contains a high factor τ^{-1}. In addition, the amplitude of "fast" oscillations grows substantially with time since $\Gamma\propto\tau^{-1}$ [see Eq. (7.2)]. Therefore, after a relatively small number of oscillations their amplitude gets to be higher than E_a and the series of oscillations appear to cease. It is believed, therefore, that fast oscillations are relatively harder to reveal than "slow" ones. Indeed, the papers listed at the beginning of this section contain reference only to experimentally observed low-frequency oscillations (at $w\ll 1$, as a rule, when their amplitude is sufficiently high). The only paper reporting the observation of high-frequency oscillations of electric field and temperature is that by Shimamoto (1974).

As is known, DeGennes and Matricon (1964) predicted the possibility of Abrikosov vortex lattice oscillations in Type-II superconductors. It would appear that this mechanism could provide an alternative interpretation to the E and T oscillations observed in the course of experiment. However, Zebouni *et al.* (1964) have shown that such an interpretation is inconsistent with the experimentally measured strong temperature dependence of the oscillation period in Nb-Zr samples (see, Fig. 26). At the same time, this dependence finds a natural interpretation within the framework of the present theory. The data provided by Zebouni *et al.* (1964) are, unfortunately, insufficient to show which one of the above-discussed approximations is closest to their experimental situation. It is clear, however, that slow oscillations were observed. As can be readily shown from the results presented in Sec. IV, the period of slow oscillations grows rapidly with temperature for the approximations natural to the low-temperature region.

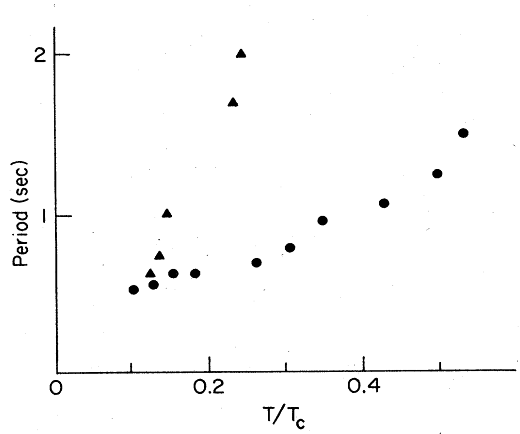

FIG. 26. Experimentally measured period of oscillations in Nb-Zr (▲) and Nb (●) samples as a function of temperature (Zebouni *et al.*, 1964).

Note further that the oscillation effects described above may occur both in normal-metal-clad superconductors and in composite superconductors (Maksimov and Mints, 1979, 1980).

B. Oscillations occurring under stationary external conditions

1. Calculation of the oscillation amplitude in a linear approximation

A stationary background electric field E_a can be generated in the sample by passing through it a current I higher than the critical current I_c. When this is done, the field E_a induced by external emf is on the order of $(I-I_c)/S\sigma$, where S is the cross-sectional area of the conductor and σ is the conductivity. If a small increment of perturbation rise is ensured by proper selection of sample parameters and external conditions, a series of many E and T oscillations will be observed in the sample. Whether this series turns to a flux jump, starts to decay, or arrives at a stationary limiting cycle depends upon nonlinear effects (see next section).

In the case of $\tau \leqslant 1$, no oscillations occur in the perturbation spectrum at $I \geqslant I_c$ in the $\beta < \beta_c$ region (see Sec. IV). In the case of $I \geqslant I_c$, oscillations occur only at sufficiently high values of τ, $w\tau$, which is characteristic of liquid-helium-cooled superconducting composites. Nonzero eigenfrequencies are present in the spectrum of eigenvalues of λ at $I \geqslant I_c$ in the case of hard superconductors clad with a normal-metal layer d exceeding some value d_0. We shall confine ourselves to the consideration of a hard superconductor with cladding.

Let a sample present a plane plate of a hard superconductor having a thickness of 2b, clad on both sides with normal metal to a thickness d, with the transport current $I > I_c = 2bj_c$. We shall assume the difference $I - I_c$ to be sufficiently great for the plate to be in the flux flow mode. As to the material of the cladding, it is assumed that $\sigma_n \gg \sigma_s$, $\kappa_n \gg \kappa_s$, $\nu_n < \nu_s$, and $\tau_1 \gg 1$. This case corresponds, for example, to a cladding of adequately pure ($\sigma_n \gtrsim 10^{20}\ s^{-1}$) copper. We shall consider here only the case of good external cooling, $w \gg 1$.

The spectrum of eigenfrequencies of T and E perturbations, as well as the stability criterion, can be found by the methods described in Sec. IV. Calculations are facilitated by the fact that for the conditions under consideration thermal processes in the cladding are inessential.

The appropriate expressions for oscillation frequencies and values of β_0, β_c have the simplest appearance in the limiting case $\sqrt{\tau}\,\tau_1 \ll 1$. In this case, ω, β_c, and β_0 depend only on the value of τ_1. The stability criterion has the form $\beta < \beta_c\,(\tau_1, d)$. The perturbation rise increment λ_2 turns to zero at $\beta = \beta_0\,(\tau_1, d)$. If $d \geqslant d_0 = 0.7b[(1+2.3/\tau_1)^{1/2}-1]$, then $\beta_0 < \beta_c$, and in the region $\beta_0 < \beta < \beta_c$ oscillations occur with a frequency $\omega = \lambda_1 t_\kappa^{-1}$. The dependence of the parameters ω, β_0, and β_c upon d is shown qualitatively in Figs. 27 and 28, where

$$\omega_c = 3\tau_1 t_\kappa^{-1}, \quad \beta_0 = \frac{\pi^2}{4} - \frac{2}{\tau_1}, \quad \beta_c = \frac{\pi^2}{4} - \frac{1}{\tau_1}.$$

The oscillation amplitude is readily found with the aid of the Laplace transformation of the appropriate linear system of differential equations, which has the form of Eq. (4.1) for superconducting material and the form of curl curl $E = 4\pi\sigma_n \dot{E}/c^2$ for normal metal. For example, let the current $I(t)$ increase monotonically from I_c to $I = I_c + \Delta I$. In this case, the solutions of $T(t)$ and $E(t)$ at $\omega t \gg 1$ have the form (Mints and Rakhmanov, 1979a)

$$T = \begin{cases} T_0 + \Delta T_0\left(1 - \tilde{x}^2 - ae^{\Gamma t}\cos\frac{\pi\tilde{x}}{2}\cos\omega t\right), & |\tilde{x}| < 1 \\ T_0, \quad |\tilde{x}| > 1 \end{cases} \tag{7.5}$$

$$E = E_p(1 - f(\tilde{x})e^{\Gamma t}\cos(\omega t + \varphi)),$$

where

$$\Delta T_0 = \frac{3\Delta I}{2b\,|dj_c/dT\,|(3\tau_1 d/b - 1)}, \quad E_p = \frac{2\kappa_s \Delta T_0}{j_c b^2}.$$

In deriving Eq. (7.5), it was assumed that $\Gamma \ll \omega$. The values of $f(\tilde{x})$, a, and φ depend considerably on the oscillation frequency ω. In the region $d - d_0 \ll d_0$, $\omega \propto (d-d_0)/d_0$. For a sufficiently small difference $d - d_0$, the oscillations are slow: $\omega t_\kappa \ll 1$. Here $f(\tilde{x}) = \text{const} \sim 1$, $\varphi \simeq 0$, $a \sim 1$ to the main approximation in $\omega t_\kappa \ll 1$.

With an increase in the frequency ω, the relative oscillation amplitude decreases while the inhomogeneity of the electric field grows, reaching the maximum at $\omega = \omega_c$. The expressions for $f(\tilde{x})$, a, and φ at $\omega = \omega_c$ to the main approximation in $\omega t_\kappa \gg 1$ have the form

$$f(x) = \begin{cases} 1.5\tau_1 a\cos\frac{\pi x}{2b}, & |x| < b \\ 3a\exp[(b-x)/d_0], & |x| > b \end{cases}$$

$$a = 16\tau_1 \frac{|\Delta I(\omega_c)|}{\Delta I}\exp(-d/d_0), \quad \varphi = \frac{\pi}{2}, \quad \Gamma = \frac{\omega_c}{2}[\beta - \beta_0(\tau_1)],$$

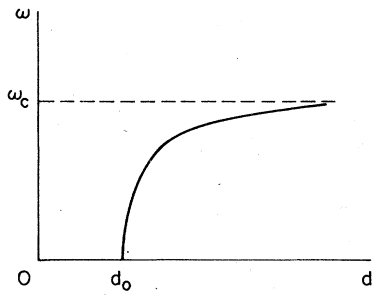

FIG. 27. Oscillation frequency ω as a function of normal cladding thickness.

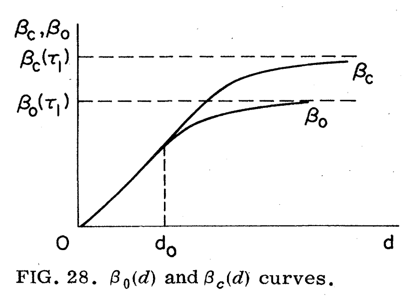

FIG. 28. $\beta_0(d)$ and $\beta_c(d)$ curves.

$$\Delta I(\omega)=t_\kappa^{-1}\int_0^\infty [I(t)-I_c]e^{i\omega t}dt\,.$$

Let us now estimate the maximum values of the oscillation amplitude E_m and frequency ω_c. In so doing, we shall proceed from the ratio between E_p and $\Delta T(0)$ (temperature variation in the sample center). Inasmuch as $E_p > E_m$, assuming that $\Delta T(0)=1$ K [$\Delta T(1)\ll 1$ K for the case of $\omega t_\kappa \gg 1$], then, on using the standard values for the parameters involved, $\Delta I/I_c=10^{-3}$, $j_c=10^5-10^6$ A cm^{-2}, $\kappa_s=10^4$ erg cm^{-1} K^{-1} s^{-1}, $\nu_s=10^4$ erg cm^{-3} K^{-1}, $\tau_1=1-10$, one has $E_m\sim 10^{-3}-10^{-6}$ V cm^{-1}, $d_0\sim(1-0.1)b$.

In addition to electric field and temperature oscillations under conditions of fixed transport current, there may apparently exist T and I oscillations under conditions of fixed voltage at the sample ends. However, this problem is essentially two dimensional.

As can readily be seen, it is more difficult to detect oscillations under stationary conditions than under variable ones. Indeed, in the latter case, the external field variation causes the parameter β to vary continuously from zero to β_c and is sure to get into the $\beta_0<\beta<\beta_c$ range (if $\beta_0<\beta_c$). In the case of stationary external conditions, the values of β, β_0, and β_c should vary owing to variation of some additional parameter (external magnetic field, temperature, external cooling, thickness of normal cladding, and the like). In so doing, β should get into the narrow region near β_c. In the case considered above, this region is on the order of

$$\frac{\beta_c-\beta_0}{\beta_c}=\frac{4}{\pi^2\tau_1}\lesssim 0.1.$$

2. Nonlinear effects

As can be seen from the foregoing, in the linear theory stationary oscillations are only possible at the point $\beta=\beta_0$, i.e., this mode is unstable in the linear theory. Clearly, at $\mathrm{Re}\lambda>0$, the behavior of oscillations over large times depends on nonlinear effects. These effects may turn out to be substantial even in the region of low-amplitude oscillations following a sufficiently large number of cycles due to accumulation of small nonlinear perturbation (see Bogoliubov and Mitropolskii, 1974).

A qualitative analysis of nonlinear oscillations, provides certain arguments in favor of the conclusion that as $t\to\infty$ oscillations lend to a stationary solution independent of the initial conditions, i.e., a limiting cycle.

For a more rigorous analysis of the effect of nonlinearity, one has to solve a nonlinearized system of heat and Maxwell equations for a hard superconductor. If θ and $\varepsilon\ll 1$ (low-amplitude oscillations), the solution of the nonlinear equations can be found analytically. We have not succeeded, however, in finding a case where a limiting cycle of low amplitude would exist at the temperature dependences actually possible for the sample parameters. Rather exotic dependences $\nu(T)$, $j_c(T)$ are required for the existence of such a cycle. The problem of the behavior of ε and θ solutions at other than low oscillation amplitude calls for further study.

VIII. THE INFLUENCE OF THERMOMAGNETIC EFFECTS UPON THE CRITICAL STATE STABILITY AND DYNAMICS

A. Derivation of basic equations

This section will be devoted to a discussion of the influence exerted upon the critical state stability and dynamics by transverse thermomagnetic Nernst and Ettingshausen effects (Gurevich and Mints, 1979, 1981). The inclusion of these effects causes the emergence of terms proportional to ∇T and E, respectively, in the expressions for electric current density and heat flux. These terms may prove significant owing to high-temperature gradients occurring with a magnetic flux jump and the relatively low heat conductivity of hard superconductors.

The physical processes caused by transverse thermomagnetic effects have already been discussed briefly in Sec. II. The term in Eq. (2.7) for current density, proportional to $S^*\nabla T$, results from thermoelastic stresses in the vortex lattice. These stresses are capable of either increasing or decreasing the value of j when an appropriate fluctuation occurs. The term in the heat flux density, proportional to E [see Eq. (2.8)] is associated with the transfer of entropy by vortex lines. For the sake of illustration, it can be rewritten as S^*nvT, where $v=cE/H$ is the rate of vortex movement and n is the vortex density. Also associated with this term is the additional heat released as a result of variation in the length of the vortex lines in the course of their movement. For example, in a current-carrying wire [Fig. 10(b)], the heat produced in the volume $dV=2\pi r\,dr\,dz$ per unit time amounts to $2\pi r q_1 dr\,dz=2\pi TS^*nv\,dr\,dz$, inasmuch as $2\pi vn\,dr\,dz$ is the total variation of the length of all the vortices passing through the volume dV per unit time. Therefore, the additional (apart from jE) heat released q_1 is equal to STE/r. In the case of a plane geometry, the length of the vortex lines does not vary with their movement and $q_1=0$.

Let us now derive the equation for small perturbations taking into account these thermomagnetic effects. As usual, we consider Bean's model of the critical state for the sake of simplicity. The equations of interest can be readily derived from the Maxwell equation (3.3) and the balance equation for the entropy density:

$$T\frac{\partial S}{\partial t}=-\mathrm{div}\mathbf{q}+\mathbf{j}\cdot\mathbf{E}\,, \tag{8.1}$$

where $\mathbf{q}$ is the heat flux density. The transport entropy S^* of the vortex lines is associated with the presence of excitations localized in the vortex core (Caroli *et al.*, 1964). Therefore, one can assume that the transport entropy of the entire vortex lattice is additive up to $H\sim H_{c_2}$, where the cores of the vortex lines start overlapping. Thus, $S=S_0+nS^*$, where S_0 is the entropy density neglecting the transport entropy of the vortices.

In the case of a plane geometry (see Fig. 5), using Eqs. (8.1) and (3.3) and the expression for S, we derive

$$\varepsilon'' = -\lambda(\beta\theta + 2\mu\theta') + \lambda\tau\varepsilon\,, \qquad \lambda\theta = \theta'' + \varepsilon\,, \tag{8.2}$$

where $\mu = 2\pi Sbj_c/c^2\nu$, and the time dependence of ΔT and E is selected in the form of Eq. (4.2). In deriving Eqs. (8.2), we have also used the Maxwell equation $\mathrm{curl}\mathbf{E} = -(1/c)\dot{\mathbf{H}}$ which serves as the continuity equation for the vortex lines. By eliminating ε from the system of equations (8.2), we derive (Gurevich and Mints, (1979)

$$\theta'''' - \lambda(1+\tau)\theta'' - 2\mu\lambda\theta' - \lambda(\beta - \lambda\tau)\theta = 0\,. \tag{8.3}$$

In determining the eigenvalue spectrum of λ, appropriate boundary conditions should be imposed on Eq. (8.3). The electrodynamic conditions, as well as the conditions on the surfaces of discontinuity of the current density j, remain unchanged (see Sec. IV). The surface cooling condition now has the form

$$w\theta(\pm 1) \pm \theta'(\pm 1) = -\alpha_1[\lambda\theta(\pm 1) - \theta''(\pm 1)]\,, \tag{8.4}$$

where $\alpha_1 = ST/j_c b$. The term on the right-hand side of Eq. (8.4), equal to $-\alpha_1\varepsilon(\pm 1)$, is associated with the term proportional to E in the expression for the heat flux density.

B. Simplified theory

In the general case, the equation for λ turns out to be rather complicated and can only be solved with the aid of a computer. An appropriate numerical analysis shows that in the range of parameters characteristic of hard superconductors (τ, α_1, $\mu \ll 1$) the instability develops rapidly: $\lambda_c \gg 1$, however, $\lambda_c\tau \ll 1$. The condition of $|\lambda| \gg 1$ helps reduce the basic equation (8.3) to second order. Indeed, at $|\lambda| \gg 1$, we derive from Eq. (8.2) to the desired accuracy

$$\varepsilon = \lambda\theta\,, \qquad \varepsilon'' + 2\mu\varepsilon' + \beta\varepsilon = 0\,. \tag{8.5}$$

The problem of the boundary condition for Eq. (8.5) deserves special attention. In the limiting case of interest to us, the current distribution in the sample to within an accuracy of up to $|\lambda|\tau \ll 1$ depends on the temperature distribution. As already noted in Sec. III, in the discussion of the qualitative theory of flux jumps, for $w \gg 1$ a sharp temperature drop occurs over a distance on the order of a "thermal" length $l_T \sim b/|\lambda|^{1/2}$. Accordingly, the current density increases near the surface, which can be interpreted as the emergence of surface current having a density of $j_p \sim |\partial j_c/\partial T|\Delta T_m$. Then, the value of $I_p = \int_{b-l_T}^{b} j_p dx$ is of the order of

$$I_p \sim \left|\frac{\partial j_c}{\partial T}\right| \Delta T_m b/|\lambda|^{1/2}\,.$$

The presence of this surface current results in a variation of the derivative E' in a layer of thickness l_T near the sample surface. Therefore, Eq. (8.5) is only true in the region $|1-\tilde{x}| < |\lambda|^{-1/2}$. From the Maxwell equations it follows that, at $\mu = 0$,

$$E'(1) - E'(1 - l_T/b) = \frac{4\pi b}{c^2}\frac{\partial I_p}{\partial t}\,. \tag{8.6}$$

Inasmuch as $E'(1) = 0$,

$$E'(1 - l_T/b) \sim \frac{4\pi\lambda\kappa}{c^2\nu b} I_p \sim \beta E|\lambda|^{-1/2} \ll E\,.$$

Thus, at $S = 0$, both the electric field and its derivative vary only slightly near the sample surface, and the boundary condition for Eq. (8.5) at $\mu = 0$ has the form $E'(\pm 1) = 0$ cited in Sec. IV.

However, if $S \neq 0$, one should take into account the variation of the boundary condition, inasmuch as this variation makes a contribution to the stability criterion of the same order of magnitude as the term $\mu\varepsilon'$ in Eq. (8.5). A corresponding boundary condition can be shown to have the form (Gurevich and Mints, 1981)

$$\frac{\varepsilon'(1)}{\varepsilon(1)} = -\frac{\alpha_1\beta(1 + w/\alpha_1\lambda)(1 + 2\mu\gamma\sqrt{\lambda}/\beta)}{1 + w\gamma|\lambda|^{-1/2}}\,, \tag{8.7}$$

where γ is a number on the order of unity.

Let us now find the stability criterion for a plane sample in the simplest case of $I = 0$, $\Delta H = H_p$. The solution of Eq. (8.5), satisfying the boundary condition $\varepsilon(0) = 0$, has the form

$$\varepsilon = Ae^{-\mu x}\sin\sqrt{\beta}\tilde{x}\,.$$

If w is so great that $w \gg \alpha_1|\lambda|, \gamma|\lambda|^{1/2}$ we derive from Eq. (8.7)

$$\frac{\varepsilon'(1)}{\varepsilon(1)} \simeq -\frac{\beta}{\gamma\sqrt{\lambda}}(1 + 2\mu\gamma\sqrt{\lambda}/\beta) \simeq -2\mu\,.$$

Then, on substituting the solution $\varepsilon(\tilde{x})$ into the boundary condition, we find that the stability criterion has the form

$$\sqrt{\beta} < \frac{\pi}{2} + \mu_0\,, \tag{8.8}$$

where $\mu_0 = \mu\beta^{-1/2}$ and where allowance is made for the fact that $\mu \ll 1$.

If $w = 0$, analogous reasoning leads to the boundary condition $\varepsilon'(1)/\varepsilon(1) = -\alpha_1\beta$. The stability criterion has the form

$$\sqrt{\beta} < \frac{\pi}{2} + \alpha_0 - \mu_0\,, \tag{8.9}$$

where $\alpha_0 = \alpha_1\beta^{-1/2}$ and α_1, $\mu \ll 1$.

With good surface cooling, as seen from the criterion (8.8), the thermomagnetic effects enhance the critical state stability, which is due to the presence near the sample surface of a "heat barrier" precluding the entry of new vortices into the superconductor bulk upon development of a perturbation. With poor cooling, as seen from Eq. (8.9), the thermomagnetic effects may either increase or decrease the stability, depending on the ratio between α_1 (defining the heat barrier on the surface) and μ_0 [which defines, as follows from Eq. (2.7), the decrease of current density in the bulk of the sample].

Let us now estimate the values of α_1 and μ. Selecting the temperature dependence of the parameters ν, S, and j_c in the form $\nu = \nu_c(T/T_c)^3$,

$$S = S_0(T/T_c)(1 - T/T_c)\,,$$

and $j_c = j_0(1 - T/T_c)$, we find

$$\alpha_0 = 2a\left(\frac{T}{T_c}(1 - T/T_c)\right)^{1/2}, \quad \mu_0 = a\left(\frac{T}{T_c}(1 - T/T_c)^3\right)^{1/2},$$

where

$$a=\frac{S_0}{c}(\pi T_c/\nu_c)^{1/2}.$$

The temperature dependence $H_j(T)$ obtained with due regard for α_0 and μ_0 is shown in Fig. 29. As can be seen from the figure, thermomagnetic effects influence the stability most strongly at low temperatures. For typical values of S_0, ν_c, and T_c, the value of the parameter a turns out to be of the order of $10^{-1}-10^{-2}$. The influence of thermomagnetic effects on the stability is pronounced at $T/T_c \lesssim a^2$, where $\mu_0 \sim 1$. The only exception can be made in cases when the stability criterion is defined by a small parameter τ (for example, a wire with transport current $I=I_c$, where $\beta_c \sim \tau$).

C. Dynamics of instability development

Consider now the dynamics of instability development (with due regard for thermomagnetic effects) for perturbations with $|\lambda| \gg 1$, based on the exact dispersion equation for λ. We shall seek a solution to Eq. (8.3) in the form

$$\theta=\sum A_i e^{k_i x},$$

where the k_i are found from the equation

$$k^4-\lambda k^2(1+\tau)-2\mu\lambda k-\lambda(\beta-\lambda\tau)=0.$$

At $|\lambda| \gg 1$ the dispersion equation can be expanded in a series in the small parameter $|\lambda|^{-1/2}$. As a result, we obtain

$$\tan p=-p\,\frac{w(\sqrt{\lambda}-\mu)+\lambda(1-2\alpha_1\mu)+\sqrt{\lambda}(2\mu-\alpha_1\beta)}{w(\mu\sqrt{\lambda}+p^2)+2\alpha_1\mu\lambda^{3/2}+\lambda(\alpha_1\beta-\mu)}, \quad (8.10)$$

$$p^2=\beta-\lambda\tau-\beta^2/\lambda.$$

From the conditions $(\partial\lambda/\partial\beta)_{\lambda_c}=\infty$, $w \gg \sqrt{\lambda_c}$, and $\alpha_1, \mu \ll 1$ we derive

$$\beta w+\lambda_c(2\mu-\alpha_1\beta)-\lambda_c^{3/2}(w\tau+4\alpha_1\mu)=0. \quad (8.11)$$

It follows from Eq. (8.11) that, at $w \gg 1$, λ_c is determined by thermomagnetic effects if

$$1 \ll w \ll w_k \sim \alpha_0\mu_0/\tau. \quad (8.12)$$

The characteristic value of w for liquid helium cooling is $10-10^2$. At helium temperatures, $\alpha_0\mu_0 \lesssim 10^{-2}$. The influence of thermomagnetic effects upon the dynamics of instability development may be considerable if $\tau \lesssim 10^{-4}$. In such cases,

$$\lambda_c=\left(\frac{\beta w}{4\alpha_0\mu_0}\right)^{2/3}.$$

If the condition (8.12) is not satisfied, thermomagnetic effects upon the dynamics and stability of the critical state are significant only if $\mu_0 \gtrsim 1$.

With weak cooling ($w \ll 1$), thermomagnetic effects upon the dynamics of instability development are observable over a wider range of parameters. Thus, if $\tau^{3/4} \ll \alpha_0\mu_0 \ll 1$, it can be easily found from Eq. (8.11) that

$$\lambda_c=\beta^{4/3}/(2\alpha_0\mu_0)^{2/3}.$$

It is apparent, therefore, that transverse thermomagnetic effects, while relatively weakly affecting the critical state stability (if $\beta \sim 1$), can determine the characteristic time of magnetic flux jumps.

D. Wire with transport current $I=I_c$

Let us consider now the case of $I=I_c$. In so doing, we shall omit the details of cumbersome calculations. Thus, for a wire with current $I=I_c$ [see Fig. 10(b)], the equation for the maximum wire radius R_m at $w \gg 1$ has the form (Gurevich and Mints, 1981)

$$\left(\frac{R_m}{R_0}\right)^3-\frac{16}{3}\left(\mu_0-\frac{\alpha_0}{2}\right)\frac{R_m^2}{R_0^2}-16\alpha_0\mu_0\frac{R_m}{R_0}-16\tilde{\tau}=0,$$

where

$$R_0^2=\frac{c^2\nu}{4\pi j_c|dj_c/dT|}, \quad \tilde{\tau}=\frac{4\pi\sigma_n\kappa}{c^2\nu}\frac{H_{c2}}{H_I}, \quad H_I=\frac{2\pi R_0 j_c}{c}$$

(we believe the sample to be in the flux flow mode), and, as usual, $\sigma=\sigma_n H/H_{c_2}$. If $\tilde{\tau} \ll \mu_0^3$, the critical state stability largely depends on thermomagnetic effects. Thus at $\alpha_0 \gg \mu_0$ one can readily find $R_m=6\mu_0 R_0$.

Assuming that the self-magnetic field of the current, H_I, is much lower than the external field transverse to the wire axis (but $I=I_c$, $w \gg 1$), the expression for R_m can be found in the general form (Gurevich and Mints, 1981)

$$R_m=\sqrt{8\tau}\left[1+\frac{H_I}{H_a}\left(\mu_0-\frac{\alpha_0}{2}\right)\right].$$

With the growth of H_a, the importance of thermomagnetic effects decreases.

IX. THERMOMAGNETOMECHANICAL INSTABILITY IN SUPERCONDUCTORS AND THE TRAINING PHENOMENON

We shall now study, in more detail than in Sec. III.E, the critical state stability in superconductors in the presence of high stresses causing plastic yield of the material. Unlike the thermomagnetic instability discussed in the preceding chapters, mechanical effects in the critical state have not been studied until recently. At the same time, they present nowadays one of the most urgent problems related to the development of large magnets. Indeed, in a magnet a superconductor is affected by high mechanical stresses due to ponderomotive interaction of a high-density current (10^4 to

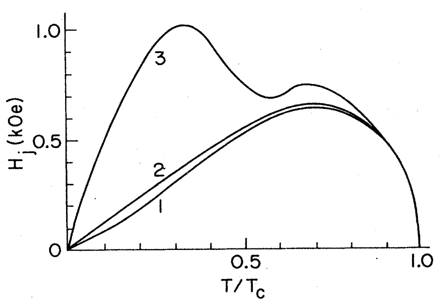

FIG. 29. $H_j(T)$ dependence for $\tau=0$, $\nu(T_c)=10^4$ erg cm^{-3} K^{-1}: curve 1, $S_0=0$; curve 2, $S_0=2\times10^{-7}$; curve 3, $S_0=10^{-6}$ erg cm^{-1} K^{-1}.

10^6 A cm^{-2}) and a magnetic field (10^4 to 10^5 Oe). This is accompanied by the emergence of stresses of up to 10^{10} dyn cm^{-2} which are of the order of the yield point for appropriate materials and, consequently, cause marked plastic strain. Associated with the presence of high mechanical stresses is a phenomenon undesirable in magnets, namely training, which has been observed in numerous experiments in both short samples and superconducting coils. The relationship between plastic strain and training of short samples has been reliably proven experimentally. Training in coils may have various causes. The presence of mechanical stresses causes additional dissipation of energy in the system owing, for instance, to movement of winding turns in the magnetic field, cracking of the material used to impregnate the windings, etc. (see Wilson *et al.*, 1970; Edwards *et al.*, 1975; Smith and Coyler, 1975; Kuroda, 1975; Anashkin *et al.*, 1977; Schmidt and Turk, 1977). The heat evolving in this case may prove sufficient for local heating of the superconductor up to $T > T_c$, or it may serve as a "primer" for the emergence of a flux jump. We do not discuss here problems related to magnet manufacturing technology. However, in many cases, the training of coils is caused by the training of the superconducting material proper.

A. Thermomagnetomechanical instability theory

1. Basic equations

In this section, we derive equations describing the development of small perturbations of temperature and electric field in the critical state, in the presence of mechanical stresses causing a plastic yield of the material (Mints, 1980; Maksimov and Mints, 1981). The instabilities of interest present a plastic strain jerk and a magnetic flux jump, developing jointly. Both hard and composite superconductors will be considered below. Here, as before, a superconducting composite is regarded as a homogeneous anisotropic superconducting material. The physical and mechanical properties of such a medium depend on the local properties of the superconductor and the normal matrix, averaged over the composite cross section.

Inasmuch as the instability under discussion is associated with macroscopic strain jerks, we shall proceed from the phenomenological description of plastic yield, in which the plastic strain rate $\dot{u}$ is a function of temperature T, of stress $\hat{\sigma}$ (or elastic strain u_e related to the latter by Hooke's law), and of plastic strain u, i.e., $\dot{u}=\dot{u}(T, \hat{\sigma}, u)$. Then, in a linear approximation (in perturbations of temperature, electric field, and plastic strain δu) the equations have the form (Mints, 1980)

$$\nu \frac{\partial}{\partial t}(\Delta T) = \kappa\nabla^2(\Delta T) + j_c E + \hat{\sigma}\delta\dot{u}\,, \qquad \nabla^2 \mathbf{E} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}\,, \tag{9.1}$$

in the region of current flow, and

$$\nu \frac{\partial}{\partial t}(\Delta T) = \kappa\nabla^2(\Delta T) + \hat{\sigma}\delta\dot{u} \tag{9.2}$$

in the sample region where the current is absent. When deriving Eqs. (9.1) and (9.2), the fluctuations of the stress $\hat{\sigma}$ included in these equations is disregarded because $\dot{u}$ involves a considerably stronger dependence on $\hat{\sigma}$. In addition, it is assumed for simplicity that the sample is only strained in one direction. Note the omission from the derived equations of any heat release due to an electric field induced upon the sample movement in the magnetic field. This term, as can be readily shown, is small in proportion to $H^2/4\pi\hat{\sigma} \ll 1$ as compared with an analogous term directly describing strain losses $\hat{\sigma}\delta\dot{u}$.

Equations (9.1) and (9.2) must be complemented by an equation describing the evolution of total strain $\delta(u+u_e)$. However, as shown by Mints and Petukhov (1980), for characteristic values of the parameters the inclusion of elastic strain fluctuations leads only to insignificant corrections of the obtained results.

Therefore, the system of Eqs. (9.1) and (9.2), at a preset dependence of critical current density and rate of plastic strain upon T, H, $\hat{\sigma}$, and u, helps us to study the critical state stability in relation to thermomagnetomechanical instability. It is assumed that $\partial j_c/\partial u$, $\partial j_c/\partial\hat{\sigma}=0$. In addition, Bean's model of the critical state is used for simplicity. Then,

$$\frac{\partial}{\partial t}\delta\dot{u} = \frac{\partial \dot{u}}{\partial T}\frac{\partial}{\partial t}\Delta T + \frac{\partial \dot{u}}{\partial u}\frac{\partial}{\partial t}\delta u\,. \tag{9.3}$$

The solutions for ΔT and E will be sought, as usual, in the form of Eq. (4.2). Analogously, $\delta u = b\delta u_0(x/b) \times\exp(\lambda t/t_\kappa)$. Then, on eliminating δu_0 from Eqs. (9.2)–(9.3), we derive

$$\lambda\theta = \nabla^2\theta + \varepsilon + \frac{\alpha_T\lambda}{\lambda+\lambda_u}\theta\,, \qquad \nabla^2\boldsymbol{\varepsilon} = \lambda\tau\boldsymbol{\varepsilon} - \lambda\beta\mathbf{e}\theta\,, \tag{9.4}$$

in the region of current flow, and

$$\lambda\theta = \nabla^2\theta + \frac{\lambda\alpha_T}{\lambda+\lambda_u}\theta\,, \tag{9.5}$$

in the region where there is no current. The notations used are

$$\lambda_u = \left|\frac{\partial\dot{u}}{\partial u}\right| t_\kappa\,, \quad \alpha_T = \hat{\sigma}\frac{\partial\dot{u}}{\partial T}\frac{b^2}{\kappa}\,, \quad \mathbf{e} = \boldsymbol{\varepsilon}/\varepsilon\,. \tag{9.6}$$

Estimates show that in the materials of interest to us the values of λ_u are such that $0 < \lambda_u \ll 1$.

The eigenvalue spectrum λ is found from the condition that nontrivial solutions exist to the system of Eqs. (9.4) and (9.5) with appropriate thermal and electrodynamic boundary conditions.

Note further that in the presence of plastic yield a superconductor is already nonuniformly heated in the unperturbed state. Therefore, when deriving the linearized equations (9.1) and (9.2), one should, generally speaking, take into account the coordinate dependence of the initial temperature.

Of principal interest for further discussion, however, are situations in which the characteristic time for magnetic flux jumps to develop and that for plastic strain jerks to develop are of the same order of magnitude. In this case, the interaction between them appears maximum. As has been shown in a number of papers (see Sec. III), the time for development of plastic strain jerk, t_p, is great—$t_p \gg t_\kappa$. We have already discussed corresponding cases when $t_j \gg t_\kappa$. This is true primarily of composite superconductors and hard superconductors in which $w \ll 1$ and $\tau \gtrsim \tau_c$.

In the case of superconducting composites, even those which are liquid-helium-cooled, one can assume that $w \ll 1$. As can be readily shown, at $w \ll 1$ and with a heat release uniform over the cross section, the dependence of the unperturbed sample temperature on the coordinates may be neglected.

2. Critical state stability in a plane plate

Consider by way of example the critical state stability in a plane plate of a hard or composite superconductor with transport current $I = iI_c$ and $H_a = 0$, in the presence of plastic yield of the material. External heat transfer will be assumed to be weak, i.e., $w \ll 1$. Note that in the case of a plate in an external field $H_a \neq 0$ parallel to the plate surface, the stability criteria can be derived from those found below by simple substitution of $c(H_a - H_I)/4\pi b j_c$ for I/I_c.

In the case of a plane geometry, on eliminating ε from Eqs. (9.4), we derive

$$\theta'''' - (\lambda\tau + \tilde{\lambda})\theta'' - \lambda(\beta - \tilde{\lambda}\tau)\theta = 0,$$

$$\tilde{\lambda} = \lambda\left(1 - \frac{\alpha_T}{\lambda + \lambda_u}\right), \tag{9.7}$$

$$\beta = \frac{4\pi b^2 j_c}{c^2 \nu}\left|\frac{dj_c}{dT}\right|.$$

For convenience, we have now defined β in such a manner that this parameter is independent of I/I_c.

The boundary conditions and the conditions for joining solutions at $|x| = b(1 - I/I_c)$ take the form

$$\theta'(\pm 1) \pm W\theta(\pm 1) = 0,$$

$$\lambda\theta'(\pm 1) - \theta'''(\pm 1) = 0, \tag{9.8}$$

$$\tilde{\lambda}\theta[\pm b(1-i)] - \theta''[\pm b(1-i)] = 0.$$

In addition, θ and θ' are continuous at $x = \pm b(1 - I/I_c)$.

The equation for temperature in the region $|x| < b(1 - I/I_c)$ has the form $\theta'' - \tilde{\lambda}\theta = 0$, and its solution (which is symmetrical with respect to the Ox axis) is $\theta = A\cosh[(\tilde{\lambda})^{1/2}\tilde{x}]$.

The solution to Eq. (9.7) in each one of the two regions can be written as

$$\theta = C_1\cosh(\bar{k}_1\tilde{x}) + C_2\sinh(\bar{k}_1\tilde{x}) + C_3\cos(\bar{k}_2\tilde{x}) + C_4\sin(\bar{k}_2\tilde{x}),$$

where

$$\bar{k}_{1,2}^2 = \left(\lambda\beta + \frac{(\tilde{\lambda} - \lambda\tau)^2}{4}\right)^{1/2} \pm \frac{\tilde{\lambda} + \lambda\tau}{2}. \tag{9.9}$$

The dispersion equation for λ is derived from the condition that the determinant of the appropriate linear system of equations relative to C_i vanish.

a. Adiabatic boundary conditions

In this case, the dispersion equation has the form

$$\bar{k}_1(\tilde{\lambda} + \bar{k}_2^2)\tanh(\bar{k}_1 i) + \bar{k}_2(\tilde{\lambda} - \bar{k}_1^2)\tan(\bar{k}_2^2 i) + (\tilde{\lambda})^{1/2}(\bar{k}_1^2 + \bar{k}_2^2)\tanh[(\tilde{\lambda})^{1/2}(1-i)] = 0. \tag{9.10}$$

It will be recalled that at $w = 0$ in the absence of plastic yield, depending on the value of τ, two types of $\lambda(\beta)$ spectrum are possible. At $\tau < \tau_c(i)$, $\lambda_c > 0$, while at $\tau > \tau_c(i)$, $\lambda_c = 0$, where

$$\tau_c(i) = \frac{5}{6}\frac{1}{i^2} - \frac{9}{7}\frac{1}{i} + \frac{1}{2}$$

(Maksimov and Mints, 1979).

Consider first the case of small τ: $\tau \ll 1$, $\tau < \tau_c(i)$. The evolution of the eigenvalue spectrum of $\lambda(\beta)$, depending on the variation of the parameter α_T, found with the aid of numerical calculations, is shown in Fig. 30. Naturally, the dependence $\lambda(\beta)$ suffers the most pronounced variation in the region of slow ($\lambda \ll 1$) perturbations.

Up to $\alpha_T = \alpha_2$ (curve 5 in Fig. 30), the critical state stability depends on rapid perturbations. In this region of parameters, while expanding Eq. (9.10), one can derive for $\beta(\lambda)$

$$\beta = \frac{\pi}{4i^2} - \alpha_T + \lambda\tau + \left(\frac{\pi^2}{4i^2} - \alpha_T\right)\frac{\pi^2}{4}\lambda^{-1},$$

whence

$$\lambda_c = \frac{\pi^2}{4i^2}\left(\frac{1 - 4i^2\alpha_T/\pi^2}{\tau}\right)^{1/2},$$

$$\beta_c = \beta_2 = \frac{\pi^2}{4i^2}(1 + 2\sqrt{\tau}) - \alpha_T. \tag{9.11}$$

Inasmuch as in the region of parameters of interest $\alpha_T \lesssim \lambda_u \ll 1$ it follows from Eq. (9.11) that the characteristics of the eigenvalue spectrum in the region of high λ vary only slightly. The heat release associated with plastic yield has no time to substantially affect the dynamics of such perturbations.

Let us now study the region of low λ. By expanding the dispersion equation (9.10) at $\lambda \ll 1$, one can derive

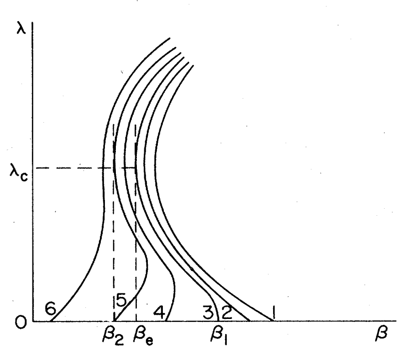

FIG. 30. Evolution of the $\lambda(\beta)$ spectrum at $\beta > \beta_c$ upon variation of the parameter α_T for $w = 0$ and $\tau < \tau_c(i)$: Curve 1, $\alpha_T = 0$; curve 2, $0 < \alpha_T < \alpha_1$; curve 3, $\alpha_T = \alpha_1$; curve 4, $\alpha_1 < \alpha_T < \alpha_2$; curve 5, $\alpha_T = \alpha_2$; curve 6, $\alpha_T > \alpha_2$.

the value of β at which λ vanishes. It is

$$\beta=\beta_1=\frac{3}{i^3}\left(1-\frac{\alpha_T}{\lambda_u}\right). \tag{9.12}$$

The value of $\alpha_T=\alpha_2$ at which the value of λ_c varies from $\lambda_c \gg 1$ to $\lambda_c=0$ stems from the equality $\beta_1=\beta_2$, whence

$$\alpha_2=\lambda_u\frac{1-\frac{1}{12}\pi^2 i(1+2\sqrt{\tau})}{1-\frac{1}{3}\lambda_u i^3}.$$

Analogously, one can find $\alpha_T=\alpha_1$ at which the nature of the spectrum varies in the region $\lambda \ll 1$ (see curve 3 in Fig. 30): $\alpha_1=0.4i^2[\tau_c(i)-\tau]\lambda_u^2 \ll \lambda_u$. On comparing α_1 and α_2, it is clear that $\alpha_1/\alpha_2 \ll 1$. In virtue of this, $\lambda_c=0$ in the region $\alpha \gtrsim \alpha_2$ and the stability criterion has the form

$$\beta<\beta_1. \tag{9.13}$$

For $\alpha_T \sim \lambda_u$, the interaction of plastic strain jerks and flux jumps considerably decreases the critical state stability criterion.

Thus the equation of the curve describing the boundary of critical state and plastic yield stability on the plane of parameters (β, α_T) follows from Eqs. (9.11) and and (9.13) and can be written as

$$\alpha_T+\beta=\frac{\pi^2}{4i^2}(1+2\sqrt{\tau}), \quad \alpha_T<\alpha_2$$

$$\frac{\alpha_T}{\lambda_u}+\frac{\beta i^3}{3}=1, \quad \alpha_T>\alpha_2.$$

The corresponding curve is shown in Fig. 31.

Consider now composite superconductors $(\tau \gg 1)$. In the range of parameters $\tau<\tau_c(i)$ the curves $\lambda(\beta, \alpha_T)$ behave qualitatively like those for the case of $\tau \ll 1$ (see Fig. 30). Calculations analogous with those performed above yield

$$\beta_c=\frac{3.8}{i^2}\sqrt{\tau}\left(1-\frac{\alpha_T}{2\lambda_0}\right),$$

where $\lambda_0=2.5/i^2\tau$ is the value of λ_c at $\alpha_T=0$ (see Sec. IV D 3). Inasmuch as $\alpha_T\tau i^2 \ll 1$ for characteristic values of the parameters, the interaction between plastic strain jerks and flux jumps affects the stability criterion only slightly. The value of α_2 in the case under consideration is equal to

$$\alpha_2=\lambda_u\frac{1-1.3i\tau^{1/2}}{1-0.25\lambda_u i^3\tau^{3/2}}.$$

The value of β_1 at which $\lambda=0$ is defined by Eq. (9.12) for any τ. We now derive for the curve separating the regions of stability and instability on the (β, α_T) plane

$$(\alpha_T\tau^{3/2}/1.3)+\beta=3.8\sqrt{\tau}/i^2, \quad \alpha_T<\alpha_2$$

$$\alpha_T/\lambda_u+\beta i^3/3=1, \quad \alpha_T>\alpha_2.$$

The separating curve $\beta(\alpha_T)$ has qualitatively the same appearance as that shown in Fig. 31.

At $\tau>\tau_c(i)$ for any values of α_T, the parameter $\lambda_c=0$. As a result, the equation for the curve bounding the stability region on the plane (β, α_T) has the form

$$\frac{i^3\beta}{3}+\frac{\alpha_T}{\lambda_u}=1. \tag{9.14}$$

This curve is shown in Fig. 32. Thus, for $\tau \gg 1$, the portion of the $\beta(\alpha_T)$ curve parallel to the X axis first appears under condition $\tau_c(i)=\tau$. Note further that, at $\tau>\tau_c(i)$, the interaction between flux jumps and plastic strain jerks has a more substantial effect on the stability criterion.

b. Weak external cooling ($\lambda_u \ll w \ll 1$)

We shall discuss here only the limiting case of $\tau \gg 1$, when slow perturbations are characteristic. With a view to applying the obtained results to the case of liquid-helium-cooled superconducting composites, we shall regard the value of i to be not too small so that $\tau_c(i) \ll \tau$, $w\tau i^2 \gg 1$.

The curves $\lambda(\beta, \alpha_T)$ in this situation have the appearance shown in Fig. 33. Numerical calculations are required for finding the stability criterion in the general case. However, in a number of limiting cases it is possible to obtain analytical results.

We shall now demonstrate how the stability criterion varies with an increase in α_T. From the dispersion equation in the approximation

$$1/i^2\tau \ll \lambda \ll w, \tag{9.15}$$

one can derive $(\lambda_u \ll w)$

$$\lambda\tau+(w-\alpha_T)\tau-i\beta+\frac{(w-\alpha_T)\sqrt{\tau}}{i\sqrt{\lambda}}=0,$$

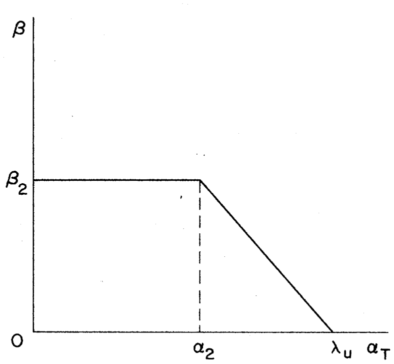

FIG. 31. Boundary of the stable region in the (β, α_T) plane for $w=0$ and $\tau<\tau_c(i)$.

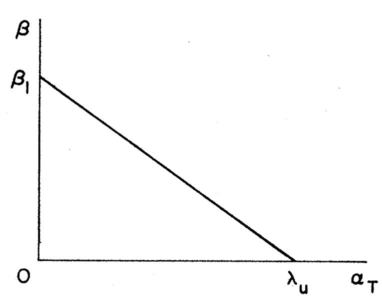

FIG. 32. Boundary of the stable region in the (β, α_T) plane for $w=0$ and $\tau>\tau_c(i)$.

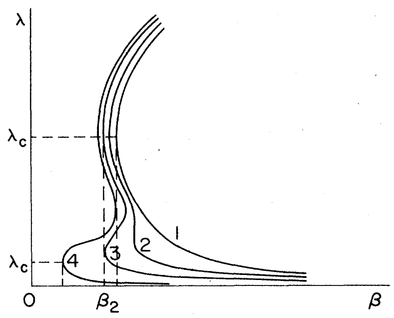

FIG. 33. Evolution of $\lambda(\beta)$ curves with a variation of α_T ($w>0$), $\tau>\tau_c(i)$: curve 1, $\alpha_T=0$; curve 2, $\alpha_T<\alpha_2$; curve 3, $\alpha_T=\alpha_2$; curve 4, $\alpha_T>\alpha_2$.

whence

$$\lambda_c=\tau^{-1/3}\left(\frac{w-\alpha_T}{2i}\right)^{2/3},\qquad \beta_c=\frac{(w-\alpha_T)\tau}{i}\left(1+\frac{1.9}{(w-\alpha_T)^{1/3}i^{2/3}\tau^{1/3}}\right). \tag{9.16}$$

For both inequalities (9.15) to be valid at $\tau\gg1$ it is necessary and sufficient that $1\ll\lambda_c i^2\tau$ or $(w-\alpha_T)i^2\tau\gg1$. With an increase in α_T, this inequality is disturbed and, in the case of $0<(w-\alpha_T)i^2\tau<1$, the dependence $\lambda(\beta,\alpha_T)$ is found from the equation

$$0.4i^2[\tau-\tau_c(i)]\lambda^2-\left(\frac{\beta i^3}{3}-1\right)\lambda+w-\alpha_T=0,$$

whence

$$\lambda_c=\left(\frac{w-\alpha_T}{0.4i^2[\tau-\tau_c(i)]}\right)^{1/2},$$

$$\beta_c=\frac{3}{i^3}(1+2\{0.4i^2(w-\alpha_T)[\tau-\tau_c(i)]\}^{1/2}).$$

And, finally, for $\alpha_T>w$, λ_u should be taken into consideration. For sufficiently large values of α_T, where it can be assumed that

$$\left(\frac{\alpha_T}{w}-1\right)^3\gg\frac{0.4i^2\lambda_u^2[\tau-\tau_c(i)]}{w},$$

we derive the following relationship for finding λ:

$$1-i^3\beta/3+w/\lambda-\alpha_T/(\lambda+\lambda_u)=0$$

from which we obtain

$$\lambda_c=\frac{\lambda_u}{(\alpha_T/w)^{1/2}-1},$$

$$\beta_c=\frac{3}{i^3}\left\{1-\frac{w}{\lambda_u}\left[1-\left(\frac{\alpha_T}{w}\right)^{1/2}\right]^2\right\}.$$

Solving for α_T, we find $\alpha_T=\alpha_c(\beta)$, where

$$\alpha_c=\left\{\left[\lambda_u\left(1-\frac{\beta i^3}{3}\right)\right]^{1/2}+\sqrt{w}\right\}^2. \tag{9.17}$$

On combining the results obtained in this section, we find that on the plane (β,α_T) the stability region is restricted by a curve:

$$\frac{i\beta}{w\tau}+\frac{\alpha_T}{w}=1,\quad 1\ll i^2\tau(w-\alpha_T)$$

$$\frac{i^3\beta}{3}-2\{0.4i^2(w-\alpha_T)[\tau_c-\tau_c(i)]\}^{1/2}=1,\quad 0<i^2\tau(w-\alpha_T)\ll1$$

$$\frac{i^3\beta}{3}+\frac{w}{\lambda_u}\left[\left(\frac{\alpha_T}{w}\right)^{1/2}-1\right]^2=1,\quad w<\alpha_T<\alpha_c.$$

This curve is shown in Fig. 34. Note further that Eq. (9.17) for the limiting value of $\alpha_T=\alpha_c$ is true for any $w\ll1$.

One can study analogously the case of quasi-adiabatic cooling $0<w\ll\lambda_u$ (Maksimov and Mints, 1981).

Thus we have shown that flux jumps and plastic strain jerks interact strongly with each other if the characteristic time for the development of each instability is of the same order. Accordingly, the criterion for thermomagnetomechanical instability may turn out to be lower than the criterion for each of the instabilities developing separately [compare the results presented in this section with those of Secs. III and IV, as well as with the criterion for plastic strain jerk found, for instance, by Malyghin (1975), Petukhov and Estrin (1975), Petukhov (1977), Mints and Petukhov (1980)]. This circumstance, as already noted in Sec. III E, helps us to interpret the training phenomenon as successive strain hardening of the superconductor stimulated by the heat emerging from the development of an instability. As a result, upon subsequent cycling of the current in the sample, the instability occurs at a higher value of current (or of applied mechanical stress). If the mechanical stress is not too high, training may apparently help attain the limiting value of current, where the critical state is stable with respect to magnetic flux jumps.

A method analogous to that used in the case of a plane geometry for $w\ll1$ may be used to study cylindrical samples (Maksimov and Mints, 1981). Note further that all of the stability criteria obtained here included as a parameter the sample temperature at the moment when instability emerged $T_1=T_1(\hat{\sigma},w,\dot{u})>T_0$. Consequently, in order to determine, say, the maximum permissible mechanical stress at a preset value of transport current, one should first calculate T_1 on the basis of the known dependence $\dot{u}=\dot{u}(u,T,\hat{\sigma})$.

B. Simplified theory

An analytical study of the critical state stability under conditions of plastic yield of the material can only be performed at $w\ll1$ when the temperature in the unperturbed state is practically uniform. Even in this

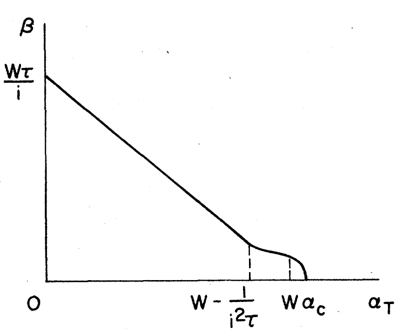

FIG. 34. Boundary of the stable region in the (β,α_T) plane for $\lambda_u\ll w\ll1$, $\tau>\tau_c(i)$.

case, however, considerable calculational difficulties arise. It is therefore desirable that simpler methods be found for studying stability, for example, methods analogous to those used in Sec. IV.C.1 for $\tau \ll 1$ or in Sec. IV.D.2 for $\tau \gg 1$. The development of such a theory for the case of τ, $w\tau \gg 1$ presents no special problems. Indeed, as follows from the results presented in the preceding section, $\lambda_u \ll \lambda_c \ll 1$, $\lambda_c \tau \gg 1$. Therefore one can assume that the instability develops with frozen-in magnetic flux and that $\partial j/\partial t = 0$. The latter condition permits us to reduce, as we did before, the system of differential equations (9.4) being studied from fourth order to second order.

In the approximation under consideration, the relationship between the electric field and the temperature perturbations is defined by Eq. (3.13). Accordingly, in a linear approximation, the heat released during perturbation development and associated with the electric field is equal to

$$\frac{j_c \Delta T}{\sigma}\left|\frac{dj_c}{dT}\right| . \tag{9.18}$$

The corresponding term for heat release associated with plastic strain is

$$\hat{\sigma}\Delta T \frac{\partial \dot{u}}{\partial T} . \tag{9.19}$$

In the case under discussion the only manifest effect of plastic yield of the material upon the critical state stability is the release of additional heat.

If $w \ll 1$ and the temperature T_1 is practically uniform, we readily find

$$\nabla^2 \theta + (\beta/\tau + \alpha_T)_{T_1}\theta = 0 . \tag{9.20}$$

Using this equation, one can, for example, obtain the stability criterion (3.19) and define the parameter γ_3 for different situations. In particular, for the case of a plane geometry this criterion naturally coincides, in the main ($\tau \gg 1$) approximation, with Eq. (9.16), the value of γ_3 in this case being equal to w/i.

If the heating due to stationary plastic yield is low, i.e., $T_1(x) = T_0[1 + \theta_1(x)]$, where $\theta_1 \ll 1$, the coordinate dependence of heat conductivity and critical current density may be neglected and they can both be regarded as functions of $T_0 = \text{const}$.

In numerous materials at low temperatures the temperature dependence of $\dot{u}$ is exponential in nature:

$$\dot{u} = \dot{u}_0 \exp[f(T)] , \tag{9.21}$$

where $\dot{u}_0 = \text{const}$, and $f(T) \gg 1$. Then the dependence of α_T upon $\theta_1(x)$ cannot be neglected. The equation for a small perturbation θ [Eq. (9.20)] assumes the form

$$\nabla^2 \theta + [\beta/\tau + \alpha_T(\tilde{x})]\theta = 0 . \tag{9.22}$$

Let us consider the simplest case of a plane plate having a thickness of $2b$ with transport current $I = I_c$. We first find the initial temperature distribution. Assume $T_1 = T_0[1 + \theta_1(\tilde{x})]$, where $\theta_1 \ll 1$ but $[df(T_0)/dT_0]T_0\theta_1 \gtrsim 1$. Then

$$\theta_1'' + \hat{\sigma}\frac{b^2 \dot{u}_0}{\kappa T_0}\exp\left(f(T_0) + \frac{df(T_0)}{dT_0}T_0\theta_1\right) = 0 . \tag{9.23}$$

Denote the function

$$\varphi = \frac{df(T_0)}{dT_0}T_0\theta_1 . \tag{9.24}$$

After that, we derive

$$\varphi'' + \alpha_0 e^{\varphi} = 0 , \tag{9.25}$$

where $\alpha_0 = \alpha_T(T_0)$; the boundary condition on Eq. (9.25) has the form

$$(\varphi' \pm w\varphi)_{\tilde{x}=\pm 1} = 0 . \tag{9.26}$$

The solution of Eq. (9.25), satisfying the boundary condition (9.26) and symmetrical to the OX axis, is

$$\varphi = \ln\left(\frac{2A^2}{\alpha_0}\cosh^{-2}(A\tilde{x})\right), \tag{9.27}$$

where the constant A is found from the relationship

$$\alpha_0 = \frac{2A^2}{\cosh^2 A}\exp\left(-\frac{A\tanh A}{w}\right). \tag{9.28}$$

It follows from Eqs. (9.27) and (9.28) that $\varphi_{\max} \sim 1$ since $\alpha_0 < 1$. Consequently, $A \lesssim 1$, and $\theta_1 \sim 1/f' \sim f(T_0) \ll 1$.

On substituting the initial temperature distribution, found above, in Eq. (9.22), we derive a solution which is symmetric with respect to the OX axis:

$$\theta = C\{(\beta/\tau)^{1/2}\cos[(\beta/\tau)^{1/2}\tilde{x}] - A\sin[(\beta/\tau)^{1/2}\tilde{x}]\tanh(A\tilde{x})\} .$$

By substituting the latter solution into the boundary condition, we find that the stability criterion has the form

$$\beta < \beta_c ,$$

where β_c is defined by the equation

$$\frac{\beta_c}{\tau}\tan\left(\frac{\beta_c}{\tau}\right)^{1/2} + A\left(\frac{\beta_c}{\tau}\right)^{1/2}\tanh A + \frac{A^2\tan\left(\frac{\beta_c}{\tau}\right)^{1/2}}{\cosh^2 A} = w\left[\left(\frac{\beta_c}{\tau}\right)^{1/2} - A\tan\left(\frac{\beta_c}{\tau}\right)^{1/2}\tanh A\right], \tag{9.29}$$

while the parameters A and α_0 are related to each other by Eq. (9.28).

For $w \ll 1$, we obtain from Eq. (9.29) $2A^2 = w - \beta/\tau$ to the desired degree of accuracy. On substituting this expression into Eq. (9.28), we derive

$$\frac{\alpha_0}{\alpha_c} = \left(1 - \frac{\beta_c}{w\tau}\right)\exp\left(\frac{\beta_c}{w\tau}\right), \tag{9.30}$$

where $\alpha_c = w/e$.

C. Experimental studies of thermomagnetomechanical instability and training

The training phenomenon in superconducting coils was discovered in the early sixties (Leblanc, 1961). However, the possibility that training was related to the inherent properties of superconducting materials (namely, to plastic yield instability) was first proposed by Evans in 1973 [for a review of the literature see Pasztor and Schmidt (1978)]. And it was only in the mid-seventies that studies of training were begun in short superconductor samples, under the effect of high mechanical loads. Indeed, it was under such condi-

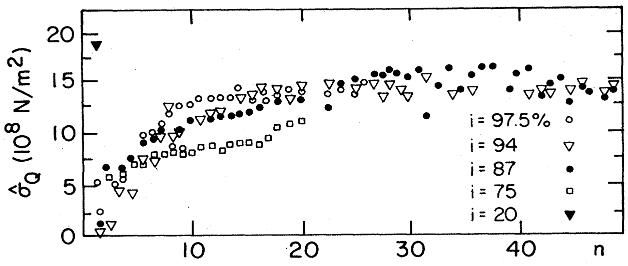

FIG. 35. Dependence of $\hat{\sigma}_Q$ upon n for a liquid-helium-cooled Nb-Ti wire, at different values of the I/I_c ratio (Pasztor and Schmidt, 1978).

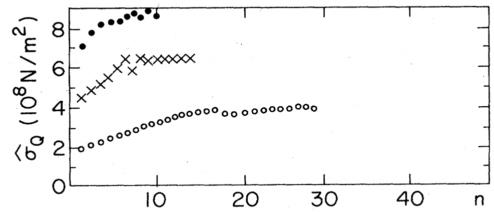

FIG. 37. The value of $\hat{\sigma}_Q(n)$ for a superconducting composite under different conditions of external cooling, $I/I_c = 95\%$: •, sample placed in liquid helium; ○, sample under vacuum; ×, heat-insulated sample in liquid helium (Pasztor and Schmidt, 1978).

tions that training was discovered (Anashkin *et al.*, 1975; Schmidt, 1976; Schmidt and Pasztor, 1977). Further studies confirmed the relationship between training and plastic yield of the material, in particular, plastic strain jerks (Pasztor and Schmidt, 1978; Anushkin *et al.*, 1977, 1979).

Shown in Figs. 35 and 36 is the dependence of stress $\hat{\sigma}_Q$ at which the transition to normal state occurs upon the number of loading cycles n for a hard superconductor (Fig. 35) and a superconducting composite (Fig. 36) at different I/I_c ratios. (Pasztor and Schmidt, 1978). As follows from theory, the higher the current the lower the stress corresponding to quench. Accordingly, an increase in mechanical stress is accompanied with a decrease in the current at which the instability occurs.

Figure 37 plots the values of $\hat{\sigma}_Q(n)$ taken from the same paper for a superconducting composite under different external cooling conditions. As can be seen from the figure, the stability increases appreciably with increased external heat transfer, which is in agreement with theoretical results. An analogous effect is observed in the case of hard superconductors.

As is known, preloading of the sample leads to mechanical hardening. The value of $\hat{\sigma}_Q$ at which a plastic strain jerk occurs increases accordingly. Such a hardening effect should naturally occur in the case of thermomagnetomechanical instability as well. Indeed, it was observed in the papers cited above by Schmidt, Schmidt and Pasztor, and Keilin and co-workers. The respective values of $\hat{\sigma}_Q(n)$ obtained by Pasztor and Schmidt (1978) prior to and after preloading are presented in Fig. 38(b) for a hard superconductor and in Fig. 38(a) for a superconducting composite.

The study of sound emissions has proven to be an interesting method of investigating the dynamics of instability development in the critical state, both in the presence (Pasztor and Schmidt, 1978) and in the absence (Pasztor and Schmidt, 1979) of plastic yield. Plotted in Fig. 39 as a function of $\hat{\sigma}$ are the number of surges of sound activity exceeding a certain power level (in the present case, 92 dB) at a frequency of 100–300 Hz for a number of cycles of mechanical loading of a Nb-Ti sample (Pasztor and Schmidt, 1978).

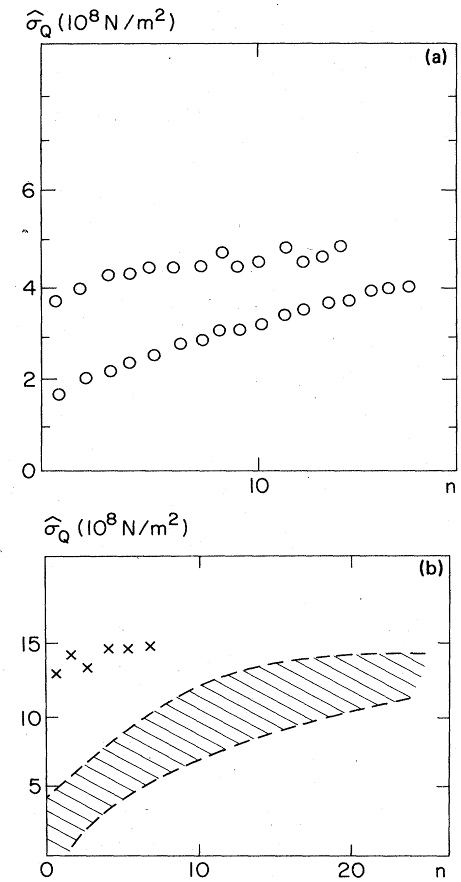

FIG. 38. Effect of prestrain upon training and stability (Pasztor and Schmidt, 1978). (a) superconducting composite, with the bottom points obtained for a sample without prestrain and the top points after five cycles of loading from 3.6×10^8 N m^{-2} to 12×10^8 N m^{-2}; (b) Nb-Ti, with the hatched area containing training curves for samples that have not been prestrained while the top points are for a sample after five loading cycles.

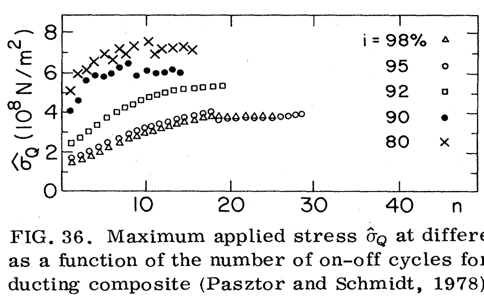

FIG. 36. Maximum applied stress $\hat{\sigma}_Q$ at different values of I/I_c as a function of the number of on-off cycles for a superconducting composite (Pasztor and Schmidt, 1978). The sample was under vacuum.

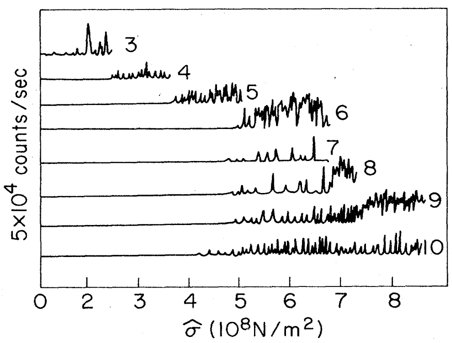

FIG. 39. Intensity of acoustic emission from Nb-Ti wire as a function of stress after several loading cycles (Pasztor and Schmidt, 1978).

As follows from the figure, a marked increase in emission occurs at $\hat{\sigma}$ values corresponding to the quench in the preceding cycle.

X. CONCLUSIONS

In this section, we shall briefly discuss a series of problems associated with the study of critical state stability in a number of superconducting materials.

A. Interaction of thermomagnetic and thermomechanical instabilities

The theoretical and experimental study of this problem has in fact begun only recently. A quantitative comparison of the available experimental and theoretical results, followed with a study into possible interaction mechanisms of magnetic flux jumps and plastic strain jerks, would be of considerable interest here. For example, Gurber (1977), Rupp (1977), and Ziegler (1978) have shown experimentally that plastic strain affects considerably both the value of j_c and the shape of the current-voltage characteristic. Naturally, it should also affect the development of thermomagnetomechanical perturbations.

B. Ultrasonic emission and attenuation

As already noted, sound emission and variation in ultrasonic attenuation have been observed while studying the development of magnetic flux jumps and the training phenomenon in superconductors. One can expect that further acoustic studies of superconductors in the critical state will provide additional information on the development of perturbations of a diverse physical nature (Pasztor and Schmidt, 1978, 1979; Nomura *et al.*, 1980). Such papers hold considerable applied interest as well (Nomura *et al.*, 1977, 1980; Pasztor and Schmidt, 1978). For example, in large superconducting systems ultrasonic diagnostic techniques may prove convenient for determining the stability threshold. Up to now, ultrasonic attenuation in hard superconductors has been studied only in the absence of instabilities (Neuringer and Shapira, 1966, 1967; Missell *et al.*, 1976a, b; Missell, 1979), while ultrasonic emission has not thus far been studied.

C. Instability development in time

The present review discusses only the initial stages of development of various instabilities, when perturbations may be regarded as small. In such discussions, naturally, one cannot determine the final state to which the superconductor will pass as a result of instability. The only exception is the case of small limited magnetic flux jumps when perturbations decrease due to the oscillatory nature of the eigenvalue spectrum (see Sec. VII).

Theoretical studies of the time evolution of instability must deal with several evident difficulties. Since one has to solve a system of two nonlinear diffusion equations, analytic results are difficult to obtain. Note that analogous problems arise in other fields of physics and chemistry, such as the theory of combustion, chemical kinetics, biophysics, etc.

The statement of a nonlinear problem immediately gives rise to the question (Neuringer and Shapira, 1966; Wipf, 1967) of whether slight nonlinearity leads to stabilization of the critical state if the stability criterion found in a linear approximation is violated or, in other words, whether only a minor shift of the magnetic flux occurs upon violation of the stability criterion. This problem has been studied both experimentally and theoretically. The experimental study in this case is rendered difficult by the fact that the magnetic flux enters a hard superconductor in the form of bundles containing a large number (up to 10^4–10^5) of vortices (Anderson and Kim, 1964). As a result, it appears rather difficult to draw a line between the movement of large bundles of vortex lines and small flux jumps. However, at least under certain conditions, no such minor shifts of magnetic flux occur (Urban, 1970).

Wipf (1967) made an attempt at obtaining an analytical criterion for determining the amplitude of a flux jump. According to his result, both large and small flux jumps may occur. However, in order to simplify the calculations, he made a number of assumptions whose applicability range is unclear.

Wipf and Lubell (1965) and Swartz and Bean (1968) have assumed that a flux jump can take place until almost complete disappearance of superconductivity, provided the energy of the superconducting currents is sufficient for heating the sample up to $T \gtrsim T_c$. It should be noted that this assertion lacks adequate proof in the papers cited above.

In a series of papers by Morton (1968a, b) and Morton and Darby (1973), nonlinear electrodynamic and heat equations for the case of hard superconductors in the critical state were solved numerically. Unfortunately, it is hard to derive any general conclusions from these papers because of the small number of cases calculated and the absence of analytical results.

Naturally, in the experiments the development of a flux jump is traced from its beginning to its end. Relatively few papers, however, study the effect of experimental conditions on the final state to which the superconductor passes. In practice, flux jumps accompanied by a complete disappearance of superconducting current and partial flux jumps when the superconducting current drops in value but does not disappear have both been experimentally observed (Urban, 1970; McFarlane and Dew-Hughes, 1970; Boyer *et al.*, 1971). Boyer *et al.* (1971) found the magnitude of partial flux jumps to increase with the deterioration of cooling. According to Chikaba *et al.* (1968) and Urban (1970), the number of limited flux jumps in a given external magnetic field range grows, while their magnitude drops, with an increase in the rate of variation of the external field $\dot{H}_a$. It should be noted that the reason for such effects in these experiments is not yet clear.

D. Heat solitons

The development of one or another instability in the critical state is accompanied by intensive local heat release. This makes possible the formation and subsequent propagation (or localization) of normal phase regions of finite dimensions—resistive domains (Volkov and Kogan, 1974; Skocpol *et al.*, 1974; Mints, 1979; Gurevich and Mints, 1980; 1981). Experimentally, this problem has not yet been properly studied. There exist only indications of the possible existence of resistive domains and periodic resistive structures (Iwasa and Williams, 1968; Wipf and Soel, 1972).

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