LOCALIZATION OF NONLINEAR WAVES IN RANDOMLY INHOMOGENEOUS MEDIA

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It is shown that localization of the interface in randomly inhomogeneous bistable media is possible. The equation describing slow dynamics of the interface is obtained. Conditions of the localization are considered. It is shown that the localization may lead to formation of stochastic dissipative structures.

In the present paper we consider motion and localization of an interface in a randomly inhomogeneous bistable medium. We assume that the interface is described by a single one-dimensional reaction-diffusion equation

$$\nu \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial \psi}{\partial x} - f(\psi, \beta, k_1(x), \dots k_n(x)) . \qquad (1)$$

Here $f(\psi)$ is an N-type function (fig. 1), $\nu(\psi) > 0$, $D(\psi) > 0$, β is the parameter defining the influence of external conditions on the system, $k_1(x), \dots k_n(x)$ are the intrinsic parameters depending on x in the inhomogeneous medium. For k_i = const. the values ψ_1, ψ_3 of the variable ψ correspond to two stable states ("phases") of the system $[f(\psi_{1,3}) = 0]$.

Eq. (1) is widely applied in different fields. It describes for example: the selforganization processes [1], nonequilibrium phase transitions in chemical systems [2,3], dissipative structures formed by Joule heating [4], propagation of an interface in first-order phase transitions [5,6], nonlinear waves in nonequilibrium superconductors [7], electric domains in semiconductors [8], optic gas discharge [9], nerve impulses propagation [10] etc.

In the homogeneous medium (k_i = const.) the interface is described by the solution of eq. (1): $\psi = \psi(x - vt)$ with $\psi(-\infty) = \psi_3$, $\psi(+\infty) = \psi_1$, corresponding to the kink-type nonlinear wave. The velocity v of the wave is constant and determined by the parameter β (β may contain for example: temperature, intensity of illumination, electric current, etc.).

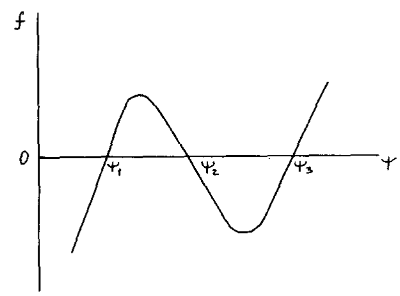

Fig. 1. The dependence of f on ψ.

Let us consider the dynamics of an interface in a randomly inhomogeneous medium. It is an important problem since all real systems are inhomogeneous to some extent. If the propagation velocity v of the wave is sufficiently large then small fluctuations of the parameters of the medium lead to diffusive-type motion of the interface in the coordinate system moving with its average velocity [11]. In this letter we show that at small v the dynamics of the interface change drastically due to the localization in the random potential formed by the inhomogeneities. (For isolated inhomogeneities a similar localization has been considered in ref. [4].)

Let us derive an equation describing slow motion of the interface in a weakly inhomogeneous medium, when $|\delta k_i(x)| \equiv |k_i(x) - \langle k_i \rangle| \ll |\langle k_i \rangle|$. [The symbol $\langle\ \rangle$ denotes averaging over all realizations of the random process $k_i(x)$.] We take $\psi(x, t)$ in the form

$$\psi(x, t) = \psi_0(x - u(t)) + \delta\psi(x, u(t)) , \qquad (2)$$

where $\psi_0(x)$ is the solution of eq. (1) corresponding to the static ($v = 0$) wave in the homogeneous medium, $u(t)$ is the coordinate of the wave front ($\psi(u(t), t) = \psi_2$). The function $\delta\psi(x, u(t))$ describes the change of the wave profile, which is due to the motion and interaction with inhomogeneities.

As it is shown below the case of small velocities $v(\beta)$ is most essential for weakly inhomogeneous media ($|\delta k_i| \ll |\langle k_i\rangle|$). The corresponding region of the parameter β is $|\delta\beta| \ll |\beta_p|$, where $\delta\beta = \beta - \beta_p$, $v(\beta_p) = 0$, $v(\beta)$ is the velocity of the interface in the homogeneous medium. The value of β_p is determined by the equation $S(\psi_3, \beta_p) = 0$ (see, for example, refs. [4,8,9]), where

$$S(\psi, \beta) = \int_{\psi_1}^{\psi} D(\psi')f(\psi', \beta, \langle k_1\rangle, ..., \langle k_n\rangle)\, d\psi' . \qquad (3)$$

The conditions $|\delta\beta| \ll |\beta_p|$ and $|\delta k_i| \ll |\langle k_i\rangle|$ lead to the inequality $\delta\psi \ll \psi_0$, which allows one to write the equation for $\delta\psi$ in form

$$\frac{\partial^2}{\partial x^2}(D\delta\psi) - \left(\frac{\partial f}{\partial\psi}\right)_{\psi_0}\delta\psi = -v\frac{du}{dt}\frac{\partial\psi_0}{\partial x} + \left(\frac{\partial f}{\partial\beta}\right)_{\beta_p}\delta\beta + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial k_i}\right)_{\langle k_i\rangle}\delta k_i(x) , \qquad (4)$$

retaining only the first-order terms in the small parameters $\delta\beta$, δk_i, du/dt. Inserting $\delta\psi = \phi\partial\psi_0/\partial x$ into (4) we obtain

$$\frac{\partial}{\partial x}\left[\left(D\frac{\partial\psi_0}{\partial x}\right)^2\frac{\partial\phi}{\partial x}\right] = \left[-v\frac{du}{dt}\frac{\partial\psi_0}{\partial x} + \left(\frac{\partial f}{\partial\beta}\right)_{\beta_p}\delta\beta + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial k_i}\right)_{\langle k_i\rangle}\delta k_i\right] D\frac{\partial\psi_0}{\partial x} . \qquad (5)$$

To find the equation defining the dynamics of the interface we integrate eq. (5) over x from $-\infty$ to $+\infty$. After that the left-hand side of the obtained equation is equal to zero and consequently we get

$$du/dt = v(\beta) - F(u) , \qquad (6)$$

$$v(\beta) = -\delta\beta\left(\frac{\partial S}{\partial\beta}\right)_{\beta_p,\psi_3}\Bigg/\left(\sqrt{2}\int_{\psi_1}^{\psi_3} vS^{1/2}\, d\psi\right)_{\beta_p} , \qquad (7)$$

$$F(u) = \sum_{i=1}^{n}\int_{-\infty}^{+\infty} G_i(x-u)\delta k_i(x)\, dx , \qquad (8)$$

$$G_i(x) = D(\psi_0) \times \frac{\partial\psi_0}{\partial x}\left(\frac{\partial f}{\partial k_i}\right)_{\psi_0,\beta_p,\langle k_i\rangle}\Bigg/\left(\sqrt{2}\int_{\psi_1}^{\psi_3} vS^{1/2}\, d\psi\right)_{\beta_p} . \qquad (9)$$

In eqs. (7)–(9) we have taken into account that $(D\partial\psi_0/\partial x)^2 = 2S(\beta_p, \psi_0)$. It is eq. (6) that describes the slow nonlinear dynamics of the interface in an arbitrary weakly inhomogeneous medium.

For $|v| \gg |F(u)|$ one may treat $F(u)$ in eq. (6) as a perturbation and set $F = F(vt)$. Then after averaging of (6) one obtains that $\langle u^2(t)\rangle = \tilde{D}t$ in the coordinate system moving with the velocity v, where the diffusion coefficient is given by [11]

$$\tilde{D} = \lim_{t\to\infty}\frac{1}{t}\int_0^t d\tau_1\int_0^t d\tau_2\,\langle F(v\tau_1)F(v\tau_2)\rangle .$$

In the opposite case $|F(u)| \sim |v|$ the dynamics of the interface changes drastically. Let us consider, for simplicity, the situation corresponding to $v > 0$. Then it follows from eq. (6) that the presence of regions with $F(u) > v$ leads to the localization of the interface on their boundaries where $F(u) = v(\beta)$ and $\partial F/\partial u > 0$. Thus the wave propagates only within the regions where $F < v(\beta)$. The mean length of such regions l_+ is the mean free-path length of the interface. The mean distance l between the neighbour localized positions of the wave is equal to $l_+ + l_-$, where l_- is the mean length of regions with $F(u) > v$.

To obtain the values of l_+, l_-, l we consider a part of the system with the length L. Let L_+ and L_- be the total lengths of the regions with $F < v$ and $F > v$ respectively and N be the number of solutions of the equation $F(u) = v(\beta)$ within this part. Then l, l_+, l_- are given by

$$l = 2\lim_{L\to\infty} L/\langle N\rangle_F , \quad l_\pm = 2\lim_{L\to\infty}\langle L_\pm\rangle_F/\langle N\rangle_F , \qquad (10)$$

where the symbol $\langle\ \rangle_F$ denotes averaging over realizations of $F(u)$. The expressions for L_+ and N can be written in the form

$$\langle L_+\rangle = \int_0^L \langle\theta(v - F(u))\rangle_F\, du ,$$

$$\langle N\rangle = \int_0^L \langle\delta(v - F(u))|\Phi|\rangle_F\, du , \qquad (11)$$

where $\Phi \equiv \partial F/\partial u$, $\theta(x)$ is the step function. After averaging of (11) one obtains

$$l = 2\left(w(v) \int_{-\infty}^{\infty} |\Phi| w_1(\Phi)\, \mathrm{d}\Phi \right)^{-1} ,$$

$$l_{\pm} = \left(2 \int_{-\infty}^{v} w(F)\, \mathrm{d}F\right) \left(w(v) \int_{-\infty}^{\infty} |\Phi| w_1(\Phi)\, \mathrm{d}\Phi\right)^{-1} , \tag{12}$$

where $w(F)$ and $w_1(\Phi)$ are the distribution functions of the random variables F and Φ respectively. If the parameters $k_i(x)$ are distributed by the Gauss law then it follows from (12) that

$$l = 2\pi(\sigma_F/\sigma_\Phi) \exp[v^2(\beta)/2\sigma_F^2] , \tag{13}$$

$$l_{\pm} = \pi(\sigma_F/\sigma_\Phi)[1 \pm \mathrm{erf}(v(\beta)/\sigma_F\sqrt{2})] \times \exp[v^2(\beta)/2\sigma_F^2] , \tag{14}$$

where $\sigma_F^2 = \langle F^2 \rangle$, $\sigma_\Phi^2 = \langle \Phi^2 \rangle$. From (8) we find

$$\sigma_F^2 = \sum_{i,l=1}^{n} \int_{-\infty}^{\infty} G_i^*(\omega) G_l(\omega) B_{il}(\omega) \frac{\mathrm{d}\omega}{2\pi} , \tag{15}$$

$$\sigma_\Phi^2 = \sum_{i,l=1}^{n} \int_{-\infty}^{\infty} G_i^*(\omega) G_l(\omega) B_{il}(\omega) \omega^2 \frac{\mathrm{d}\omega}{2\pi} , \tag{16}$$

where $B_{il}(\omega)$ and $G_i(\omega)$ are the Fourier transforms of the functions $B_{il}(\tau) = \langle k_i(x+\tau) k_l(x) \rangle$ and $G_i(\tau)$ respectively.

In the case when $v(\beta) \gtrsim \sigma_F$ the free-path length l_+ increases exponentially due to the fast decreasing of the interface localization probability. Nevertheless if $L > l_+$ then the localization occurs even for $v^2 \gg \sigma_F^2$. Hence, strictly speaking, the results of ref. [11] are valid only for $L < l_+$. If $L < l_+$, then one may get also a correction to the average velocity of the interface $\langle v \rangle$. Setting $u(t) = \langle v \rangle t + \delta u(t)$ and expanding (6) up to terms $\sim \delta u^2$ one obtains after averaging

$$\langle v \rangle = v(\beta) - \sigma_F^2/v(\beta) , \quad (v^2(\beta) \gg \sigma_F^2) . \tag{17}$$

The range of the parameters $v(\beta) \lesssim \sigma_F$ corresponds to a regime of strong localization. Here two cases are possible depending on the relation between the width of the wave front l_w and the correlation radius r_c of $k_i(x)$. From (13)–(16) one may obtain that for $l_\mathrm{w} \gtrsim r_\mathrm{c}$ the mean free-path length $l_+ \sim l \lesssim l_\mathrm{w}$ and in the opposite case $l_\mathrm{w} \ll r_\mathrm{c}$ respectively $l_+ \sim l \sim \sqrt{l_\mathrm{w} r_\mathrm{c}} \gg l_\mathrm{w}$. Thus propagation of the interface is impossible for $v^2 \lesssim \sigma_F^2$ in spite of an average homogeneity of the medium ($\langle F(u) \rangle = 0$). This may lead to the existence of metastable stochastic structures which consist of alternating domains of two phases $\psi = \psi_1$ and $\psi = \psi_3$. The characteristic minimal lengths of the domains are l_- and l_+ respectively. The boundaries of the domains are the localized interfaces considered above. Obviously an infinite number of such structures may arise in an infinite medium. Hence the specific distribution $\psi(x)$ in each concrete case is defined by the prehistory of the system and by the character of the external perturbations.

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