Phase Retrapping in a Pointlike φ Josephson Junction: The Butterfly Effect

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We consider a φ Josephson junction, which has a bistable zero-voltage state with the stationary phases $\psi = \pm\varphi$. In the nonzero voltage state the phase "moves" viscously along a tilted periodic double-well potential. When the tilting is reduced quasistatically, the phase is retrapped in one of the potential wells. We study the viscous phase dynamics to determine in which well ($-\varphi$ or $+\varphi$) the phase is retrapped for a given damping, when the junction returns from the finite-voltage state back to the zero-voltage state. In the limit of low damping, the φ Josephson junction exhibits a butterfly effect—extreme sensitivity of the destination well on damping. This leads to an impossibility to predict the destination well.

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The term *butterfly effect* is widely used to denote the extreme sensitivity of complex dynamical systems to initial conditions [1,2]. The effect puts a clear distinction between determinism and predictability. For example, due to the butterfly effect it is not possible to predict the weather reliably for more than 3–5 days in advance. Although the original work [3] was related to simulation of atmospheric processes, it was discovered later on that in quite a few problems of nonlinear physics, a tiny perturbations of initial conditions might lead to completely different final states. In particular, the butterfly effect was observed in simulations of a long Josephson junction subjected to an oscillating magnetic field [4].

Consider now a pointlike φ Josephson junction (φ JJ) proposed theoretically [5–7] and recently demonstrated experimentally [8]. This φ JJ has a doubly degenerate ground state phase $\psi = \pm\varphi$, which is a result of the unusual Josephson energy profile:

$$U_J(\psi) = 1 - \cos(\psi) + \frac{\Gamma_0}{4}[1 - \cos(2\psi)]. \qquad (1)$$

The energy $U_J(\psi)$ has a form of a 2π-periodic double-well potential with wells at $\psi = \pm\varphi$; see Fig. 1. The ground state phase $\varphi = \arccos(-1/\Gamma_0)$. The parameter $\Gamma_0 < 0$ defines the depth of the wells [7,9]. The potential has two wells per period for $\Gamma_0 < -1$. Application of a bias current γ tilts the potential as shown in Fig. 1.

Since in the zero-voltage state the φ JJ is bistable, it is interesting to understand in which of these two states the phase is retrapped when the φ JJ returns from the finite-voltage state to the zero-voltage state upon the quasistatic decrease of the tilt (bias current density) γ. Note that for conventional 0 or for π JJs with only a single energy minimum per period of Josephson energy, such a question does not arise. Earlier [10] we have naively suggested that upon returning from the positive-voltage state to the zero-voltage state, the phase is retrapped in the $+\varphi$ state. This is indeed true for rather large damping [8]. However, at lower damping the behavior is nontrivial and often experimentally seems to be nondeterministic [8].

Here we study the retrapping process of the Josephson phase in a pointlike φ JJ and demonstrate that at low damping the system exhibits the butterfly effect.

The dynamics of a φ JJ can be described by the equation of motion for the phase $\psi(t)$ (see the Supplemental Material [11])

$$\ddot{\psi} + \frac{\partial U_J}{\partial \psi} = \gamma - \alpha\dot{\psi}, \qquad (2)$$

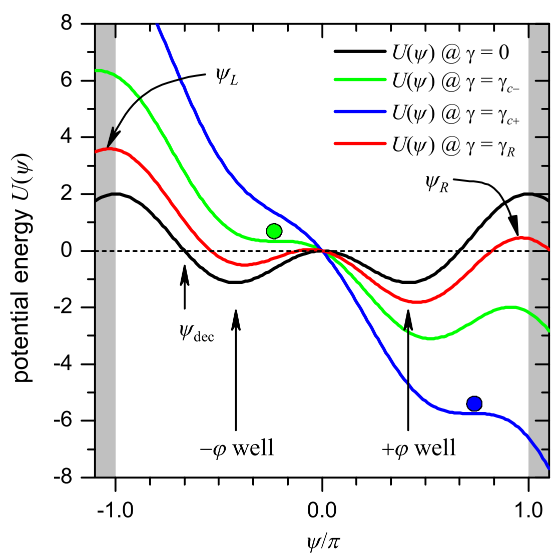

FIG. 1 (color online). Tilted periodic double-well (Josephson) potential $U(\psi)$ given by Eq. (3) for $\Gamma_0 = -4$ and different values of bias current (tilt) γ.

where $\Gamma_0 < -1$ is a parameter of the potential defining its depth, γ is the bias current density normalized to the average critical current density $\langle j_c(x)\rangle$, and α is the dimensionless damping coefficient [also normalized using $\langle j_c(x)\rangle$]. This model describes well a φ JJ made out of a 0-π JJ with facet lengths $L_0 \lesssim \lambda_{J,0}$ and $L_\pi \lesssim \lambda_{J,\pi}$ [5–7,9], where $\lambda_{J,0}$ and $\lambda_{J,\pi}$ are the Josephson lengths in the 0 and π parts, accordingly. For 0-π JJs with somewhat longer facets, like in experiment [8], it holds qualitatively. For experimental parameters the double-well potential $U(\psi)$ calculated numerically is not so deep as in the model, Eq. (1), as can be seen in Fig. 1 of Ref. [8]. This leads only to a quantitative rather than qualitative difference in the results obtained.

Equation (2) describes a phase (a pointlike particle of a unit mass with the coordinate ψ) moving viscously (term $-\alpha\dot{\psi}$) in a tilted 2π-periodic double-well potential:

$$U(\psi) = U_J(\psi) - \gamma\psi; \tag{3}$$

see Fig. 1. γ sets the tilt.

The main process that we are interested in here is the dynamics of switching from the finite-voltage state to the zero-voltage state. At $\gamma = 0$ the phase is trapped in one of the wells of the potential $U(\psi)$, i.e., at $\psi = -\varphi$ or at $\psi = +\varphi$. Upon an increase of the bias current γ, the potential $U(\psi)$ tilts and, at some value of the bias current γ, the phase escapes because the corresponding well disappears. For $\gamma > 0$ the phase escapes from the $-\varphi$ well at $\gamma = \gamma_{c-}$ and from the $+\varphi$ well at $\gamma = \gamma_{c+} > \gamma_{c-}$ as found earlier [7,8]; see Fig. 1. After escape, the phase slides viscously along the periodic potential. The voltage across the junction is proportional to the velocity of the phase motion $\dot{\psi}(t)$. Further, we start decreasing the bias current density (tilt) γ *quasistatically*. At some $\gamma = \gamma_R$, which depends of the damping α, the phase is retrapped in one of the wells. It is this retrapping process that is the main subject of this study.

Note that, in general, the damping α is a function of temperature T. However, the temperature is also responsible for thermal fluctuations that can be added as an additional stochastic current to the rhs of Eq. (2). In the following we assume that such fluctuations are negligible (zero) and the only effect of temperature is the change in α. At the end we discuss shortly the effect of these fluctuations on our results.

To analyze the retrapping process, first, we search the value of the tilt γ_R (retrapping current) at a given damping α. The retrapping situation corresponds to the trajectory, on which the phase starts with zero velocity at the main maximum of the potential $U(\psi)$ situated at $\psi = \psi_L$ (see Fig. 1) slides down viscously, passes two minima and one maximum, and arrives to $\psi_R = \psi_L + 2\pi$ with zero velocity. The value of ψ_L is one of the roots of the equation $\partial U/\partial\psi = 0$, i.e., from Eq. (3),

$$\left.\frac{\partial U_J}{\partial \psi}\right|_{\psi=\psi_L} = \gamma, \tag{4}$$

corresponding to the maximum of $U(\psi)$.

Since ψ_L depends on γ, it is more convenient to fix the tilt γ and look for the critical value of $\alpha_R(\gamma)$ at which retrapping occurs, rather than looking for $\gamma_R(\alpha)$. To find $\psi(t)$, Eq. (2) was solved for fixed γ and α with initial conditions $\psi(0) = \psi_L + \epsilon$ and $\dot{\psi}(0) = 0$ (typically we use $\epsilon \sim 10^{-6}$). The solution was calculated up to the point where either $\psi(t) > \psi_L + 2\pi$ or where $\dot{\psi} < 0$. In the first case the particle is *not* trapped for given γ and α and moves to the next period of the potential. In the second case the particle is trapped. By varying α we repeat the simulation to find the boundary values $\alpha_R(\gamma)$ between the above two cases with a given accuracy of 10^{-6}. The resulting plots of already inverted $\gamma_R(\alpha)$ dependences for different values of Γ_0 are shown in Fig. 2. One can see that the dependences $\gamma_R(\alpha)$ are almost linear.

In the limit of $\alpha \to 0$ and $\gamma \to 0$ one can use a simple perturbation theory (PT) to obtain the slope of this linear dependence. We assume that α and γ are perturbations. Without perturbations ($\alpha = \gamma = 0$) the phase dynamics is governed by the equation

$$\ddot{\psi} + \left[\sin(\psi) + \frac{\Gamma_0}{2}\sin(2\psi)\right] = 0, \tag{5}$$

which has the first integral

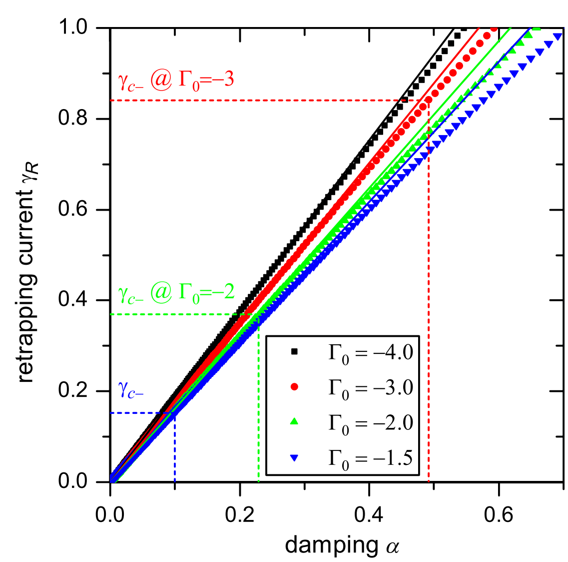

FIG. 2 (color online). The dependence $\gamma_R(\alpha)$ for different values of Γ_0. Symbols represent the results of direct numerical simulation. Lines show PT results given by Eq. (10) for $\alpha \to 0$. The horizontal dashed lines show the values of the depinning current γ_{c-} for given Γ_0, i.e., the current, at which the $-\varphi$ well disappears. For $\gamma > \gamma_{c-}$ the potential has only one $+\varphi$ well and the phase is retrapped there. The vertical dashed line shows the corresponding value of $\alpha_R(\gamma_{c-})$. For JJ with $\alpha > \alpha_R(\gamma_{c-})$ the retrapping current $\gamma_R(\alpha) > \gamma_{c-}$ and potential has only one $+\varphi$ well where the phase is retrapped.

$$\dot{\psi}^2 = C + \left[2\cos(\psi) + \frac{\Gamma_0}{2}\cos(2\psi)\right], \tag{6}$$

where $C = 2 - (\Gamma_0/2)$ is determined from the initial condition for the retrapping trajectory: $\dot{\psi}(-\infty) = 0$, $\psi(-\infty) = \psi_L$. Now, if we turn on the small damping α and the bias γ, they will lead to dissipation and driving, correspondingly. The dissipated energy Q along the "critical" path from $\psi_L = -\pi$ to $\psi_L + 2\pi = +\pi$ (for $\gamma = 0$) is

$$Q = \alpha \int_{-\pi}^{+\pi} \dot{\psi}\, d\psi = \alpha I(\Gamma_0), \tag{7}$$

where, using Eq. (6), we define

$$I(\Gamma_0) = \int_{-\pi}^{+\pi} \sqrt{2[1+\cos(\psi)] - \Gamma_0 \sin^2(2\psi)}\, d\psi, \tag{8}$$

which can be calculated numerically for any Γ_0.

The energy input due to the tilt γ is

$$E_\gamma = \gamma[\psi_L + 2\pi - \psi_L] = 2\pi\gamma. \tag{9}$$

In the case of retrapping trajectory, E_γ compensates Q and brings the particle exactly to the position $\psi_L + 2\pi$ with zero velocity. Thus, from $E_\gamma = Q$ we get

$$\gamma_R(\alpha) = \frac{I(\Gamma_0)}{2\pi}\alpha. \tag{10}$$

We note that in the limit $\Gamma_0 \to 0$ we have $I(\Gamma_0) \to 8$ and obtain a well-known result [12] valid for conventional JJ with sinusoidal CPR, namely $\gamma_R = (4/\pi)\alpha$. The lines corresponding to $\gamma_R(\alpha)$ dependences, Eq. (10), are shown in Fig. 2 and agree well with numerical data for $\alpha \to 0$.

Knowing the dependence $\alpha_R(\gamma)$ we now take various values of γ, take the corresponding $\alpha_R(\gamma)$, put the phase at the turning point $\psi = \psi_R(\gamma) - \epsilon$ (see Fig. 1), and follow its time evolution. The ultimate goal is to see in which well ($-\varphi$ or $+\varphi$) the phase is trapped. The decision about trapping is taken when the velocity $\dot{\psi}$ changes the sign two times in a row on the *same* side relative to the energy barrier separating the two wells. Examples of the trajectories on the phase plane ($\dot{\psi}$, ψ) are shown in Fig. 3. Thus we get the destination well vs $\alpha_R(\gamma)$ dependence.

Figure 4 shows the destination well ($-\varphi$ or $+\varphi$) as a function of α. One can see that, indeed, for large α the phase is trapped in the $+\varphi$ well, as predicted [7] and demonstrated experimentally [8]. However, as α decreases, the destination well changes from $+\varphi$ to $-\varphi$ then back to $+\varphi$ and so on. The intervals of α, corresponding to the retrapping in a particular well, become smaller and smaller even on a logarithmic scale; see Fig. 4. Thus, in the limit of small α the destination well is extremely sensitive to the initial conditions—a tiny variation (or fluctuation) of α or γ (thermal or electronic noise) results in a global effect—retrapping in a different well. Thus, our φ JJ exhibits the butterfly effect.

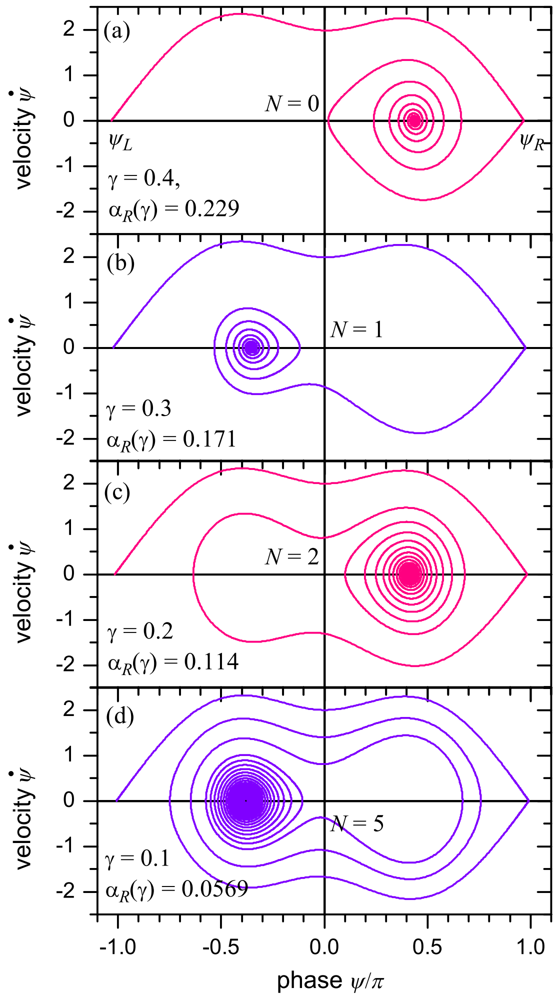

FIG. 3 (color online). The retrapping trajectories in the phase plane ($\dot{\psi}$, ψ) for $\Gamma_0 = -3$ and different tilt γ. The phase starts at ψ_L, where $U(\psi)$ has a maximum, with $\dot{\psi} = 0$ and arrives to $\psi_R = \psi_L + 2\pi$ [the next $U(\psi)$ maximum] with $\dot{\psi} = 0$. Then the phase falls back and is trapped in one of the minima of the $U(\psi)$.

In our case, the butterfly effect prevents one from forecasting, in which well the phase will be retrapped in an experiment in the limit of small α. In fact, already in the first experimental work [8] on φ JJs it was seen that the retrapping is not deterministic at low damping (temperatures ~300 mK). In the experiment, due to the inevitable presence of noise, the destination well vs the α curve will be smeared. If we assume a low frequency Gaussian electronic noise of the amplitude σ_γ in the bias circuitry, one can calculate the probability P_- to find the system in the $-\varphi$ state by making a convolution of the $\pm\varphi(\alpha)$ curve with the Gaussian distribution of width σ_γ. The resulting $P_-(\alpha)$ is also shown in Fig. 4. One can see that the noise smears the fast switchings and $P_- \to 1/2$ at $\alpha \to 0$.

However, the presented model is oversimplified. If one includes a stochastic (instrumental noise or thermal fluctuations) current term in the rhs of Eq. (2), it will also affect

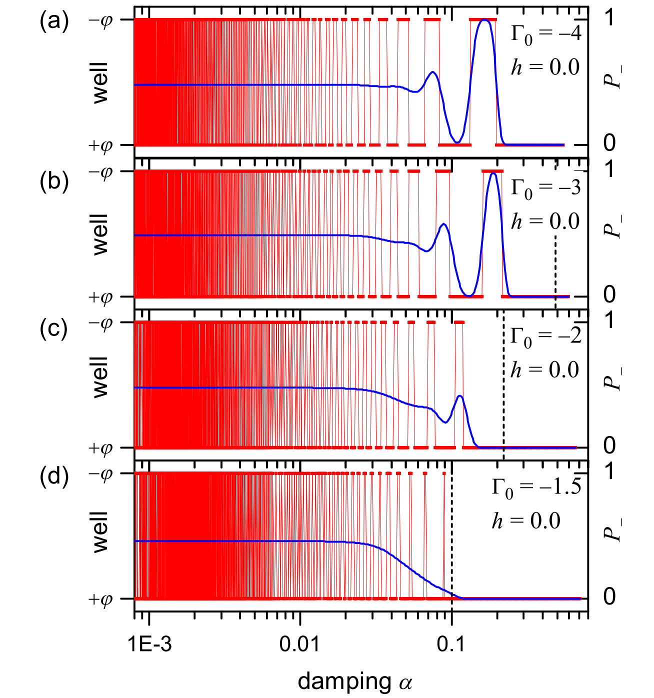

FIG. 4 (color online). The well in which the particle is trapped at $\gamma_R(\alpha)$ as a function of α (symbols). Vertical dashed line show the value $\alpha_R(\gamma_{c-})$. For $\alpha > \alpha_R(\gamma_{c-})$ the potential has only a single $+\varphi$ well. In (a) $\alpha_R(\gamma_{c-}) \approx 1.327$ and is not visible. The lines show the effect of the low frequency Gaussian noise with $\sigma_\gamma = 0.02$ in the bias current circuitry. In this case the right vertical axis represents the probability P_- to find the system in the $-\varphi$ state.

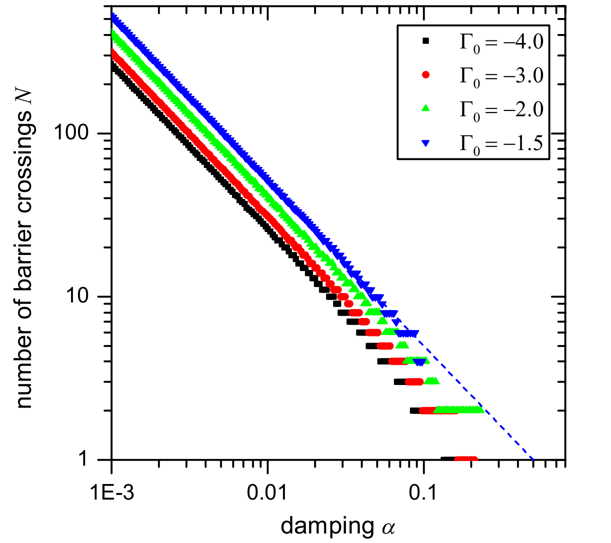

FIG. 5 (color online). The dependence $N(\alpha)$ for different values of Γ_0. At low N one sees that N is an integer. At high N it is a straight line on this double log scale, i.e., it corresponds to $N \propto 1/\alpha$.

the value of the retrapping current $\gamma_R(\alpha)$ making it not well defined (smeared) with the ensemble average $\langle\gamma_R(\alpha)\rangle$ larger than $\gamma_R(\alpha)$ calculated above [13,14]. A rigorous treatment of noise will be presented elsewhere.

At the end we would like to mention an interesting detail. When simulating a retrapping dynamics, we also have counted how many times N the phase crossed the barrier separating the wells before being trapped in one of the wells. In fact, the well ($+\varphi$ or $-\varphi$) plotted in Fig. 4 is just $\varphi[1 - 2(N \bmod 2)]$. Figure 5 shows $N(\alpha)$ plots for different values of Γ_0. Note that N is an integer so it changes stepwise, as it is well visible for large α. For small α (large N) the dependence looks almost continuous and can be well approximated by $N \approx C_\alpha/\alpha$. The coefficient C_α depends on Γ_0. In the Supplemental Material [11] it is proven analytically that $N = C_\alpha/\alpha$ for *any* potential $U(\psi)$ in the PT limit $\alpha \ll 1$. In our case, using $\psi_{\rm dec} = \arccos[-(2+\Gamma_0)/\Gamma_0]$ (see Fig. 1) and $\psi(E_{\max}) = \pi$ we get

$$C_\alpha^{\rm PT} = \int_{\psi_{\rm dec}}^{\pi} \frac{\partial U(\psi_{\rm st})}{\partial \psi_{\rm st}} \frac{d\psi_{\rm st}}{W(\psi_{\rm st})}, \tag{11}$$

where

$$W(\psi_{\rm st}) = \int_{-\psi_{\rm st}}^{+\psi_{\rm st}} \sqrt{2[U(\psi_{\rm st}) - U(\psi)]}\,d\psi. \tag{12}$$

The values of C_α obtained from direct simulations as well as $C_\alpha^{\rm PT}$ calculated using Eq. (11) are summarized in Table I together with other key numbers.

In conclusion, we have studied the retrapping of the phase in a pointlike φ JJ upon transition to zero-voltage state. For given damping α, we have calculated the retrapping current γ_R and the destination well, where the phase is trapped. For large α it is always a deeper well ($+\varphi$ for $\gamma > 0$). However, as α decreases, the dependence of the destination well on α is an oscillating function, with oscillations (switching of the destination well) happening faster and faster even on the logarithmic scale; see Fig. 4. Thus, at $\alpha \to 0$ a tiny variation of α or γ (noise) leads to a different destination well, i.e., to a butterfly effect. Detailed treatment of the noise will be presented elsewhere.

The butterfly effect at small damping does not allow us to manipulate the φ JJ by means of the bias current as described earlier [8,10]. Simultaneously, in this regime one can use a φ JJ as a random number generator (coin or dice) giving the output of $-\varphi$ or $+\varphi$ randomly. The extreme sensitivity may also be exploited in amplifiers or detectors as well as for the investigation of the fine details of the JJ

TABLE I. The values of key quantities.

Γ_0	γ_{c-}	$\alpha_R(\gamma_{c-})$	$C_\alpha(\Gamma_0)$	$C_\alpha^{\rm PT}(\Gamma_0)$
−1.5	0.153	0.100	0.521	0.522
−2.0	0.369	0.229	0.412	0.412
−3.0	0.840	0.492	0.313	0.314
−4.0	1.327	1.327	0.263	0.264

dynamics itself. In the quantum regime the dynamics described here may lead to extremely strong mixing or entanglement of the states $|-\varphi\rangle$ and $|+\varphi\rangle$.

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