J. Phys. D: Appl. Phys. 20 (1988) 826–830. Printed in the UK

The current-carrying capacity of twisted multifilamentary superconducting composites

R G Mints and A L Rakhmanov

Institute of High Temperatures, 127412, Moscow, USSR

Received 28 May 1987

Abstract. The superconducting state stability in twisted multifilamentary wire is investigated. The case of varying transport current and transverse magnetic field is considered. It is shown that the current-carrying capacity I_m of multifilamentary superconducting composites increases with the decrease of the twist pitch L and I_m attains its maximum value at L less than some critical value L_c.

1. Introduction

The current-carrying capacity of superconductors is limited by the superconducting state stability relative to perturbations of a different physical nature in many cases of practical interest. If the intensity of these perturbations is not too high, then the maximum transport current I_m is determined by the superconducting state stability with respect to small disturbances (Mints and Rakhmanov 1981, 1984). The problem has already been considered by the present authors for the case when the twisting of the superconducting wire is neglected (Mints and Rakhmanov 1982, 1984). It has been shown that to explain the high values of $I_m \sim I_s$ (where I_s is the critical current) one has to take into account the behaviour of the current–voltage characteristics of a hard superconductor in the region of low values of the electric field E. At low E this I–V characteristic may be presented in the form (Polak *et al* 1973)

$$j = j_s + j_1 \ln(E/E_0) \tag{1}$$

where j is the current density, $j_s = j(E_0)$ is the critical current density and the ratio j_1/j_s is usually of the order of (1–3) $\times 10^{-2}$. Then, the differential conductivity of the superconductor is $\sigma(E) = j_1/E$. Assuming that $j_1 = 10^7$ A m^{-2} and $E = 10^{-5}$ V m^{-1} one finds that $\sigma(E) = 10^{12}\ \Omega^{-1}$ m^{-1}, which is much higher than the normal matrix conductivity. Since the superconducting state stability increases with the increase of longitudinal conductivity (Mints and Rakhmanov 1981), then the value of I_m depends on the electric field E induced by external sources (Mints and Rakhmanov 1982, 1984).

In this paper the maximum transport current I_m for a twisted multifilamentary superconducting wire placed in a transverse magnetic field $B_a(t)$ is found. In § 2 the stability criterion for a twisted wire is obtained. In § 3 the electric field distribution in a twisted composite with a transport current is discussed and § 4 investigates the dependence of I on the twist pitch L. Finally, in § 5 the equation for I_m is found for the case of a wire with a sufficiently small twist pitch L.

2. Stability criterion

In order to find the stability criterion in the case of a twisted composite superconductor the standard method of stability investigation may be used (Mints and Rakhmanov 1984).

Let us consider a twisted multifilamentary wire of radius R_0 in a transverse magnetic field $B_a(t)$. It is known (Carr 1983) that in this situation the current density $\boldsymbol{j}$ has both a longitudinal component $j_\parallel$ and a transverse component $j_\perp$. Taking into account the current in the normal matrix one can write

$$\begin{aligned} j_\parallel &= j_s + j_1 \ln(E/E_0) + \sigma_\parallel E_\parallel \\ j_\perp &= \sigma_\perp E_\perp \end{aligned} \tag{2}$$

where $E_\parallel$ and $E_\perp$ are the longitudinal (along the filaments) and transverse components of the field $\boldsymbol{E}$ and $\sigma_\parallel$, $\sigma_\perp$ are the longitudinal and transverse conductivities of the composite. In practical situations one may suppose that $\sigma_\parallel E_\parallel$, $\sigma_\perp E_\perp \ll j_1 \ll j_s$.

We consider here only extended disturbances (such as $\dot{B}_a$) and therefore can neglect heat conduction along the composite in the following derivation of the stability criterion. The considered instability causes temperature and electromagnetic field perturbations to increase in a correlated manner. Each one of these

0022-3727/88/050826 + 05 $02.50

processes is characterised by its respective diffusion coefficient, namely the thermal diffusion coefficient $D_t = \kappa/\nu$ and the magnetic diffusion coefficient $D_m = (\mu_0\sigma)^{-1}$, where ν is the heat capacity and κ is the heat conductivity of the composite. Let us introduce the parameter τ:

$$\tau = \frac{D_t}{D_m} = \mu_0 \frac{\sigma\kappa}{\nu}. \tag{3}$$

For superconducting composites $\tau \gg 1$. Accordingly, fast heating of composites occurs under conditions of frozen-in magnetic flux. Then, one can assume that in the initial stage of the instability development the perturbation of the current density $\boldsymbol{\delta j}$ is equal to zero, i.e.

$$\delta j = \frac{\partial j_\parallel}{\partial E_\parallel}\delta E_\parallel + \frac{\partial j_\perp}{\partial E_\perp}\delta \boldsymbol{E}_\perp + \frac{\partial \boldsymbol{j}}{\partial T}\delta T = 0 \tag{4}$$

where $\delta \boldsymbol{E}$ and δT are infinitesimally small perturbations of the field $\boldsymbol{E}$ and temperature T. Equation (4) allows us to find the relation between $\delta \boldsymbol{E}$ and δT. Using this relation and equations (1) and (2) to obtain the Joule heating term $\delta T \partial(\boldsymbol{E}\boldsymbol{j})/\partial T$, we find the heat equation in a linear approximation as

$$\nu \frac{\partial}{\partial t}\delta T = \kappa \nabla^2 \delta T + \frac{j_s}{j_1}\left|\frac{\partial j_s}{\partial T}E_\parallel\right|\delta T. \tag{5}$$

The heat conductivity of the composite is usually high and $W = 2W_0R_0/\kappa < 1$, where W_0 is the heat transfer coefficient. In this case one can assume that δT is uniform over the cross section of the wire. Then, integrating equation (5) over the cross section of the conductor and using the boundary conditions

$$\kappa_\perp(\boldsymbol{e}_R \cdot \nabla \delta T) = -W_0 \delta T \qquad R = R_0 \tag{6}$$

we have

$$\nu/\delta \dot{T} = -\frac{2W_0}{R_0}\delta T + \left\langle \frac{j_s}{j_1}\left|\frac{\partial j_s}{\partial T}E_\parallel\right|\right\rangle \delta T \tag{7}$$

where $\boldsymbol{e}_R$ is the unit vector in the radial direction; the brackets $\langle\,\rangle$ indicate the value of the enclosed function averaged over the cross section. The superconducting state is stable ($\delta \dot{T} < 0$) if

$$\left\langle \frac{j_s}{j_1}\left|\frac{\partial j_s}{\partial T}E_\parallel\right|\right\rangle < \frac{2W_0}{R_0}. \tag{8}$$

In the range of high magnetic fields $B_a \gg \mu_0 j_s R_0$ one may suppose that $j_s = j_s(B_a), j_1 = j_1(B_a)$ and the criterion (8) may be rewritten in the form

$$\langle |E_\parallel| \rangle < E_c = \frac{2W_0 j_1}{R_0 j_s |\partial j_s/\partial T|}. \tag{9}$$

The stability criterion (9) is similar to the analogous criterion obtained for the untwisted wire (Mints and Rakhmanov 1982).

Thus, to find I_m one has to calculate the longitudinal component of the electric field.

3. Electric field distribution

The electric properties of twisted multifilamentary composites have been studied extensively (Carr 1983). It has been shown that in the case of the wire placed in a varying transverse magnetic field the cross section of the conductor may be divided into two regions: an external or 'saturated' region in which $E_\parallel$ is non-vanishing and an interior one in which $E_\parallel = 0$. To find the analytical solution for $E_\parallel$ in the 'saturated' region in a general case is impossible. The purpose of the present calculations is to find $E_\parallel$ in some limiting cases being, however, of practical interest.

Let us suppose that the value of $\dot{B}_a$ is not too high:

$$a = \frac{\dot{B}_a \tau_0}{B_p} \ll 1 \qquad \frac{\dot{B}_a \tau_0}{B_m} \ll 1 \tag{10}$$

where a is the effective field variation, $\tau_0 = \mu_0 L^2 \sigma_\perp / 8\pi^2$ is the characteristic time of the resistive current decay in a twisted wire (Carr 1983), $B_p = 2j_s\mu_0R_0/\pi$ is the magnetic flux penetration field and B_m is the amplitude of the $B_a(t)$ variation in the considered process. Under conditions (10) one can write Maxwell's equation in the form (Carr 1983)

$$\operatorname{curl} \boldsymbol{B} = \dot{\boldsymbol{B}}_a. \tag{11}$$

In the cylindrical coordinates R, φ, z, where the axis of the wire is along the z direction, the solution of equation (11) may be presented in the form

$$\boldsymbol{E} = -\dot{B}_a R \sin\varphi\, \boldsymbol{e}_z + \frac{L\dot{B}_a}{2\pi}\nabla(f(R,\varphi) - R\cos\varphi) \tag{12}$$

where φ is measured from the direction of $\boldsymbol{B}_a$, taken to be along the x axis, $\boldsymbol{e}_z$ is the unit vector in the z direction and $f(R, \varphi)$ is a continuous function giving the distribution of $E_\perp$. As the radial component of the current $j_R = 0$ at $R = R_0$, one finds by means of equation (12) the boundary condition in the form

$$\frac{\partial f}{\partial R} = \cos\varphi \qquad R = R_0. \tag{13}$$

The unit vector $\boldsymbol{e}_\parallel$ parallel to the filaments is given by

$$\boldsymbol{e}_\parallel = \left(\boldsymbol{e}_z + \frac{2\pi R}{L}\boldsymbol{e}_\varphi\right)/[1 + (2\pi R/L)^2]^{1/2} \tag{14}$$

where $\boldsymbol{e}_\varphi$ is the unit vector in the φ direction. By means of equations (12) and (14) one finds

$$E_\parallel = \frac{\dot{B}_a}{[1 + (2\pi R/L)^2]^{1/2}}\frac{\partial f}{\partial \varphi}. \tag{15}$$

If $R = R_1(\varphi)$ is the equation for the interface between 'saturated' and internal regions, then

$$\frac{\partial f}{\partial \varphi} = 0 \qquad R = R_1(\varphi). \tag{16}$$

Substituting formulae (2), (12) and (15) into the current continuity equation div $\boldsymbol{j}(\boldsymbol{E}) = 0$ we obtain the equation for $f(R, \varphi)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} = \left(\frac{2\pi R_0}{L}\right)^2\left(r\cos\varphi + \frac{\sigma_\parallel}{\sigma_\perp}\frac{\partial^2\phi}{\partial r^2}\right) - \left(\frac{2\pi R_0}{L}\right)^2 \frac{j_1}{\sigma_\perp \dot{B}_a R_0}\frac{\partial^2\phi/\partial\varphi^2}{\partial\phi/\partial\varphi} \tag{17}$$

where

$$\phi = \frac{f}{R_0} \qquad r = \frac{R}{R_0} \qquad r_1(\varphi) = \frac{R_1(\varphi)}{R_0}.$$

The boundary conditions (13) and (16) may be rewritten in the form

$$\frac{\partial\phi}{\partial\varphi} = 0 \qquad r = r_1(\varphi) \qquad \frac{\partial\phi}{\partial r} = \cos\varphi \qquad r = 1. \tag{18}$$

Note that the non-linearity of the I–V characteristic of the composite (2) results in the non-linear term in the right-hand side of equation (17).

First, let us consider the case when $I = 0$. In the situation under consideration the 'saturated' region is small ($1 - r_1(\varphi) \ll 1$) and one can write (Carr 1983)

$$r_1(\varphi) = 1 - \frac{4a}{\pi}|\sin\varphi|. \tag{19}$$

The power series of $\phi(r, \varphi)$ in the 'saturated' region at $1 - r < 1 - r_1 \ll 1$ may be written in the form

$$\phi(r,\varphi) = r\cos\varphi + \Psi(\varphi) + \sum_{n=2}^{\infty} b_n(\varphi)(1-r)^n \tag{20}$$

where Ψ and b_n are functions of φ. Expression (20) satisfies the boundary condition at $r = 1$ automatically.

By means of equations (15) and (20) one finds

$$E_\parallel = \frac{\dot{B}_a R_0}{[1 + (2\pi R/L)^2]^{1/2}} \times \left(-r\sin\varphi + \frac{d\Psi}{d\varphi} + \sum_{n=2}^{\infty}\frac{db_n}{d\varphi}(1-r)^n\right). \tag{21}$$

Substituting expression (20) into the boundary conditions (18) at $r = r_1$ we obtain the relation between $d\Psi/d\varphi$ and $r_1(\varphi)$:

$$\frac{d\Psi}{d\varphi} = r_1\sin\varphi - \sum_{n=2}^{\infty}\frac{db_n}{d\varphi}(1-r_1)^n. \tag{22}$$

Using equations (21) and (22) one finds $E_\parallel$ in the form

$$E_\parallel = \frac{\dot{B}_a R_0}{[1 + (2\pi R/L)^2]^{1/2}} \times \left((r - r_1)\sin\varphi + \sum_{n=2}^{\infty}\frac{db_n}{d\varphi}[(1-r)^n - (1-r_1)^n]\right). \tag{23}$$

To find the recurrence relations for b_n we have to substitute expansions (22) and (23) into equation (17). Omitting the details of rather complicated calculations we may state:

(i) the expansions (20)–(23) converge rapidly;

(ii) the terms in the right-hand side of equation (17) may be neglected, since the contribution of the first term in the solution ϕ is of the order of $(2\pi R_0/L)^2 \ll 1$ and the contribution of the second one is of the order of $j_1/j_s \ll 1$;

(iii) the first term in expansion (23) is of the order of a and the sum of the other terms is of the order of a^2 and in the first approximation this sum may be omitted.

Then, in the case under consideration we have

$$E_\parallel = \dot{B}_a \sin\varphi(R_1(\varphi) - R). \tag{24}$$

By making use of relations (19) and (24) one can find the averaged value $E_\parallel$ and the stability criterion, as given by the inequality (9), becomes

$$\langle|E_\parallel|\rangle = \frac{64a^2}{3\pi^3}\dot{B}_a R_0 < \frac{2W_0 j_1}{j_s/|\partial j_s/\partial T|R_0} \tag{25}$$

or by substituting the effective field variation $a = \dot{B}_a L^2\sigma_\perp/16\pi j_s R_0$ from (10)

$$L < L_c = \left(\frac{24\pi^5 j_1 j_s W_0}{\sigma_\perp^2 \dot{B}_a^3|\partial j_s/\partial T|}\right)^{1/4}. \tag{26}$$

Note that the value of L_c is independent of the wire radius. Supposing that $W_0 = 10^2$ W m^{-2} K^{-1}, $j_1 = 10^7$ A m^{-2}, $j_s/|\partial j_s/\partial T| = 3$ K, $\sigma_\perp = 5\times10^9\ \Omega^{-1}$ m^{-1} and $\dot{B}_a = 1$ T s^{-1}, one finds that $L_c = 0.03$ m.

Equation (25) is valid at $a \ll 1$. It is not too difficult to show that in the opposite case, $a \gg 1$, the twisting does not affect the superconducting state stability since the distribution of $E_\parallel$ is independent of L in this situation and to find I_m one can use the results obtained for an untwisted wire.

4. A wire with a small transport current

Let us consider a wire carrying a current $I \ll I_s = \pi R_0^2 j_s$ and suppose that $\dot{B}_a = 0$ at $t = 0$ and at $t > 0$ the value of B_a increases monotonically with the rate $\dot{B}_a$. The transport current flows initially within the region $R_0(1 - i)^{1/2} < R < R_0$ where $i = I/I_s$. At $t > 0$ the variable magnetic field induces the electric field $\boldsymbol{E}$ in the wire. The direction of the component $E_\parallel$ of this field coincides with the direction of the transport current at $0 < \varphi < \pi$ and $E_\parallel$ has the opposite direction at $\pi < \varphi < 2\pi$ (see figure 1). In the case under consideration i, $a \ll 1$ and one can conclude that the 'saturated' region is small, i.e. $1 - r_1(\varphi) \ll 1$. In this situation the boundary $R_1(\varphi)$ may be found by the method proposed by Carr (1974). According to this method the longitudinal current flowing in the 'saturated' region is considered as the surface current. Taking into account

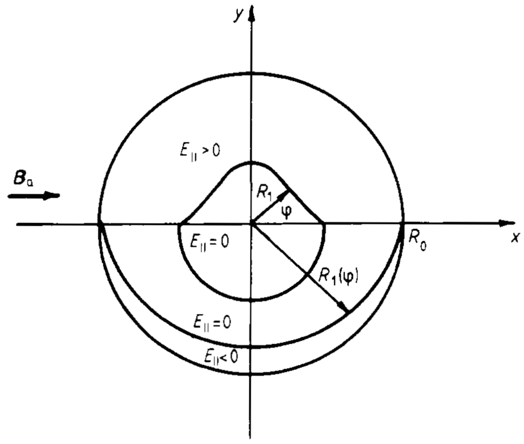

Figure 1. Current and electric field distributions in a twisted wire with a transport current.

that $\dot{I} \neq 0$ and $\partial B/\partial t = \dot{B}_a$ it may be verified that the calculations described in detail in the paper by Carr (1974) give the following result for R_1:

$$R_1(\varphi) = R_0\left(1 - \frac{i}{2} - \frac{4a}{\pi}\sin\varphi\right) \qquad 0 < \varphi < \pi \tag{27}$$

$$R_1(\varphi) = R_0\left(1 + \frac{2a}{\pi}\sin\varphi\right) \qquad \pi < \varphi < 2\pi. \tag{28}$$

Expression (28) is valid if $R_1(\varphi) > R_0(1-i)^{1/2}$ or

$$\frac{4a}{\pi} < i \ll 1. \tag{29}$$

It ought to be emphasised that at $\pi < \varphi < 2\pi$ the field $E_{\parallel}$ exists in the layer $R_1 < R < R_0$ where $j_{\parallel} = -j_s$ and $E_{\parallel} = 0$ in the region $R_0(1-i)^{1/2} < R < R_1$, where $j_{\parallel} = j_s$.

To find $\langle|E_{\parallel}|\rangle$, equation (24), with $R_1(\varphi)$ defined by expressions (27) and (28), may be used. The result of the calculations has the form

$$\langle|E_{\parallel}|\rangle = \frac{R_0\dot{B}_a}{4\pi}\left(i^2 + 4ai + \frac{160}{3\pi^2}a^2\right). \tag{30}$$

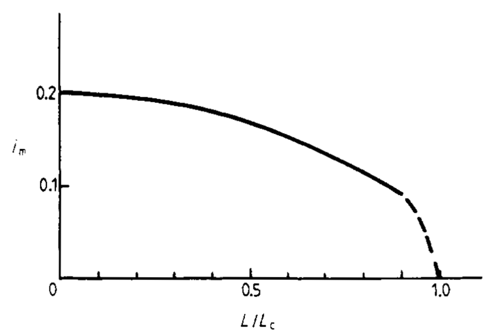

Figure 2. The dependence of i_m on L/L_c at $4\pi E_c/R_0\dot{B}_a = 0.04$.

Substituting equation (30) for the stability criterion (9), and neglecting small terms, one can find the value of the maximum transport current

$$I_m = 2I_s\left(\frac{\pi E_c}{R_0\dot{B}_a}\right)^{1/2}\left[1 - \frac{3^{1/2}\pi}{8}\left(\frac{L}{L_c}\right)^2\right]. \tag{31}$$

The dependence $I_m(L/L_c)$ in equation (31) is shown in figure 2. The current I_m attains its maximum value $2I_s(\pi E_c/R_0\dot{B}_a)^{1/2}$ at $L \ll L_c$. Equation (31) is valid if the inequalities in (29) are fulfilled. Expression (31) allows one to write these inequalities in the form

$$L/L_c \leqslant 0.9 \qquad E_c/R_0\dot{B}_a \ll 1. \tag{32}$$

5. A wire with a current $I \sim I_s$

For a wire with a transport current $I \sim I_s$ the current-carrying capacity may be found in the case of $a \ll 1$. To determine I_m we shall calculate the value of $E_{\parallel}$ in the main approximation with respect to $a \ll 1$. First, we shall consider the situation when $\partial I/\partial t = 0$. Since $I/I_s \sim 1$ and $a \ll 1$ a good approximation is $R_1(\varphi) = R_0(1-i)^{1/2}$ at $0 < \varphi < \pi$ and $R_1 = R_0$ at $\pi < \varphi < 2\pi$ in analogy with the results of the previous section. The value of $E_{\parallel}$ may be calculated by means of equation (17). Following the procedure described in § 3 it may be shown that the terms in the right-hand side of this equation are negligible at $a \ll 1$. Thus we have

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0. \tag{33}$$

To obtain the value of $E_{\parallel}$ one has to find the solution of equation (33) which satisfies the boundary conditions (18). Substituting this solution into equation (15) we have, to a first approximation

$$E_{\parallel} = -\frac{\dot{B}_a\sin\varphi}{(2-i)}\frac{R^2 - R_0^2(1-i)}{R}. \tag{34}$$

From criterion (9) and relation (34) we find the equation for I_m at $L \ll L_c$

$$\frac{4\pi E_c}{R_0\dot{B}_a} = \frac{8}{3}\frac{1.5i_m + (1-i_m)^{3/2} - 1}{1 - 0.5i_m}. \tag{35}$$

The values of $i_m = I_m/I_s$ given by equations (31) and (35) coincide at $L \ll L_c$ and $4\pi E_c/R_0\dot{B}_a \ll 1$. It follows from equation (35) that $I_m = I_s$ at $1.5\pi E_c \geqslant R_0\dot{B}_a$. The dependence of i_m on $E_c/R_0\dot{B}_a$, calculated with the help of equation (35), is shown in figure 3 (curve A).

For the case $\dot{I} \neq 0$ the calculations analogous to the previous ones give the equation for I_m in the form

$$\frac{\mu_0\dot{I}}{8\pi E_c}g_1(i_m) + \frac{R_0\dot{B}_a}{4\pi E_c}g_2(i_m) = 1 \tag{36}$$

where

$$g_1(i_m) = -2[i_m + \ln(1 - i_m)]$$
$$g_2(i_m) = \frac{8}{3}\frac{1.5i_m + (1-i_m)^{1/2} - 1}{1 - 0.5i_m}. \tag{37}$$

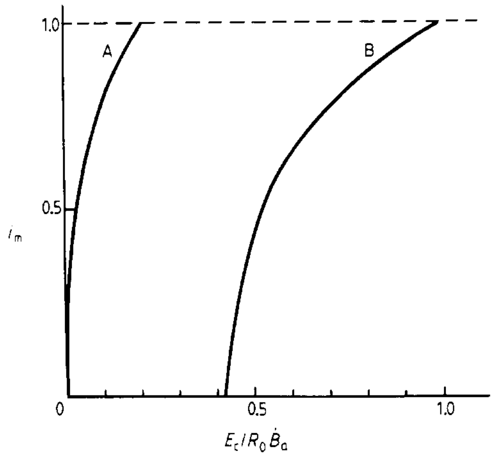

Figure 3. The dependence of i_m on $E_c/R_0\dot{B}_a$ at $\dot{I} = 0$. Curves: A, twisted wire with $L \ll L_c$, B, untwisted wire.

The boundaries of the stable regions in the $(\dot{I}, \dot{B}_a)$ plane are shown in figure 4 for different values of I_m. The dependence of i_m on the parameter $\mu_0\dot{I}(1+A)/4\pi E_c$ is shown in figure 5 for different values of $A = 2\pi R_0\dot{B}_a/\mu_0\dot{I}$.

To compare the current-carrying capacities of the twisted and untwisted wires let us consider the situation in which $\dot{I} = 0$ and B_a is high enough to penetrate the untwisted wire completely. In this case in the untwisted wire the value of $E_\parallel$ is given by the expression

$$E_\parallel = e_z\dot{B}_a(y - y_0) \tag{38}$$

where y_0 is defined by the equation

$$i = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{y}{R_0}\right) + \frac{y_0}{R_0}\left(1 - \frac{y_0^2}{R_0^2}\right)^{1/2}\right]. \tag{39}$$

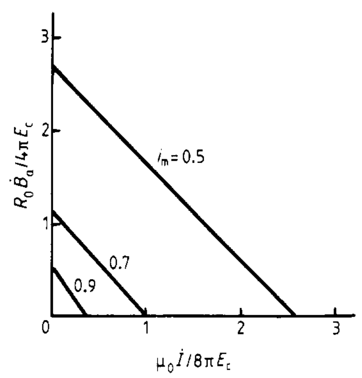

Figure 4. The boundary of the stable region in the $(\dot{I}, \dot{B}_a)$ plane for different I_m.

Using equations (38) and (39) and the criterion (9) one can find the equation for i_m in the form

$$i_m\frac{y_0(i_m)}{R_0} + \frac{4}{3\pi}\left(1 - \frac{y_0^2(i_m)}{R_0^2}\right)^{3/2} = \frac{E_c}{R_0\dot{B}_a}. \tag{40}$$

The dependence of i_m on $E_c/R_0\dot{B}_a$ for the untwisted wire is shown in figure 3 by curve B. The comparison of curves A and B shows that the twisting results in an appreciable increase of the current I_m.

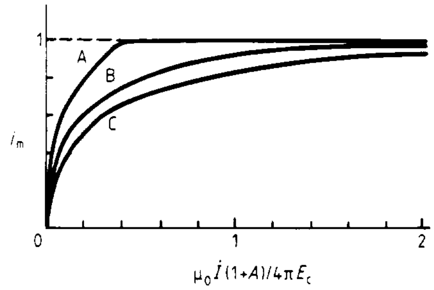

Figure 5. The dependence of i_m on $\mu_0\dot{I}(1+A)/4\pi E_c$ for different values of A. Curves: A, $A = 0$; B, $A = 1$; C, $A \gg 1$.

6. Conclusions

(i) The twisting of the multifilamentary superconducting composites leads to a considerable increase of the current-carrying capacity.

(ii) The critical value L_c of the twist pitch L is obtained: at $L \ll L_c$ the current-carrying capacity attains its maximum.

(iii) The equation allowing one to calculate the maximum transport current at $L \ll L_c$ is found.

References

Carr W J 1974 *J. Appl. Phys.* **45** 929–34

—— 1983 *AC Loss and Macroscopic Theory of Superconductors* (New York: Gordon and Breach)

Mints R G and Rakhmanov A L 1981 *Rev. Mod. Phys.* **53** 551–92

—— 1982 *J. Phys. D: Appl. Phys.* **15** 2297–306

—— 1984 *Neustoichivosti v sverhprovodnicah* (Moscow: Nauka)

Polak M, Hlasnik I and Krempasky L 1973 *Cryogenics* **13** 702–11