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MOTION OF A KINK IN A BISTABLE MEDIUM WITH HYSTERESIS

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The propagation velocity of a kink through a bistable medium with hysteresis is treated. The analysis of a basis system of two nonlinear diffusion equations shows that two types of kinks exist. The stability of the kinks is studied. An explicit expression is derived for the kink's propagation velocity; it is demonstrated that hysteresis may substantially reduce the kink velocity.

1. Introduction

This paper treats the propagation of nonlinear waves in a homogeneous bistable medium. Assume that these waves are described by a one-dimensional nonlinear diffusion equation.

$$\tau \frac{\partial \psi}{\partial t} = l^2 \frac{\partial^2 \psi}{\partial x^2} - f(\psi, \beta), \tag{1}$$

which is made dimensionless such that τ and l are the characteristic time and length, respectively. The parameters τ and l are related by the formula $l^2 = D\tau$, where D is the corresponding diffusion coefficient. Depending on the specifics of the problem, $\psi(x, t)$ may be the temperature of the medium, order parameter, concentration of a chemical agent, etc. (e.g., see [1]); β is a parameter describing the external factors affecting the system, and $f(\psi)$ is an N-shaped function ($f(\psi_{1,3}) = 0$, fig. 1(a)). In order to simplify the presentation, we will assume hereafter that τ and l are independent of ψ.

The transition from one stable state ('phase'), $\psi = \psi_1$, of a bistable medium into the other, $\psi = \psi_3$, can be realized by steady-state propagation of

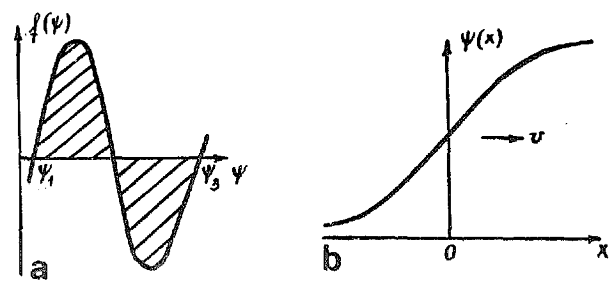

Fig. 1. The function $f = f(\psi)$ for $\beta = \beta_p$ (a) and the distribution $\psi(x)$ in the kink (b).

a kink (switching wave). This kink is described by a self-similar solution of eq. (1) of the type $\psi = \psi(x - vt)$ (fig. 1(b)), where the velocity v is a function of β. When β takes a specific value $\beta = \beta_p$ that satisfies the familiar 'equal areas rule' [1],

$$\int_{\psi_1}^{\psi_3} f(\psi, \beta_p)\, d\psi = 0, \tag{2}$$

the kink is at rest ($v = 0$). If $\beta \neq \beta_p$, the kink is in motion, with one of the phases replacing the other (metastable one) and occupying the entire space. This kink is the unique stable inhomogeneous solution of eq. (1). The theory of kink motion was elaborated (e.g., see [1]) for the case of single-valued $f(\psi)$ (fig. 1a).

In this paper we consider the situation in which $f(\psi)$ is a multivalued function (fig. 2) that has in the interval $\psi_- < \psi < \psi_+$ two stable (the upper and the lower) and one unstable (intermediate) branches. In this situation the transitions between the branches of $f(\psi)$ in response to the changes in ψ are accompanied with a hysteresis, so that treating the behavior of the kink in terms of a single equation (1) is mathematically incorrect. It will be shown below that a kink always exists in a bistable medium with hysteresis, and a method of correct description will be given. The paper analyzes the stability of the kink and derives an explicit formula for the kink's propagation velocity; it is also shown that hysteresis produces a number of specific features in the kink front structure and also a dependence $v(\beta)$.

2. Basis set of equations

One of the possible reasons for the multivaluedness of the function $f(\psi)$ is the fact that a bistable system can be described in terms of not one but several variables, $\psi(x,t)$, $\varphi_1(x,t),\dots,\varphi_N(x,t)$, each of them satisfying a nonlinear diffusion equation:

$$\tau_\psi \frac{\partial\psi}{\partial t} = l_\psi^2 \frac{\partial^2\psi}{\partial x^2} - f(\psi,\varphi_1,\dots,\varphi_N),$$

$$\tau_{\varphi_1} \frac{\partial\varphi_1}{\partial t} = l_{\varphi_1}^2 \frac{\partial^2\varphi_1}{\partial x^2} - f_1(\psi,\varphi_1,\dots,\varphi_N),$$

$$\dots .$$

The variables $\varphi_1,\dots,\varphi_N$ can be removed from the analysis by applying the procedure of adiabatic elimination [1], provided the characteristic time and space scales, τ_{φ_i} and l_{φ_i}, are much smaller than the corresponding scales of variation of the 'slowest' variable ψ, that is, $\tau_{\varphi_i} \ll \tau_\psi$, $l_{\varphi_i} \ll l_\psi$. In this case, one can neglect the derivatives with respect to coordinate and time in the differential equations for φ_i, thereby reducing them to local relations of the type $f_i(\psi,\varphi_1,\dots,\varphi_N) = 0$. Expressing in them $\varphi_i = \varphi_i(\psi)$ and substituting into the equation for ψ, we obtain a single equation of the type (1) with a function $f(\psi,\varphi_1(\psi),\dots,\varphi_N(\psi))$. The procedure of adiabatic elimination of the rapid variables φ_i described above is mathematically correct if the function obtained $f(\psi,\varphi_1(\psi),\dots,\varphi_N(\psi))$ is single-valued.

In the opposite case, the solutions $\varphi_i(x)$ that describe a kink may contain jumps, that is, regions where $\varphi_i(x)$ varies considerably over a scale of order l_{φ_i} (fig. 4). In this situation, it is necessary to work with the complete set of equations, even though $\tau_{\varphi_i} \ll \tau_\psi$ and $l_{\varphi_i} \ll l_\psi$ (e.g., see [2–4]).

In this paper we treat the simplest case in which the hysteresis arises in two one-dimensional diffusion equations:

$$\tau_\psi \frac{\partial\psi}{\partial t} = l_\psi^2 \frac{\partial^2\psi}{\partial x^2} - F(\psi,\varphi,\beta), \qquad (3)$$

$$\tau_\varphi \frac{\partial\varphi}{\partial t} = l_\varphi^2 \frac{\partial^2\varphi}{\partial x^2} - R(\psi,\varphi,\beta), \qquad (4)$$

in which the nonlinear functions $F(\psi,\varphi)$ and $R(\psi,\varphi)$ are single-valued, and $\tau_\varphi \ll \tau_\psi$, $l_\varphi \ll l_\psi$. Eqs. (3), (4) form the basis set for the description of a wide range of problems in physics, chemistry, and biology. The variables ψ and φ may stand for temperatures of electrons and ions in plasma, electron and phonon temperatures in semiconductors [5], order parameters used to describe the dynamics of phase transitions [6], concentration of excess

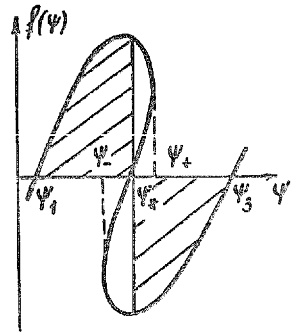

Fig. 2. The function $f = f(\psi)$ for $\beta = \beta_p$ in the case of hysteresis.

electrons and order parameter of nonequilibrium superconductor [7], concentration of chemical reactants [8] or size of biological populations in self-organization processes [1] and in the analysis of dissipative structures [9], transmembrane potential and concentration of ions when nervous impulses are considered [10], and so forth. As an example of a particular system, we can choose the equations

$$\tau_\psi \frac{\partial \psi}{\partial t} = l_\psi^2 \frac{\partial^2 \psi}{\partial x^2} - \psi + \varphi + \beta,$$

$$\tau_\varphi \frac{\partial \varphi}{\partial t} = l_\varphi^2 \frac{\partial^2 \varphi}{\partial x^2} + \alpha\varphi - \varphi^3 + \psi$$

that describe the phase transition dynamics. Here $\varphi(x, t)$ is the real order parameter satisfying the Ginzburg–Landau equation and interacting with the field $\psi(x, t)$; α and β are parameters. This system is treated in more detail in Section 6.

The bistability of the medium described by the basis set of eqs. (3), (4) implies the existence of two homogeneous stable states ('phases' 1 and 3, fig. 3) that satisfy the equations

$$F(\psi_{1,3}, \varphi_{1,3}, \beta) = 0,$$

$$R(\psi_{1,3}, \varphi_{1,3}, \beta) = 0.$$

In this case the null-clines $\psi = \psi_F(\varphi)$ and $\psi = \psi_R(\varphi)$ have three intersection points where $F(\psi_F, \varphi) = 0$, $R(\psi_R, \varphi) = 0$. Let us consider now the conditions under which this medium manifests hysteresis. In the framework of the approximation $\tau_\varphi \ll \tau_\psi$, $l_\varphi \ll l_\psi$, eq. (4) can be replaced with a local function $\varphi = \varphi_R(\psi)$, where $R(\psi, \varphi_R) = 0$; the substitution of this relation into eq. (3) yields a single equation for ψ with nonlinear function $F[\psi, \varphi_R(\psi)]$. Therefore, if $\varphi_R(\psi)$ is single-valued (fig. 3(a)), the procedure of elimination of φ is mathematically correct. If $\varphi_R(\psi)$ is not single-valued (fig. 3(b)) then $F(\psi)$ is non-single-valued too, and the interval of hysteresis coincides with the region of the non-single-valuedness of the function $\varphi_R(\psi)$: $\psi_- < \psi < \psi_+$ (see figs. 2, 3(b)). In this case, the complete system of eqs. (3), (4) must be considered to achieve correct description, even though $\tau_\varphi \ll \tau_\psi$, $l_\varphi \ll l_\psi$.

Thus, the kink propagation in a hysteretic medium can be described by two nonlinear diffusion equations. The various nonlinear solutions of eqs. (3), (4) were investigated previously, such as stationary structures, moving and oscillating pulses, kinks etc. (e.g., see [3, 4, 9] and references therein). In the present paper we use the adiabatic elimination procedure to find the kink's propagation velocity in a hysteretic medium. It has been shown that there is an interval of β where the hysteresis substantially reduces the kink velocity.

3. Kink in a medium with hysteresis

Consider a kink propagation at a velocity $v(\beta)$. The corresponding solutions $\psi(y), \varphi(y)$ of eqs. (3), (4) are functions of $y = x - vt$ and satisfy the boundary conditions $\psi(-\infty) = \psi_1, \varphi(-\infty) = \varphi_1$, $\psi(+\infty) = \psi_3$, $\varphi(+\infty) = \varphi_3$, $\psi'(\pm\infty) = \varphi'(\pm\infty) = 0$, where the prime indicates differentiation with respect to y. Such solutions correspond to a phase trajectory connecting points 1 and 3 on the phase plane φ, ψ (see fig. 3(b)). In the limiting case $l_\varphi \ll l_\psi$, $\tau_\varphi \ll \tau_\psi$ this trajectory is in fact identical to the null-cline $\psi = \psi_R(\varphi)$, to an accuracy of the order of $\max(l_\varphi^2/l_\psi^2, \tau_\varphi/\tau_\psi) \ll 1$ [3].

Several types of phase trajectories (and hence, several types of kinks) are possible that meet this condition.

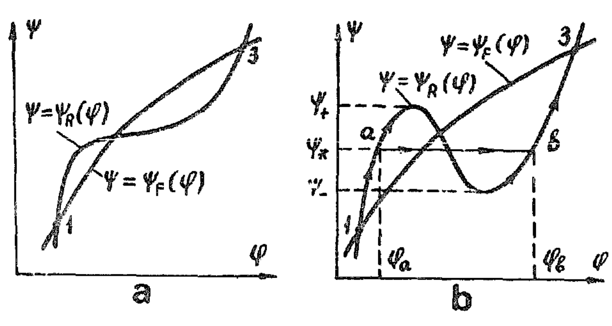

Fig. 3. Null-clines of the system of eqs. (3), (4): in the case of (a) no hysteresis and (b) medium with hysteresis.

In the first case, the phase trajectory connecting points 1 and 3 is completely identical to the $\psi = \psi_R(\varphi)$ null-cline. The corresponding distributions $\psi(y), \varphi(y)$ are shown in fig. 4(a); the dependence $\psi(y)$ is nonmonotonic.

Another type of phase trajectory involves the transition from one branch of the $\varphi = \varphi_R(\psi)$ null-cline to another at $\psi = \psi_* = \text{const.}$ (curve 1ab3 in fig. 3(b)). The transition corresponds to a jump in the distribution $\varphi(y)$ (see fig. 4(b)) by $\Delta\varphi = \varphi_b - \varphi_a$ which occurs in a narrow region of space, $\Delta y \sim l_\varphi$. Here the gradient $\varphi' \sim (\varphi_b - \varphi_a) l_\varphi^{-1}$ is large and the diffusion term $l_\varphi^2 \varphi''$ must be taken into account in eq. (4); the local relation $\psi = \psi_R(\varphi)$ does not hold. The change of $\psi(y)$ in the region of width $\Delta y \sim l_\varphi$ is small and to the accuracy of $l_\varphi / l_\psi \ll 1$ we can assume $\psi = \psi_* = \text{const.}$, which means the transition from one branch of the $\varphi = \varphi_R(\psi)$ null-cline to the other [3, 4].

In principle, one can construct a more complex phase trajectory with several transitions between the left- and right-hand branches of $\psi_R(\varphi)$ (several values of ψ_*). However, the corresponding distributions $\psi(y)$ and $\varphi(y)$ do not satisfy the stationary equations (3), (4). To show this, we derive an equation that determines the value of ψ_*. We have already mentioned that the variable $\varphi(y)$ changes abruptly in the region of the jump, $|y| \lesssim l_\varphi$, from $\varphi(+l_\varphi) \simeq \varphi_b$ to $\varphi(-l_\varphi) \simeq \varphi_a$, while the variable $\psi(y)$ is smooth: $[\psi(+l_\varphi) - \psi(-l_\varphi)]/\psi_* \sim l_\varphi / l_\psi \ll 1$; hence, eq. (4) can be rewritten in the form

$$l_\varphi^2 \varphi'' + \tau_\varphi v \varphi' - R(\psi_*, \varphi, \beta) = 0. \qquad (5)$$

Multiplying eq. (5) by φ' and integrating in y in the region $|y| \lesssim l_\varphi$, we arrive at an equation that determines ψ_* to the accuracy of the order of $l_\varphi^2 / l_\psi^2 \ll 1$:

$$\tau_\varphi v \int_{-\infty}^{+\infty} \varphi'^2 \, \mathrm{d}y - \int_{\varphi_a}^{\varphi_b} R(\psi_*, \varphi, \beta) \, \mathrm{d}\varphi = 0, \qquad (6)$$

where $\varphi(y)$ is the solution to eq. (5) with the boundary conditions $\varphi(-\infty) = \varphi_a$, $\varphi(+\infty) = \varphi_b$,

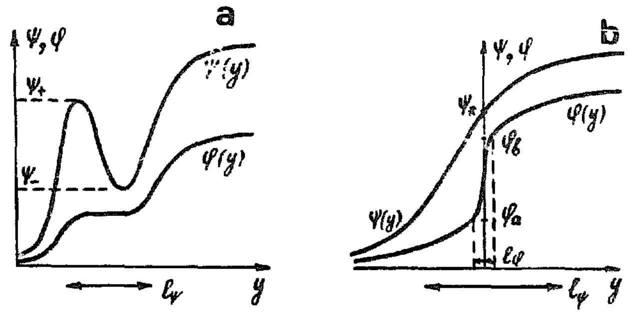

Fig. 4. The distributions $\psi(y)$ and $\varphi(y)$ at the front of (a) unstable and (b) stable kinks.

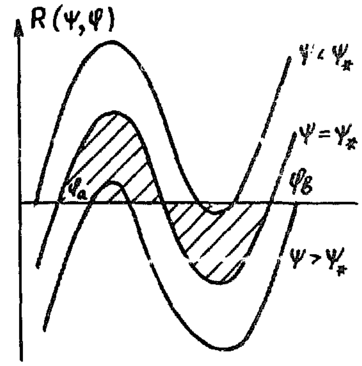

Fig. 5. The function $R = R(\psi, \varphi)$ for different values of ψ.

$\varphi'(\pm\infty) = 0$, $R(\psi_*, \varphi_{a,b}, \beta) = 0$ (see figs. 3(b), 4(b)). If $v = 0$, eq. (6) simplifies to

$$\int_{\varphi_a}^{\varphi_b} R(\psi_*, \varphi, \beta) \, \mathrm{d}\varphi = 0. \qquad (7)$$

Since $R(\varphi)$, similarly to $\psi_R(\varphi)$, is an N-shaped function, eq. (7) has a simple geometrical meaning of equality of the areas shaded in fig. 5; the equality holds for a single value of ψ_*. The function $\psi_*(\beta, v)$ is thus single-valued, and there is a single distribution $\psi(y), \varphi(y)$ with a jump in $\varphi(y)$ at the kink front. To find which of the kinks shown in fig. 4 is actually realized, it is necessary to analyze their stability with respect to small perturbations $\delta\psi(x, t)$ and $\delta\varphi(x, t)$. Here we limit the analysis to the case of $\partial F / \partial \varphi < 0$, which holds for most of the systems mentioned above. The physical meaning of this condition is that the variable $\psi(x, t)$ damps out the perturbations $\delta\varphi(x, t)$. It can be shown that in this case the kink shown in fig. 4(a) is always unstable while that in fig. 4(b) is always stable. A detailed analysis of stability is given in appendix A. Hereafter we will consider only a stable kink.

4. Kink propagation velocity

We will calculate the velocity $v(\beta)$ of kink propagation as a function of β. First we find the value of the parameter $\beta = \beta_p$ at which the kink is at rest. Correspondingly, we multiply eq. (3) for $\partial\psi/\partial t = 0$ by ψ' and integrate in y from $-\infty$ to $+\infty$; to the accuracy $l_\varphi^2/l_\psi^2 \ll 1$, we obtain

$$\int_{\psi_1}^{\psi_*} F(\psi, \varphi_a(\psi), \beta)\, d\psi + \int_{\psi_*}^{\psi_3} F(\psi, \varphi_b(\psi), \beta)\, d\psi = 0, \tag{8}$$

where $\varphi_{a,b}(\psi)$ are the branches of the null-cline $\varphi_R(\psi)$, labelled by the letters a and b, respectively, in fig. 3(b). Eqs. (7), (8) completely determine ψ_* and β_p. Relation (8) implies the equality of areas shaded in fig. 2, generalizing the 'equal areas theorem' (2) to the case of a medium with hysteresis. If $v \neq 0$, we follow the same procedure as in deriving eq. (6) and find

$$\tau_\psi v \int_{-\infty}^{+\infty} \psi'^2\, dy = S_3(\psi_*, \beta), \tag{9}$$

where

$$S_3(\psi_*, \beta) = S(\psi_3, \psi_*, \beta)$$
$$S(\psi) = \textstyle\int_{\psi_1}^{\psi} F(\psi, \psi_*, \beta)\, d\psi$$

and

$$F(\psi, \psi_*, \beta) = \begin{cases} F[\psi, \varphi_a(\psi), \beta], & \psi < \psi_*, \\ F[\psi, \varphi_b(\psi), \beta], & \psi > \psi_*. \end{cases} \tag{10}$$

An explicit expression for $v(\beta)$ can be obtained only in the neighborhood of β_p: $|\beta - \beta_p| \ll \beta_p$, where $v(\beta) \ll v_\psi, v_\varphi$; $v_\psi = l_\psi/\tau_\psi$, $v_\varphi = l_\varphi/\tau_\varphi$. We therefore expand eqs. (6), (9) in series in $\delta\beta = \beta - \beta_p$, taking into consideration that $v \ll v_\psi, v_\varphi$ and that

$$\psi_*(\beta, v) = \psi_*(\beta_p) + \delta\psi_*(\beta, v), \quad \delta\psi_* \ll \psi_*.$$

Retaining only the first terms of the expansion, we obtain

$$\tau_\psi v \int_{-\infty}^{+\infty} \psi'^2\, dy = \delta\psi_* \frac{\partial S_3}{\partial \psi_*} + \delta\beta \frac{\partial S_3}{\partial \beta}, \tag{11}$$

$$\tau_\varphi v \int_{-\infty}^{+\infty} \varphi'^2\, dy = \delta\psi_* \frac{\partial Q_b}{\partial \psi_*} + \delta\beta \frac{\partial Q_b}{\partial \beta}, \tag{12}$$

where $Q_b(\psi_*, \beta) = Q(\psi_*, \varphi_b, \beta)$ and

$$Q(\varphi) = \int_{\varphi_a}^{\varphi} R(\psi_*, \varphi, \beta)\, d\varphi.$$

The derivatives $\psi'(y)$ and $\varphi'(y)$ can be found from eqs. (3), (4) for $\partial\psi/\partial t = \partial\varphi/\partial t = 0$. We have

$$l_\psi^2 \psi'^2 = 2S(\psi); \quad l_\varphi^2 \varphi'^2 = 2Q(\varphi).$$

Using eqs. (11), (12), we obtain the following explicit formula for $v(\beta)$:

$$\sqrt{2}\, v \left(\frac{1}{v_\varphi} \frac{\partial S_3}{\partial \psi_*} \int_{\varphi_a}^{\varphi_b} Q^{1/2}\, d\varphi - \frac{1}{v_\psi} \frac{\partial Q_b}{\partial \psi_*} \int_{\psi_1}^{\psi_3} S^{1/2}\, d\psi \right)_{\beta_p} = \left(\frac{\partial Q_b}{\partial \beta} \frac{\partial S_3}{\partial \psi_*} - \frac{\partial S_3}{\partial \beta} \frac{\partial Q_b}{\partial \psi_*} \right)_{\beta_p} \delta\beta. \tag{13}$$

The expression for the velocity, (13), can be simplified in the important particular case when eqs. (3), (4) can be recast in the gradient form:

$$\tau_\psi \frac{\partial \psi}{\partial t} = -\frac{\delta G}{\delta \psi}, \quad \tau_\varphi \frac{\partial \varphi}{\partial t} = -\frac{\delta G}{\delta \varphi},$$

where

$$G = \int dy \left[\tfrac{1}{2} l_\psi^2 \psi'^2 + \tfrac{1}{2} l_\varphi^2 \varphi'^2 + U(\psi, \varphi, \beta) \right],$$

$$F(\psi, \varphi, \beta) = \frac{\partial U}{\partial \psi}, \quad R(\psi, \varphi, \beta) = \frac{\partial U}{\partial \varphi}.$$

Such equations describe, for example, the dynamics of two-component order parameter of the second-order phase transition with G being the free energy functional [6]. Hence,

$$\frac{\partial R}{\partial \psi} = \frac{\partial F}{\partial \varphi} \tag{14}$$

and therefore,

$$\frac{\partial S_3}{\partial \psi_*} = F[\psi_*, \varphi_a(\psi_*)] - F[\psi_*, \varphi_b(\psi_*)],$$

$$\frac{\partial Q_b}{\partial \psi_*} = \int_{\varphi_a}^{\varphi_b} \frac{\partial R}{\partial \psi_*} \mathrm{d}\varphi = \int_{\varphi_a}^{\varphi_b} \frac{\partial F}{\partial \varphi} \mathrm{d}\varphi = -\frac{\partial S_3}{\partial \psi_*}.$$

In this case we obtain for $v(\beta)$ the expression

$$v(\beta) = \frac{\left(\frac{\partial S_3}{\partial \beta} + \frac{\partial Q_b}{\partial \beta}\right)_{\beta_p} (\beta - \beta_p)}{\sqrt{2}\left(\frac{1}{v_\psi}\int_{\psi_1}^{\psi_3} S^{1/2} \mathrm{d}\psi + \frac{1}{v_\varphi}\int_{\varphi_a}^{\varphi_b} Q^{1/2} \mathrm{d}\varphi\right)}. \tag{15}$$

Note that in the limiting case of no hysteresis, $\varphi_a = \varphi_b$ ($Q_b = 0$), expression (15) coincides with the well-known formula for the kink velocity in bistable medium (e.g., see the review in [11]). For a medium with hysteresis, eq. (15) contains a second term in the denominator that appears because of the jump in $\varphi(y)$ at the kink front (see fig. 4(b)). As a result, the kink velocity v is determined here by the minimum of the characteristic velocities v_ψ and v_φ. If $v_\psi \ll v_\varphi$, the quantity $v(\beta) \sim v_\psi$ and we can neglect the effect of hysteresis on the kink velocity. In the opposite limiting case, $v_\varphi \ll v_\psi$, the kink velocity is determined by the characteristic velocity v_φ, that is, $v(\beta) \sim v_\varphi \ll v_\psi$. Therefore, the effect of hysteresis is seen in this case as a considerable drop in kink velocity despite the fact that $l_\varphi \ll l_\psi$, $\tau_\varphi \ll \tau_\psi$.

It should be noted in conclusion of this section that the condition $\tau_\varphi \ll \tau_\psi$ is not necessary for the derivation of formula (15). Indeed, the term $\tau_\varphi v \varphi' \sim l_\varphi v(\varphi_3 - \varphi_1)/l_\psi v_\varphi \ll (\varphi_3 - \varphi_1)$ in eq. (5) is small in the case we consider now, $v \ll \min(v_\psi, v_\varphi)$, regardless of the ratio of the times τ_φ and τ_ψ; this is why eq. (15) can be derived.

5. The case of weak hysteresis

So far we were addressing the general case of kink propagation in a medium with hysteresis. Assume now that the hysteresis is weak, that is, $\varphi_b - \varphi_a \ll \varphi_3$. In this case we can consider how to extend the region of applicability of formula (15) and derive an expression for the kink propagation velocity valid in a wider range of velocities, $v \lesssim \min(v_\psi, v_\varphi)$.

In order to analyze the weak hysteresis case, assume that $F(\psi, \varphi)$ is of a general form, and the function $R(\psi, \varphi)$ depends on a certain parameter α in such a manner that the multivaluedness vanishes when $\alpha < 0$ (fig. 3(a)) but appears when $\alpha > 0$ (fig. 3(b)). A cusp-type bifurcation of the function $\psi_R(\varphi)$ occurs at the point $\alpha = 0$ (e.g., see [1]), and the function $R(\psi, \varphi, \alpha)$ can be expanded in the neighborhood of the bifurcation point into a series; retaining only the terms linear in ψ and cubic in φ, we have

$$R(\psi, \varphi, \alpha) = -\alpha\varphi + \varphi^3 - \psi.$$

For convenience, the variables ψ and φ are measured off the original placed in the bifurcation point $\psi = \varphi = 0$; it is also assumed that eq. (4) is invariant under the transformation $\varphi \to -\varphi$, which eliminates from this expansion the terms quadratic in φ. Then eq. (5) can be written in the dimensionless form

$$l_\varphi^2 \varphi'' + \tau_\varphi v \varphi' + \alpha\varphi - \varphi^3 + \psi_* = 0. \tag{16}$$

The solution to this equation, satisfying the boundary conditions $\varphi(-\infty) = \varphi_a$, $\varphi(+\infty) = \varphi_b$ has the form [12]

$$\varphi(y) = \frac{\varphi_b + \varphi_a}{2} + \frac{\varphi_b - \varphi_a}{2} \tanh\frac{y}{2L},$$

$$L = \frac{\sqrt{2}\, l_\varphi}{(\varphi_b - \varphi_a)}. \tag{17}$$

The distribution $\varphi(y)$ in the jump region has a universal form (hyperbolic tangent). For the velocity v we find

$$v = \frac{v_\varphi}{\sqrt{2}}\left[\varphi_a(\psi_*) + \varphi_b(\psi_*) - 2\varphi_c(\psi_*)\right], \tag{18}$$

where φ_a, φ_b, and φ_c are the roots of the equation $\varphi^3 - \alpha\varphi = \psi_*$. Note that $v = 0$ if $\psi_* = \psi_*(\beta_p) = 0$. If the expression (18) is substituted into eq. (3), the system of eqs. (3), (4) reduces to a single equation that takes the following form in the reference frame moving with the kink:

$$l_\psi^2\psi'' + \tau_\psi v\psi' - F(\psi, \psi_*, \beta) = 0, \tag{19}$$

where $F(\psi, \psi_*, \beta)$ is given by eq. (10) (see also fig. 2). If the solution to eq. (19) taking into account the boundary conditions $\psi'(\pm\infty) = 0$ and the eigenvalue $\psi_*(\beta)$ are found, eq. (18) makes it possible to determine the kink velocity $v(\beta)$. The problem of finding the kink-type solution to eqs. (3), (4) is thus reduced to the standard problem of finding the solution to a single equation (19) in the case of weak hysteresis. Note that the region of applicability of eq. (18) is wider than that of eq. (15): $v \lesssim \min(v_\psi, v_\varphi)$. In the limiting case of low velocities, $v \ll \min(v_\psi, v_\varphi)$, eqs. (18), (19) can be expanded into series in powers of $\delta\beta = \beta - \beta_p$, $\delta\psi_* = \psi_* - \psi_*(\beta_p)$. Retaining the first terms of the expansions and assuming that the condition (14) is satisfied, it can be shown that eq. (18) transforms into eq. (15), namely,

$$v = \frac{C(\beta - \beta_p)}{\left(\dfrac{A}{v_\psi} + \dfrac{B(\alpha)}{v_\varphi}\right)}, \tag{20}$$

where $A, C \sim 1$, $B(\alpha) = 0$ $(\alpha < 0)$, $B(\alpha) = \sqrt{2}\int_{\varphi_a}^{\varphi_b} Q^{1/2}\,d\varphi = l_\varphi \int_{-\infty}^{+\infty}\varphi'^2\,dy$ $(\alpha > 0)$. Taking into consideration eq. (17) and also that $\psi_*(\beta_p) = 0$, we obtain $B(\alpha) = \frac{2}{3}\alpha^{3/2}$. Eq. (20) shows that if $v_\varphi \gg v_\psi$, the effect of hysteresis on the kink velocity is negligible and $v \sim v_\psi$. In the other limiting case, $v_\varphi \ll v_\psi$, the hysteresis of the medium may substantially affect the kink velocity (rather, if $v_\varphi/v_\psi \ll B(\alpha)/A$). Then the effect of hysteresis is negligible if

$$0 < \alpha \lesssim \alpha_c \sim (v_\varphi/v_\psi)^{2/3},$$

while at $\alpha \gtrsim \alpha_c$ the kink velocity v is of the order of v_φ: in other words, the medium's hysteresis results in a considerable reduction of kink velocity. It is thus possible in the case of weak hysteresis to give a universal estimate of the parameter $\alpha_c \sim (v_\varphi/v_\psi)^{2/3}$ beginning with which the hysteresis affects the kink velocity substantially. Unless otherwise specified, we will hereafter consider the case $v_\varphi \ll v_\psi$ in which the kink velocity is affected by hysteresis.

We have considered the case of low kink velocities, that is, the region of β values in the neighborhood of $\beta = \beta_p$. The theory of kink propagation in bistable media gives a qualitative form of the $v(\beta)$ function in the entire range of bistability $\beta_p < \beta < \beta_c$ [9, 11], where β_c is that value of the parameter at which the number of steady states of the system changes. When the value of β considerably deviates from β_p, the linear dependence $v \sim (\beta - \beta_p)$ (see eqs. (15), (20)) is violated; at the bifurcation value of the parameter $\beta = \beta_c$, the kink velocity reaches its maximum value, $v = v_c$, where $(\partial v/\partial\beta)_{\beta_c} = \infty$.

Assuming $l_\varphi \ll l_\psi$, the maximum kink velocity v_c is of order v_ψ, as we see from eq. (3) or from its particular case, eq. (19). On the other hand, expression (18) shows that v is limited, $|v| < v_\varphi/\sqrt{2} \ll v_\psi$, attaining the min and max values at $\psi_* = \psi_\pm$. The following reasoning removes the apparent paradox (a kink cannot move at velocities $|v| > v_\varphi/\sqrt{2}$ but at $\beta \simeq \beta_c$ the velocity $v \sim v_\psi \gg v_\varphi$). We set $\psi(y) = \psi_* = \text{const.}$ in eq. (5) under the approximation $l_\varphi/l_\psi \ll 1$. It can be shown, however, that as $\psi_* \to \psi_\pm$, the dependence of $\psi(y)$ on y becomes important despite the smallness of changes $\delta\psi/\psi_* \sim l_\varphi/l_\psi \ll 1$ in the region of the jump. Note that if the dependence $\psi(y) = \psi_* + \psi' y$ is taken into account in the next approximation in $l_\varphi/l_\psi \ll 1$, the kink's velocity sharply increases and its front structure changes as $\psi_* \to \psi_\pm$. A detailed analysis of the effect of the dependence $\psi(y)$ on jump propagation velocity is given in appendix B.

6. Numerical calculation of the kink's velocity

The results outlined are illustrated below with numerical solution of the equations

$$\tau_\psi \frac{\partial \psi}{\partial t} = l_\psi^2 \frac{\partial^2 \psi}{\partial x^2} - \psi + \varphi + \beta, \tag{21}$$

$$\tau_\varphi \frac{\partial \varphi}{\partial t} = l_\varphi^2 \frac{\partial^2 \varphi}{\partial x^2} + \alpha\varphi - \varphi^3 + \psi. \tag{22}$$

These equations have already been written above and they describe, for example, a phase transition in a system with interacting order parameter $\varphi(x, t)$ and external field $\psi(x, t)$. The value of the parameter β determines the effect exerted on the system. Eq. (22) is the real Ginzburg–Landau equation, and the value of the parameter α determines how close the critical point is: $\alpha \sim T_c - T$. The null-clines of the system of eqs. (21), (22) are plotted in fig. 6. Note that the bifurcation in α (at $\alpha = 0$), describing the onset of the hysteresis, is independent of bifurcation in β (at $\beta = \beta_c$) at which the bistability vanishes. The signs in eqs. (21), (22) are chosen in accord with the situation characterized above, namely, that a perturbation in the variable φ is damped by the variable ψ, and vice versa.

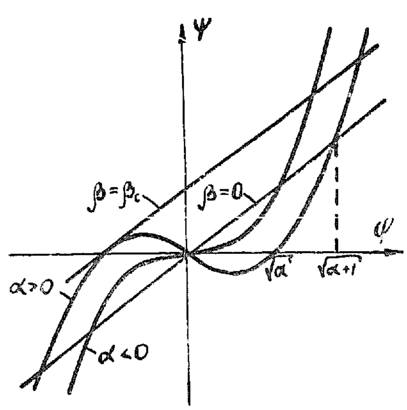

Fig. 6. Null-clines of the system of eqs. (19), (20) for different values of α and β.

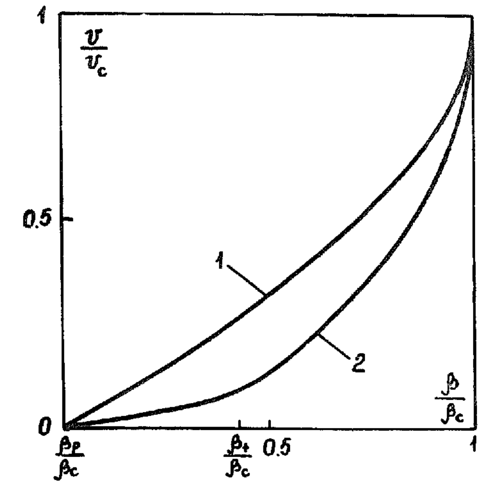

Fig. 7. The function $v = v(\beta)$ in the case of no hysteresis (curve 1: $\alpha = -0.5$, $\beta_c = 0.136$, $v_c = 0.375$), and in a medium with hysteresis (curve 2: $\alpha = 1$, $\beta_c = 1.09$, $v_c = 0.675$), $l_\varphi/l_\psi = 5 \times 10^{-2}$, $v_\varphi/v_\psi = 5 \times 10^{-2}$.

At $\beta = 0$, the phase transition is of the second order, with the kink being symmetric ($\beta_p = 0$, $\psi_*(\beta_p) = 0$) and quiescent, and at $\beta \neq 0$, the phase transition is of the first order, and the kink (interfacial surface) is moving.

Eqs. (21), (22) were solved numerically, and the kink velocity was calculated in a straightforward manner, from the propagation of the kink front, at the accuracy of at least 3%. Typical distributions

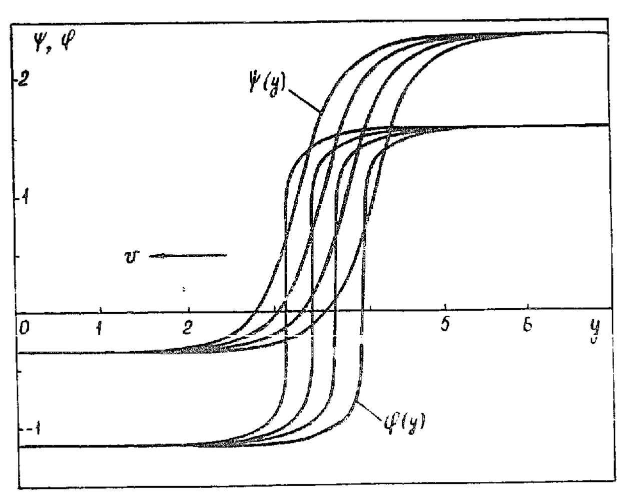

Fig. 8. The distributions $\psi(y)$, $\varphi(y)$ at the kink front; $\alpha = 1$, $\beta/\beta_c = 0.8$, $l_\varphi/l_\psi = 5 \times 10^{-2}$. The kink velocity $v/v_c = 0.43$, the time interval between the curves $\Delta t/\tau_\psi = 0.72$.

$\psi(y)$, $\varphi(y)$ obtained in this way are plotted in fig. 8. Note that $\psi(y)$ is a smooth curve with spatial change scale $\sim l_\psi$, and $\varphi(y)$ manifests a jump in a narrow spatial region $\sim l_\varphi$. Fig. 7 plots $v(\beta)$ in the cases of no hysteresis (curve 1) and with hysteresis (curve 2). In the hysteretic case, the $v(\beta)$ curve has a characteristic low-slope segment (plateau) within which the kink's velocity is substantially lower than with no hysteresis. If $\beta \sim \beta_c$, the kink's velocities have comparable values in the two cases.

7. Discussion of results

In order to explain the characteristic features of the behavior of $v(\beta)$, it is convenient to consider a kink in a medium with hysteresis as two connected kinks. The first one is described by the distribution $\psi(y)$, representing a wave of switching from the phase $\psi = \psi_1$ to $\psi = \psi_3$ with a characteristic spatial width $\sim l_\psi$ and a characteristic propagation velocity $v \sim v_\psi$. This kink is described by nonlinear equation (9) depending on β and ψ_*. The second kink is described by the distribution jump $\varphi(y)$, and constitutes a switching wave from $\varphi = \varphi_a$ to $\varphi = \varphi_b$ with a characteristic spatial width $\sim l_\varphi \ll l_\psi$ and a characteristic velocity $v \sim v_\varphi$. This kink is described by the nonlinear equation (6) incorporating ψ_* as a parameter.

The relative arrangement of these two kinks is fixed in an unambiguous manner by the quantity $\psi_* = \psi_*(\beta)$, so that $R(\psi_*, \varphi_{a,b}, \beta) = 0$. As a result of interaction between kinks, the velocity of kink propagation through the medium with hysteresis, v, is determined by a minimum of the characteristic velocities v_ψ and v_φ. We have mentioned above that if $v_\psi \ll v_\varphi$, the effect of hysteresis on the kink's velocity is negligible and $v \sim v_\psi$.

If $v_\varphi \ll v_\psi$, a sufficiently strong hysteresis ($\alpha \gtrsim \alpha_c \sim (v_\varphi/v_\psi)^{2/3}$) results in $v \sim v_\varphi \ll v_\psi$. The velocity of the $\psi(y)$ kink tunes in to the velocity of $\varphi(y)$. This takes place in a certain neighborhood of $\beta = \beta_p$. As $\beta > \beta_p$ increases, the quantity $\psi_*(\beta)$ grows until it reaches the boundary of the interval of the non-single-valuedness, $\psi_* = \psi_+$. This situation corresponds to the moment when the velocity of the $\varphi(y)$ kink reaches the value $v \sim v_\varphi$; it corresponds to the value β_+ of the parameter β determined to the accuracy $v_\varphi/v_\psi \ll 1$ by the 'equal areas theorem' (8) for $\psi_* = \psi_+$:

$$S_3(\psi_+, \beta_+) = 0.$$

In the interval $\beta_p < \beta < \beta_+$, therefore, the interaction of the kinks $\psi(y)$ and $\varphi(y)$ results in tuning of $\psi(y)$-kink velocity to $\varphi(y)$-kink velocity, and the resulting velocity $v \sim v_\varphi$. The effect of inhomogeneity $\psi_*(y)$ in the vicinity of $\beta \simeq \beta_+$ ($\psi_* \simeq \psi_+$) produces such a change in the kink front $\varphi(y)$ that its propagation velocity ceases to be limited ($|v| \lesssim v_\varphi$). As the parameter β increases further in the interval $\beta_+ < \beta < \beta_c$, the quantity $\psi_* = \psi_+$ remains unaltered. The velocity of the kink $\varphi(y)$ tunes to that of the kink $\psi(y)$ and the resulting velocity is $v \sim v_\psi$. The distribution $\psi(y)$ is described by the one equation (9) with $\psi_* = \psi_+$ up to $\beta = \beta_c$, where v reaches its maximum value $v(\beta_c) = v_c \sim v_\psi$.

Hysteresis thus produces a number of specific features in the behavior of $v(\beta)$. In the interval $\beta_p < \beta < \beta_+$, the $v(\beta)$ curve has a flat segment (a plateau), where $v \sim v_\varphi \ll v_\psi$, that is, it is much lower than the kink's velocity without hysteresis ($v \sim v_\psi$). This plateau is a result of the interaction of $\psi(y)$ and $\varphi(y)$ kinks which produces the dependence $\psi_* = \psi_*(\beta)$. The kink's velocity in the interval $\beta_+ < \beta < \beta_c$ is $v \sim v_\psi$, comparable in magnitude with the kink velocity in a medium without hysteresis, until the limiting value $v = v_c$ is reached at $\beta = \beta_c$. It can thus be said under the approximation $v_\varphi/v_\psi \ll 1$ that the hysteresis shifts the value of $\beta = \beta_p$ to the point $\beta = \beta_+$.

A similar picture is found when the parameter $\beta < \beta_p$ is decreasing. The quantity ψ_* decreases and reaches the other boundary of the non-single-valuedness interval, $\psi_* = \psi_-$, at the value of the parameter $\beta = \beta_-$ that is determined by the 'equal areas theorem' (8) for $\psi_* = \psi_-$ to the accuracy $v_\varphi/v_\psi \ll 1$:

$S_3(\psi_-, \beta_-) = 0.$

As the parameter β decreases further, the kink velocity $|v| \sim v_\psi$ reaches its minimum value at the bifurcation point.

Appendix A

Kink's stability

Let us analyze the stability of the kinks shown in fig. 4 with respect to small perturbations $\delta\psi(y,t) = \delta\psi(y)\exp(\lambda t)$, $\delta\varphi(y,t) = \delta\varphi(y)\exp(\lambda t)$. To simplify the analysis, assume $v = 0$. Then $\delta\psi(y), \delta\varphi(y)$ satisfy the equations

$$l_\psi^2 \delta\psi'' - \left(\tau_\psi \lambda + \frac{\partial F}{\partial \psi}\right) \delta\psi - \frac{\partial F}{\partial \varphi} \delta\varphi = 0, \tag{A.1}$$

$$l_\varphi^2 \delta\varphi'' - \left(\tau_\varphi \lambda + \frac{\partial R}{\partial \varphi}\right) \delta\varphi - \frac{\partial R}{\partial \psi} \delta\psi = 0, \tag{A.2}$$

and the boundary conditions $\delta\psi(\pm\infty) = \delta\varphi(\pm\infty) = 0$. The dependence of the coefficients of eqs. (A.1), (A.2) on y is determined by the steady-state distributions $\psi(y), \varphi(y)$. A kink is stable if the eigenvalues λ satisfy the condition $\operatorname{Re}\lambda < 0$.

One of the solutions to eqs. (A.1), (A.2) is $\delta\psi = \psi'(y)$, $\delta\varphi = \varphi'(y)$, $\lambda = 0$, which is implied by the translation invariance of eqs. (3), (4). It is therefore more convenient to pass to new variables $u = \delta\psi/\psi'$, $w = \delta\varphi/\varphi'$. Then

$$l_\psi^2 \frac{d}{dy}(u'\psi'^2) - \tau_\psi \lambda u \psi'^2 + (u - w)\psi'\varphi' \frac{\partial F}{\partial \varphi} = 0, \tag{A.3}$$

$$l_\varphi^2 \frac{d}{dy}(w'\varphi'^2) - \tau_\varphi \lambda w \varphi'^2 + (w - u)\psi'\varphi' \frac{\partial R}{\partial \psi} = 0. \tag{A.4}$$

For further stability analysis, assume that the condition (14) is satisfied. One can then obtain the kink's stability criterion without solving eqs. (A.3), (A.4). Indeed, eqs. (A.3), (A.4) can be obtained by varying the functional

$$\tilde{Q} = \int_{-\infty}^{+\infty} dy \left[l_\psi^2 \psi'^2 u'^2 + l_\varphi^2 \varphi'^2 w'^2 - (u - w)^2 \psi'\varphi' \frac{\partial F}{\partial \varphi} \right] \tag{A.5}$$

in ψ and φ, taking into account normalization conditions

$$\tau_\psi \int_{-\infty}^{+\infty} u^2 \psi'^2 dy = \tau_\varphi \int_{-\infty}^{+\infty} w^2 \varphi'^2 dy = 1.$$

We multiply eq. (A.3) by u and eq. (A.4) by w, add them up, and integrate in y from $-\infty$ to $+\infty$. We have

$$\lambda = -\frac{\tilde{Q}}{(\tau_\psi + \tau_\varphi)}. \tag{A.6}$$

The functions $u(y)$ and $w(y)$, minimizing the functional $\tilde{Q}[u, w]$, thus correspond to the 'most dangerous' perturbations at the maximum value of $\mathrm{Re}\,\lambda$. A kink is stable if $\tilde{Q}_{\min} > 0$, which is always satisfied if

$$\psi'(y)\varphi'(y)\frac{\partial F}{\partial\varphi} \le 0 \tag{A.7}$$

for all y. Condition (A.7) leads to the inequalities $\psi'(y)\varphi'(y) \ge 0$ for $\partial F/\partial\varphi < 0$ and $\psi'(y)\varphi'(y) \le 0$ for $\partial F/\partial\varphi > 0$. The product $\psi'(y)\varphi'(y)$ is positive for all y for the kink shown in fig. 4. This means that this kink is stable if $\partial F/\partial\varphi < 0$. This last condition has a clear physical meaning: it signifies that the variable $\psi(x, t)$ damps the perturbation $\delta\varphi(x, t)$, which does hold for most of the particular nonequilibrium systems mentioned in section 2.

For the kink shown in fig. 4(a), the product $\psi'(y)\varphi'(y)$ does not have a constant sign, and the criterion (A.7) does not allow to draw a conclusion on its stability. The stability of this kink can be analyzed using eqs. (A.5), (A.6) and test functions of the form

$$u = 0, \quad w = \exp\left[-(y - y_0)^2/l^2\right],$$

where y_0 is so chosen that $\psi'(y_0) < 0$ and $l \ll l_\psi$. Then $\psi'(y)\varphi'(y)$ in eq. (A.5) can be replaced with $\psi'(y_0)\varphi'(y_0)$ and hence,

$$\tilde{Q} = \sqrt{\pi}\left[\frac{2l_\varphi^2}{l}\varphi'^2(y_0) - l\psi'(y_0)\varphi'(y_0)\frac{\partial F}{\partial\varphi}(y_0)\right].$$

In this particular case, $\partial F/\partial\varphi < 0$ and $\psi'(y_0) < 0$, that is, $\tilde{Q}$ becomes negative if

$$l > l_c = l_\varphi\left|\frac{2\varphi'(y_0)}{\psi'(y_0)\,\partial F/\partial\varphi(y_0)}\right|.$$

The quantity $\tilde{Q}$ calculated in terms of eigenfunctions of eqs. (A.3), (A.4) is always smaller than $\tilde{Q}_{\min}$ for test functions. Hence, the increment λ for perturbations with $l > l_c$ is positive, and the kink shown in fig. 4(a) is unstable. The instability of this kink arises because it contains an interval of the unstable phase ($\psi_- < \psi < \psi_+$, $\psi'(y) < 0$), in which perturbations (e.g., those with $l > l_c$) grow and ultimately destroy the kink.

Thus there are two types of kinks in the hysteretic medium, one of these being stable and the other unstable. This situation is quite different from the well-known case of one-component nonhysteretic system, where the kink is the unique stable inhomogeneous solution of eq. (1).

Appendix B

Velocity and structure of the kink $\varphi(y)$ in the vicinity of the bifurcation point $\psi_* = \psi_\pm$

In the neighborhood of the jump, $\Delta y \sim l_\varphi$, the function $\psi(y)$ can be presented in the form $\psi(y) = \psi_* + ky$, where $k = \psi'(0)$. Eq. (16) transforms to

$$l_\varphi^2 \varphi'' + \tau_\varphi v \varphi' + \alpha\varphi - \varphi^3 + \psi_* + ky = 0. \tag{B.1}$$

If $k = 0$, eq. (B.1) has the well-known solution

$$\varphi_0(y) = \frac{\varphi_b + \varphi_a}{2} + \frac{\varphi_b - \varphi_a}{2} \tanh \frac{y}{2L}, \quad L = \frac{\sqrt{2}\, l_\varphi}{(\varphi_b - \varphi_a)}, \tag{B.2}$$

$$v_0(\psi_*) = \frac{v_\varphi}{\sqrt{2}} [\varphi_a(\psi_*) + \varphi_b(\psi_*) - 2\varphi_c(\psi_*)], \tag{B.3}$$

where $\varphi_a, \varphi_b, \varphi_c$ are the roots of the equation $\varphi^3 - \alpha\varphi = \psi_*$. The velocity $v_0(\psi_*)$ is equal to zero at $\psi_* = 0$ and it increases monotonically with increasing ψ_* up to the maximum value $v_0 = v_{c\varphi}$ at $\psi_* = \psi_+$, where

$$v_{c\varphi} = \frac{v_\varphi}{\sqrt{2}} [\varphi_b(\psi_+) - \varphi_a(\psi_+)].$$

Here φ_a, φ_b, and ψ_+ are determined by the conditions for which the stable state φ_a disappears, i.e. $\varphi_a = \varphi_c = -(\alpha/3)^{1/2}$, $\varphi_b^3 - \alpha\varphi_b = \psi_+$ and $\psi_+ = 2(\alpha/3)^{3/2}$.

The correction $|ky| \lesssim (l_\varphi/l_\psi)\psi_* \ll \psi_*$ being small, we seek the solution to eq. (B.1) in the form

$$\varphi(y) = \varphi_0(y) + \delta\varphi,$$
$$v = v_0 + \delta v.$$

Then we have for $\delta\varphi$

$$l_\varphi^2\, \delta\varphi'' + \tau_\varphi v_0\, \delta\varphi' + (\alpha - 3\varphi_0^2)\, \delta\varphi = -ky - \tau_\varphi\, \delta v\, \varphi_0'.$$

Since $\varphi_0'(y)$ is a solution of this equation for $k = \delta v = 0$, it is convenient to replace $\delta\varphi$ with

$$\delta\varphi = \varphi_0' z,$$

where z satisfies the equation

$$l_\varphi^2 \varphi_0' z'' + 2 l_\varphi^2 \varphi_0'' z' + \tau_\varphi v_0 \varphi_0' z' = -ky - \tau_\varphi\, \delta v\, \varphi_0',$$

which transforms after the change of variables $z' = g \exp(-\tau_\varphi v_0 y / l_\varphi^2)$ to the form

$$l_\varphi^2 \frac{\mathrm{d}}{\mathrm{d}y} (\varphi_0'^2 g) = (-ky - \tau_\varphi\, \delta v\, \varphi_0') \exp\left(+ \frac{\tau_\varphi v_0 y}{l_\varphi^2} \right).$$

Integrating in y from $-\infty$ to $+\infty$, we obtain

$$\delta v = -\frac{k}{\tau_\varphi} \frac{\int_{-\infty}^{+\infty} y \varphi_0'(y) \exp\left(+\tau_\varphi v_0 y / l_\varphi^2\right) \mathrm{d} y}{\int_{-\infty}^{+\infty} \varphi_0'^2(y) \exp\left(+\tau_\varphi v_0 y / l_\varphi^2\right) \mathrm{d} y} \tag{B.4}$$

As follows from eqs. (B.2), (B.3) the integral in the numerator of eq. (B.4) diverges as $\psi_* \to \psi_\pm$ like $\int y \, \mathrm{d} y$, and $\delta v \to \infty$. Really, one finds from the eqs. (B.2)–(B.4) that the value of δv at $v_0 \to v_{c\varphi}$ is given by

$$\frac{\delta v}{v_0} = \frac{k l_\varphi}{(\varphi_b - \varphi_a)^3} \left(\frac{v_{c\varphi}}{v_0 - v_{c\varphi}} \right)^2 . \tag{B.5}$$

Therefore, as a result of the dependence of $\psi(y)$ on y, the expression (B.3) ceases to hold in the neighborhood of the bifurcation point $\psi_* = \psi_\pm$. For $\delta\varphi(y)$ we find

$$\delta\varphi(y) = -\varphi_0'(y) \int^{y} \frac{\mathrm{d} y_1 \exp\left(\tau_\varphi v_0 y_1 / l_\varphi^2\right)}{\varphi_0'^2(y_1)} \int^{y_1} \left[k y_2 + \tau_\varphi \delta v \, \varphi_0'(y_2) \right] \exp\left(+ \frac{\tau_\varphi v_0 y_2}{l_\varphi^2} \right) \mathrm{d} y_2 . \tag{B.6}$$

Eq. (B.6) shows that as $\psi_* \to \psi_\pm$, the correction $\delta\varphi$ ceases to be small, $\delta\varphi \to \infty$, that is, the dependence of $\psi(y)$ on y results in a substantial change in the structure of the jump $\varphi(y)$, even though ky is small, $|ky| \ll \psi_*$.

The qualitative analysis performed in section 5 is valid if $\delta v(\psi_*) \ll v_0(\psi_*)$. This condition becomes invalid at $v_0 \to v_{c\varphi}$ where the dependence of ψ_* on y should be taken into account. As follows from the eq. (B.5) there is a crossover region $|v - v_{c\varphi}| \sim v_k$, $v_k \sim v_\varphi (l_\varphi / l_\psi)^{1/2}$ where $\delta v \sim v_0$ and the velocity v is determined by the complete set of eqs. (3), (4). In the region $v - v_{c\varphi} \gg v_k$ the value of ψ_* becomes independent of β and the velocity v is determined by the single equation (19) with $\psi_* = \psi_+$. The structure of the jump $\varphi(y)$ is determined by eq. (4) with v and $\psi(y)$ obtained above; the width of the jump increases as $\Delta y \sim l_\varphi (v / v_\varphi)$.

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