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Self-generated flux in Josephson junctions with alternating critical current density

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We study an interesting state with a self-generated magnetic flux in a long Josephson junction. The critical current density j_c is assumed to alternate randomly along the tunnel contact on a length scale much less than the local Josephson penetration depth. The phase difference is then a sum of an alternating term and a smooth term which self-consistently contribute to the formation of a stationary state with self-generated flux. Two types of Josephson vortices are found in this state, one with magnetic flux $\Phi_1 < \Phi_0/2$ and another one with magnetic flux $\Phi_2 = \Phi_0 - \Phi_1 > \Phi_0/2$. [S0163-1829(98)50306-2]

Recently Mannhart *et al.*[1] observed a self-generated magnetic flux in asymmetric 45° [001] tilt grain boundaries in $YBa_2Cu_3O_{7-x}$ films in zero applied field. The authors explained qualitatively the existence of this flux using a model[2] based on the assumption of a $d_{x^2-y^2}$ symmetry component of the order parameter and on the observation of facets with alternating orientation forming the grain boundary.[3] This model introduces a tunneling current density $j(x) \propto \sin[\varphi(x) - \alpha(x)]$, where x is along the grain-boundary line, $\varphi(x)$ is the phase difference caused by the flux, and $\alpha(x)$ is the phase difference caused by the misalignment of anisotropic $d_{x^2-y^2}$-wave superconductors. The value of $\alpha(x)$ is determined by the orientation of the facets, for asymmetric 45° grain boundaries $\alpha(x)=0$ or π, therefore $j(x) \propto \cos\alpha(x)\sin\varphi(x)$.

The self-generated flux $\Phi_s(x)$ observed by Mannhart *et al.*[1] is randomly distributed along a meandering grain-boundary line. The flux $\Phi_s(x)$ changes its sign randomly, the amplitude of $\Phi_s(x)$ variations is much less than the flux quantum Φ_0, and the average $\langle \Phi_s \rangle$ is nearly zero.

In the framework of a model relating $j(x)$ to orientation of facets,[2] a meandering grain boundary results in an alternating critical current density $j_c(x) \propto \cos\alpha(x)$. The length scales for variations in $\Phi_s(x)$ and $j_c(x)$ are of the same order. This leads to a nonzero average $\langle j \rangle$ of the tunneling current density $j(x)$ induced by an alternating flux $\Phi_s(x)$. Namely, the current density $j(x)$ is equal to $j_c(x)\sin\xi(x)$, where $\xi(x) = 2\pi\Phi_s(x)/\Phi_0$ is the alternating phase difference caused by the flux $\Phi_s(x)$. Since $j_c(x)$ and $\sin\xi(x)$ alternate on the same typical scale, their product has a nonzero average $\langle j \rangle$. This consequence, which has not been mentioned in the qualitative consideration,[1] is important for understanding the self-generated flux $\Phi_s(x)$.

In a stationary state, however, $\langle j \rangle$ has to be zero. The contradiction is resolved if a smooth phase shift $\psi(x)$ arises simultaneously with the flux $\Phi_s(x)$. The additional phase $\psi(x)$ causes an additional smooth current density $\langle j_c \rangle \sin\psi(x)$ which compensates the nonzero current induced by the fast alternating phase $\xi(x)$. The average current density $\langle j \rangle$ is then zero and a self-generated flux $\Phi_s(x)$ is established *self-consistently*. Namely, on the background of an alternating critical current density $j_c(x)$ the alternating flux $\Phi_s(x)$ causes a smooth phase shift $\psi(x)$, and the smooth phase shift $\psi(x)$ in its turn determines the alternating flux $\Phi_s(x)$.

In this paper we study this unusual stationary state with a self-generated flux $\Phi_s(x)$ in a Josephson junction with an alternating critical current density. We derive equations determining the alternating phase difference $\xi(x)$ and the smooth phase shift $\psi(x)$. From these equations we obtain a spontaneous flux $\Phi_s(x)$ occurring when the average energy density of the self-generated magnetic field exceeds that of a critical value proportional to the average critical current density. We find two types of Josephson vortices in a state with a self-generated flux: one with flux $\Phi_1 < \Phi_0/2$, and one with flux $\Phi_2 = \Phi_0 - \Phi_1 > \Phi_0/2$.

Consider a one-dimensional, infinitely long Josephson junction parallel to the x axis and assume that the critical current density $j_c(x)$ is a random function taking positive and negative values. We denote the typical length scale of $j_c(x)$ as l and define the average value of the critical current density $\langle j_c \rangle$ as

$$\langle j_c \rangle = \frac{1}{L}\int_0^L j_c(x)\,dx, \tag{1}$$

with the averaging interval $L \gg l$. The effective Josephson penetration depth Λ_J is introduced by

$$\Lambda_J^2 = \frac{c\Phi_0}{16\pi^2 \lambda \langle j_c \rangle}, \tag{2}$$

where λ is the London penetration depth.

In the case of a Josephson junction with $\lambda \ll l \ll \Lambda_J$ the phase difference $\varphi(x)$ satisfies the equation[4]

$$\Lambda_J^2 \varphi'' - \frac{j_c(x)}{\langle j_c \rangle}\sin\varphi = 0. \tag{3}$$

We write the critical current density $j_c(x)$ as

$$j_c(x) = \langle j_c \rangle [1 + g(x)], \tag{4}$$

with $\langle g(x) \rangle = 0$. The length scale of variation of $g(x)$ is of the order of l and the typical values of $\max|g(x)|$ vary from $\max|g(x)| \sim 1$ to $\max|g(x)| \gg 1$. In particular, if the average $\langle j_c \rangle$ is small compared to the typical amplitude of $j_c(x)$, then $\max|g(x)| \gg 1$.

In terms of the function $g(x)$ we rewrite Eq. (3) for the phase difference $\varphi(x)$ in the form

$$\Lambda_J^2\varphi''-[1+g(x)]\sin\varphi=0. \tag{5}$$

The idea of the following calculations is based on the assumption that $l\ll\Lambda_J$ and on a mechanical analogy. Namely, Eq. (5) for the phase difference $\varphi(x)$ coincides with the equation describing the motion of a pendulum with a vibrating pivoting point (Kapitza pendulum). In this mechanical analogy φ is the angle determining the position of the pendulum and x is the time. We treat here the case when $g(x)$ is rapidly alternating over the length of Λ_J. In terms of the mechanical analogy, this means that the dependence of the phase difference φ on x is similar to the trajectory of a pendulum with a rapidly vibrating pivoting point. In this case, the motion of the pendulum is a slow motion along a certain smooth trajectory with rapid oscillations around this trajectory; the amplitude of these oscillations is small.[5] Therefore, we write the phase difference $\varphi(x)$ as

$$\varphi(x)=\psi(x)+\xi(x), \tag{6}$$

where $\psi(x)$ is a smooth function with length scale of order Λ_J and $\xi(x)$ is a rapidly alternating function with length scale of order l. The average value of $\xi(x)$ vanishes and the typical amplitude of variations of $\xi(x)$ is small, i.e., $\langle\xi(x)\rangle=0$ and $\langle|\xi(x)|\rangle\ll 1$.

Substituting the ansatz (6) into Eq. (5) and keeping terms up to first order in $\xi(x)$ we find

$$\Lambda_J^2\psi''+\Lambda_J^2\xi''-[1+g(x)][\sin\psi+\xi\cos\psi]=0. \tag{7}$$

Two types of terms appear in Eq. (7): terms alternating over a length l and smooth terms varying over a length Λ_J. The alternating terms cancel each other, independently of the smooth terms, which also cancel each other. As a result we obtain both functions $\psi(x)$ and $\xi(x)$ from *one* equation (7).[5]

First, the alternating phase $\xi(x)$ is determined by

$$\Lambda_J^2\xi''=g(x)\sin\psi. \tag{8}$$

In the derivation of Eq. (8) we omitted two out of three alternating terms in Eq. (7) since they are proportional to $\xi(x)$ and therefore much smaller than $g(x)$.

Second, an equation for the phase $\psi(x)$ is derived by averaging Eq. (7) over lengths $a\gg l$ $(a\ll\Lambda_J)$,[5] yielding

$$\Lambda_J^2\psi''-\sin\psi-\langle g(x)\xi(x)\rangle\cos\psi=0. \tag{9}$$

Introducing the Fourier transform of $g(x)$ by

$$g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g_k e^{ikx}dk, \tag{10}$$

we find the solution of Eq. (8) in the form of

$$\xi(x)=-\frac{\sin\psi}{2\pi\Lambda_J^2}\int_{-\infty}^{\infty}\frac{g_k e^{ikx}}{k^2}dk=-\xi_g(x)\sin\psi. \tag{11}$$

The function $\xi_g(x)$ depends only on the alternating component of the critical current density $\langle j_c\rangle g(x)$ and the self-generated flux $\Phi_s(x)$, a measurable quantity, is

$$\Phi_s=\Phi_0\frac{\xi}{2\pi}=-\Phi_0\frac{\xi_g}{2\pi}\sin\psi. \tag{12}$$

This relation allows one to study $\xi_g(x)$ experimentally. In deriving Eq. (11) we ignored the dependence of $\sin\psi$ on x. This may be done since on the length scale l the variations of the smooth function $\sin\psi(x)$ are of order $l/\Lambda_J\ll 1$. Taking into account that the alternating part of the critical current density has typical wave numbers $k\sim 1/l$ we estimate $\xi(x)$ from Eq. (11) as

$$\xi(x)\sim-\sin\psi\frac{l^2}{\Lambda_J^2}g(x). \tag{13}$$

Therefore, typical values of the alternating phase difference $\xi(x)$ are small $[\langle|\xi(x)|\rangle\ll 1]$ if

$$\langle|g(x)|\rangle\ll\frac{\Lambda_J^2}{l^2}. \tag{14}$$

Next, using Eqs. (8) and (11), we calculate the average $\langle g(x)\xi(x)\rangle$ and thus obtain the equation (9) for the smooth phase shift $\psi(x)$ in the final form:

$$\Lambda_J^2\psi''-\sin\psi+\gamma\sin\psi\cos\psi=0, \tag{15}$$

where the constant $\gamma=\langle g(x)\xi_g(x)\rangle$ is given by

$$\gamma=\frac{c\lambda}{\Phi_0\langle j_c\rangle}\langle B_s^2\rangle=\frac{\langle B_s^2\rangle}{\langle B_J^2\rangle}. \tag{16}$$

Here,

$$B_s=\frac{4\pi}{c}\langle j_c\rangle\int g(x)dx=\frac{\Phi_0}{4\pi\lambda}\frac{d\xi_g}{dx} \tag{17}$$

is the magnetic field generated by the alternating component of the critical current $(\langle B_s\rangle=0)$, and

$$B_J=\frac{4\pi}{c}\langle j_c\rangle\Lambda_J. \tag{18}$$

We estimate the value of γ as

$$\gamma\sim\frac{l^2}{\Lambda_J^2}\langle g^2\rangle. \tag{19}$$

The assumption $\langle\xi(x)\rangle\ll 1$ restricts the value of γ. However, it follows from Eqs. (14) and (19) that $\langle\xi(x)\rangle\ll 1$ and $\gamma>1$ hold simultaneously if

$$\frac{\Lambda_J}{l}\ll\langle|g(x)|\rangle\ll\frac{\Lambda_J^2}{l^2}. \tag{20}$$

In particular, Eq. (20) is satisfied when the average value of the critical current density is small compared to the typical amplitude of its variations, i.e., $\langle j_c\rangle\ll\max|j_c(x)|$.

To complete the description of a stationary state with a self-generated flux we calculate the energy of a Josephson junction $\mathcal{E}$.[6] It takes the form $\mathcal{E}=\mathcal{E}_0+\mathcal{E}_\varphi$, where $\mathcal{E}_0$ is independent of the phase difference $\varphi(x)$ and

$$\mathcal{E}_\varphi=\frac{\hbar\langle j_c\rangle}{2e}\int dx\{\tfrac{1}{2}\Lambda_J^2\varphi'^2-[1+g(x)]\cos\varphi\}. \quad (21)$$

Using Eqs. (6), (11), and (16), we obtain the energy $\mathcal{E}_\varphi$ in terms of the smooth phase shift $\psi(x)$, namely,

$$\mathcal{E}_\varphi=\frac{\hbar\langle j_c\rangle}{4e}\int dx\{\Lambda_J^2\psi'^2-2\cos\psi-\gamma\sin^2\psi\}. \quad (22)$$

The solutions $\psi(x)$ of Eq. (15) correspond to the minima and maxima of the functional $\mathcal{E}_\varphi[\psi(x)]$ [Eq. (22)].

Let us now apply Eqs. (15) and (22) to two stationary states in a Josephson junction with length $\mathcal{L}\gg\Lambda_J$. The first state occurs in zero applied field. In this case the average flux inside the junction is zero and thus an alternating self-generated flux $\Phi_s(x)$ appears simultaneously with a phase shift $\psi=\text{const}$. The second state occurs in nonzero applied field. In this case the average flux inside the junction is carried by Josephson vortices. The alternating self-generated flux $\Phi_s(x)$ appears then simultaneously with a phase shift $\psi(x)$, whose spatial dependence describes the vortices.

In the stationary state with $\psi=\text{const}$ the values of ψ are determined by Eq. (15) which takes the form

$$\sin\psi(1-\gamma\cos\psi)=0. \quad (23)$$

In the case $\gamma\leqslant 1$ Eq. (23) has two solutions, $\psi=0$ and $\psi=\pi$ and thus, cf. Eq. (12), there is no self-generated flux.

In the case $\gamma>1$ there are four solutions of Eq. (23), namely, $\psi=-\psi_\gamma$, 0, ψ_γ,π, where

$$\psi_\gamma=\arccos(1/\gamma). \quad (24)$$

The energy of a Josephson junction $\mathcal{E}$ has a minimum for $\psi=\pm\psi_\gamma$ and a maximum for $\psi=0,\pi$. It follows then from Eqs. (12) and (24) that a self-generated flux

$$\Phi_s(x)=-\Phi_0\frac{\xi_g(x)}{2\pi}\sin\psi_\gamma=\mp\Phi_0\frac{\xi_g(x)}{2\pi}\frac{\sqrt{\gamma^2-1}}{\gamma} \quad (25)$$

arises in the two states with the minimum energy $\mathcal{E}$. Note that the criterion $\gamma>1$ means that the average energy density of the self-generated magnetic field $\langle B_s^2\rangle/4\pi$ is higher than $\Phi_0\langle j_c\rangle/4\pi c\lambda$.

From Eqs. (13) and (25) we estimate $|\Phi_s(x)|$ as

$$|\Phi_s(x)|\sim\Phi_0\frac{\sqrt{\gamma^2-1}}{\gamma}\frac{l^2}{\Lambda_J^2}|g(x)|\ll\Phi_0. \quad (26)$$

Note also that the average tunnel current density $\langle j\rangle$ is equal to zero for the state with $\psi=\text{const}$ as

$$\langle j\rangle=\langle j_c(x)\sin\varphi\rangle=\langle j_c\rangle\sin\psi(1-\gamma\cos\psi). \quad (27)$$

Next we consider a Josephson vortex. In this case the phase $\psi(x)$ satisfies Eq. (15) and the boundary conditions $\psi'(\pm\infty)=0$. It is convenient to write Eq. (15) as

$$\Lambda_J^2\psi''=-\frac{dU}{d\psi}, \quad (28)$$

where the function $U(\psi)$ is given by

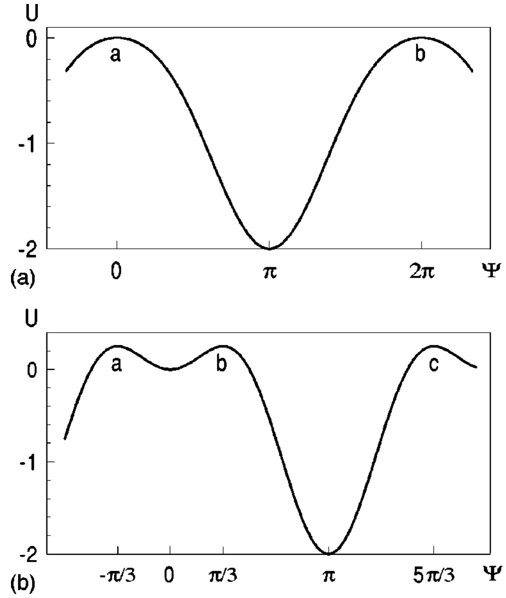

FIG. 1. The dependence of the potential U on ψ for $\gamma<1$ and $\gamma>1$. (a) $\gamma=0.5$, (b) $\gamma=2$.

$$U(\psi)=-1+\cos\psi+\frac{\gamma}{2}\sin^2\psi. \quad (29)$$

Equation (28) describes the motion of a particle with mass Λ_J^2 in a potential $U(\psi)$, where ψ and x are the coordinate of the particle and the time, respectively. This mechanical analogy allows for a qualitative analysis of the solutions of Eq. (28). In Fig. 1 we show the dependence of the potential U on ψ for $\gamma\leqslant 1$ and $\gamma>1$.

In the case $\gamma\leqslant 1$, the curve $U(\psi)$ has two maxima of equal height located at $\psi=0,2\pi$. A Josephson vortex corresponds to the trajectory of a particle motion which starts at the potential maximum $\psi=0$ [point *a* in Fig. 1(a)] at time $x=-\infty$ with zero velocity $[\psi'(-\infty)=0]$ and ends at the potential maximum $\psi=2\pi$ [point *b* in Fig. 1(a)] at time $x=\infty$ with zero velocity $[\psi'(\infty)=0]$. This solution $\psi_0(x)$ exhibits $\psi_0(\infty)-\psi_0(-\infty)=2\pi$, i.e., it describes a vortex with the flux Φ_0 localized in a region with typical size Λ_J.

In the case $\gamma>1$, the curve $U(\psi)$ has three maxima of equal height located at $\psi=-\psi_\gamma$, ψ_γ, $2\pi-\psi_\gamma$ and therefore two trajectories exist corresponding to two different Josephson vortices.

The first vortex is described by a trajectory of a particle which starts with zero velocity at $\psi=-\psi_\gamma$ [point *a* in Fig. 1(b)] at time $x=-\infty$ and ends with zero velocity at $\psi=\psi_\gamma$ [point *b* in Fig. 1(b)] at time $x=\infty$. This solution $\psi_1(x)$ exhibits $\psi_1(\infty)-\psi_1(-\infty)=2\psi_\gamma$, i.e., it describes a vortex with the flux $\Phi_1=\psi_\gamma\Phi_0/\pi<\Phi_0/2$.

The second vortex corresponds to the trajectory of a particle starting with zero velocity at $\psi=\psi_\gamma$ [point *b* in Fig. 1(b)] at time $x=-\infty$ and ending with zero velocity at $\psi=2\pi-\psi_\gamma$ [point *c* in Fig. 1(b)] at time $x=\infty$. This solution $\psi_2(x)$ exhibits $\psi(\infty)-\psi(-\infty)=2\pi-2\psi_\gamma$, i.e., it describes a vortex with the flux $\Phi_2=\Phi_0-\Phi_1=\Phi_0-\psi_\gamma\Phi_0/\pi>\Phi_0/2$ localized in a region with the size of the order of Λ_J.

The flux Φ_1 and energy $\mathcal{E}_1$ of the first vortex tend to zero when $\gamma\to 1$. An approximate solution for $\psi_1(x)$, Φ_1, and $\mathcal{E}_1$

can be found if $0<\gamma-1\ll1$. In this case, $\psi_\gamma^2\approx2(\gamma-1)$ and $U(\psi)\approx\psi_\gamma^2\psi^2/4-\psi^4/8$. It follows then from Eq. (28) that

$$\psi_1(x)\approx\sqrt{2(\gamma-1)}\tanh\left(\frac{x}{\Lambda_J}\sqrt{\frac{\gamma-1}{2}}\right), \tag{30}$$

i.e., the flux $\Phi_1=\sqrt{2(\gamma-1)}\Phi_0/\pi\ll\Phi_0$ is localized over a length $\Lambda_J/\sqrt{\gamma-1}\gg\Lambda_J$, and the energy is

$$\mathcal{E}_1\approx\frac{2\sqrt{2}}{3}\frac{\hbar\langle j_c\rangle\Lambda_J}{e}(\gamma-1)^{3/2}. \tag{31}$$

In the case of $\gamma\gg1$ we have $\psi_\gamma\approx\pi/2$ and therefore $\Phi_1\approx\Phi_2\approx\Phi_0/2$. The main contribution to the potential $U(\psi)$ (21) comes from the term $(\gamma/2)\sin^2\psi$ and both the first and second vortices are described by

$$\psi_{1,2}(x)\approx\arcsin\left[\tanh\left(\frac{x}{\Lambda_J}\sqrt{\gamma}\right)\right]. \tag{32}$$

Each of the two vortices is localized in a region of size $\Lambda_J/\sqrt{\gamma}=\Phi_0/(4\pi\lambda\sqrt{\langle B_s^2\rangle})\ll\Lambda_J$ and has an energy $\mathcal{E}_{1,2}$ which is independent of $\langle j_c\rangle$,

$$\mathcal{E}_{1,2}\approx\frac{\hbar\langle j_c\rangle\Lambda_J}{e}\sqrt{\gamma}=\frac{\Phi_0}{4\pi^2}\sqrt{\langle B_s^2\rangle}. \tag{33}$$

It follows from Eq. (12) that a self-generated flux exists in the presence of a Josephson vortex for any value of the parameter γ since $\psi\neq0$. The alternating flux $\Phi_s(x)$ is then proportional to $\xi_g(x)\sin\psi(x)$, i.e., the rapidly alternating component $\xi_g(x)$ is modulated by a smooth factor $\sin\psi(x)$ imposed by the vortex. In the case $\gamma<1$ a state with a self-generated flux can be studied experimentally *in the presence of vortices*.

Let us now illustrate our calculations by the model $j_c(x)=\langle j_c\rangle+\tilde{j}_c\sin(2\pi x/l)$. From Eqs. (11) and (16) we find $\xi_g(x)=(4l^2\tilde{j}_c/c\Phi_0)\sin(2\pi x/l)$ and

$$\gamma=\frac{2\lambda l^2}{c\Phi_0}\frac{\tilde{j}_c^2}{\langle j_c\rangle}. \tag{34}$$

As expected $\xi_g(x)$ depends only on the alternating component of $j_c(x)$ and a self-generated flux exists $(\gamma>1)$ if the amplitude is $\tilde{j}_c\gg\langle j_c\rangle$.

In conclusion, we have shown that in a long Josephson junction with an alternating critical current density, a self-generated flux arises if the average energy density of the self-generated magnetic field is higher than a critical value proportional to the average critical current density. Two types of Josephson vortices exist under these conditions: one with flux $\Phi_1<\Phi_0/2$ and one with the complementary flux $\Phi_2=\Phi_0-\Phi_1>\Phi_0/2$. The stability of these vortices remains to be checked.

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