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Thermomagnetic effects, stability and oscillations in the critical state in hard superconductors

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Abstract. The influence of thermomagnetic (Ettingshausen and Nernst) effects on the stability and dynamics of the critical state in hard superconductors has been considered. Both the criteria for stability against flux jumps and the nature of a developing instability have been investigated in some simple cases. It is shown that thermomagnetic effects lead to essential changes in the dynamics of the critical state. The extent of the regime within which thermomagnetic effects lead to oscillations of the temperature and electric field during a flux jump has been investigated in terms of characteristic parameters that depend on material properties and temperature.

1. Introduction

It is well known that the critical state instability in hard superconductors occurs due to the avalanche-type increase of the electric field $\boldsymbol{E}$ and the temperature δT fluctuations. This leads to total or partial penetration of the external magnetic field into the sample (flux jump). The corresponding stability criteria have been obtained from a linear analysis of the thermal diffusion and Maxwell equations. It has been assumed that the damping of both δT and $\boldsymbol{E}$ fluctuations depends on the normal currents in the resistive state, the heat capacity ν and the thermal conductivity κ (see for example Wilson *et al* 1970, Mints and Rakhmanov 1977).

In this paper the influence of the Ettingshausen and Nernst transverse thermomagnetic effects on the stability and dynamics of the critical state is considered. These effects are responsible for terms proportional to ∇T and $\boldsymbol{E}$ in the expressions for the electric current density $\boldsymbol{j}$ and the heat flux $\boldsymbol{q}$, respectively. These terms may be important either due to the large gradients of δT generated by the flux jump

$$(|\nabla \delta T| \sim 10^3\text{–}10^4 \ \text{deg cm}^{-1})$$

or the relatively small values of κ in hard superconductors.

We consider the conditions below the critical state stability for: (i) a plate in a constant magnetic field, (ii) a wire with the transport current in a transverse magnetic field, (iii) a tube in a time-dependent magnetic field parallel to the axis of the tube. The range of parameters has been determined where oscillations of δT and $\boldsymbol{E}$ are possible during a flux jump (Mints 1978).

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2. The basic equations

Let us consider the time evolution of E and $\delta T = T - T_0$ perturbations in the critical state in the linear approximation (T_0 is the initial temperature). The expressions for $\boldsymbol{j}$ and $\boldsymbol{q}$ may be written in the form (see for example Campbell and Evetts 1972)

$$\boldsymbol{j} = j_c \boldsymbol{e} + \sigma \boldsymbol{E} + s(\boldsymbol{n} \times \nabla T) \tag{1}$$

$$\boldsymbol{q} = -\kappa \nabla T + sT(\boldsymbol{E} \times \boldsymbol{n}) \tag{2}$$

where j_c is the critical current density, $\boldsymbol{e} = \boldsymbol{E}/E$, $\boldsymbol{n} = \boldsymbol{B}/B$, $\boldsymbol{B}$ is the magnetic induction, σ is the conductivity in the flux flow regime, $s = s^* c/\phi$, s^* is the transport entropy per unit vortex length (Campbell and Evetts 1972), c is the velocity of light and ϕ is the flux quantum. The last terms in equations (1) and (2) describe the Ettingshausen and Nernst effects respectively.

To obtain a closed set of equations for T and $\boldsymbol{E}$, we use Maxwell equations and the entropy density S balance equation (as usual $\boldsymbol{H} = \boldsymbol{B}$, where $\boldsymbol{H}$ is the magnetic field).

$$\text{curl curl } \boldsymbol{E} = -\frac{4\pi}{c^2} \frac{\partial \boldsymbol{j}}{\partial t} \tag{3}$$

$$T \frac{\partial S}{\partial t} = -\text{div } \boldsymbol{q} + \boldsymbol{j} \cdot \boldsymbol{E}. \tag{4}$$

Since s^* is conditioned by the electron excitations localised at the core of the fluxoid (Caroli *et al* 1964) the transport entropy of the fluxoid lattice is additive until the fluxoid cores begin to overlap ($B \simeq H_{c_2}$), that is, $S = S_0 + n_f s^*$; $n_f = B/\phi$ is the fluxoid density, H_{c_2} is the upper critical field and S_0 is the entropy density excluding the fluxoid transport entropy. Let us define the dimensionless coordinates as $\boldsymbol{r}_1 = \boldsymbol{r}/L$, time $t_1 = t\kappa/\nu L^2$, temperature $\theta = (T - T_0)/T_0$, electric field $\boldsymbol{\epsilon} = j_c(T_0) L^2 \boldsymbol{E}/T_0$. $\boldsymbol{r}$ and t denote the dimensional coordinates and time, L is the characteristic length (for example $L = b$ for the plate in figure 1, $L = R$ for the wire of radius R). For simplicity it will be assumed that $j_c(B, T) = j_c(T)$ (the Bean model). Thus for j_c one finds $j_c(\theta) = j_c(T_0) + T_0 j_t \theta$, where $j_t = |\partial j_c/\partial T|_{T_0}$. We attempt the solution of the linearised equations (3), (4) in the form: $\boldsymbol{\epsilon}(\boldsymbol{r}_1, t_1) = \exp(\lambda t_1)\, \boldsymbol{\epsilon}(\boldsymbol{r}_1)$, $\theta(\boldsymbol{r}_1, t_1) = \exp(\lambda t_1)\, \theta(\boldsymbol{r}_1)$. The dimensionless parameter λ is an eigenvalue to be defined. The critical state is unstable when $\lambda > 0$. The electric field has only

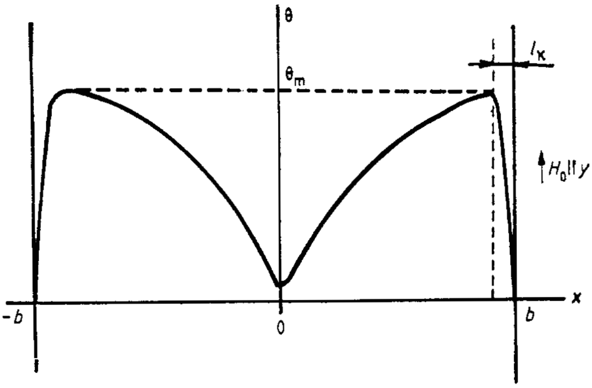

Figure 1. The distribution of temperature in the plate.

the z-component ($\varepsilon_z = \varepsilon$) in the cases of planar and cylindrical geometry. Therefore using equations (1)–(4), one obtains

$$\nabla^2\varepsilon - \lambda\tau\varepsilon = -\lambda\left[\beta\theta + 2\mu\left(n_y\frac{\partial\theta}{\partial x_1} - n_x\frac{\partial\theta}{\partial y_1}\right)\right] \tag{5}$$

$$\lambda\theta = \nabla^2\theta + \left[1 + \alpha\left(\frac{\partial n_y}{\partial x_1} - \frac{\partial n_x}{\partial y_1}\right)\right]\varepsilon \tag{6}$$

where

$$\beta = \frac{4\pi j_c j_t L^2}{j_c^2} \qquad \mu = \frac{2\pi j_c sL}{\nu c^2} \qquad \alpha = \frac{sT}{j_c L} \tag{7}$$

and τ is the ratio of the temperature and the magnetic diffusivities

$$\tau = \frac{4\pi\sigma\kappa}{\nu c^2}. \tag{8}$$

In equations (5) and (6) the vector $\boldsymbol{n} = \boldsymbol{B}/B$ is determined by the initial induction distribution in the superconductor ($\theta = \varepsilon = 0$). In deriving equation (6) we have used the Maxwell equation

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\operatorname{curl} \boldsymbol{E}.$$

We have neglected the dependence of σ, κ, ν, s on x and y because they change slowly in the range of the magnetic inductions $B - H_0 \lesssim \Delta B \sim 4\pi j_c L/c$ (H_0 is the applied magnetic field).†

If the external magnetic field penetration depth $l_0 = cH_0/4\pi j_c$ satisfies the inequality $l_0 > b$ the boundary conditions for equations (5) and (6) in the case of the plate (figure 1) are:

$$\varepsilon'(1) = 0 \qquad W\theta(1) = -\theta'(1) - \alpha\varepsilon(1) \tag{9}$$

$$\varepsilon(0) = 0 \qquad \theta'(0) = 0 \tag{10}$$

where $W = W_0 b/\kappa$, W_0 is the coefficient of heat transfer from the surface and the prime denotes the differentiation with respect to x_1 ($0 \leqslant x_1 \leqslant 1$). The general form of the dispersion curves $\lambda = \lambda(b)$ may be obtained by means of a computer (see figure 2). The critical state is unstable in the plate when $b > b_c$ (or $\beta > \beta_c$).

The last terms in equations (1) and (2) correspond to two effects. The first is associated with the interaction between the fluxoids and the temperature gradients. Indeed, when the fluctuation δT occurs in the superconductor, it generates the force $\boldsymbol{f} = -\phi j_t \times [\boldsymbol{e} \times \boldsymbol{n}]\,\delta T/c - s^*\nabla\delta T$, which acts on the fluxoid. The second term in the equation for $\boldsymbol{f}$ describes the thermoelastic stress arising in the fluxoid lattice. Thus the details of the distribution of δT (depending on the external cooling) influence the stability. In some cases the temperature gradient $\nabla\delta T$ may prevent penetration of the fluxoids into the superconductor. This effect increases the critical state stability and this increase of the stability may be essential under certain conditions (see below).

The second term in the right-hand side of equation (2) describes the heat flux $\boldsymbol{q}_1$ transported by the fluxoids. $\boldsymbol{q}_1$ may be written as $\boldsymbol{q}_1 = s^* T_0 n_f \boldsymbol{v}$, where $\boldsymbol{v} = c[\boldsymbol{E} \times \boldsymbol{B}]/B^2$.

† For $\nu(B)$, $\kappa(B)$ this is true when $\Delta B/H_{c_2} \ll 1$. For $\sigma \propto B^{-1}$ (see for example Gor'kov and Kopnin 1975) one obtains the criterion $\Delta B/H_0 \ll 1$. The dependence of s on B may be neglected if $H_0/H_{c_2} \ll 1$.

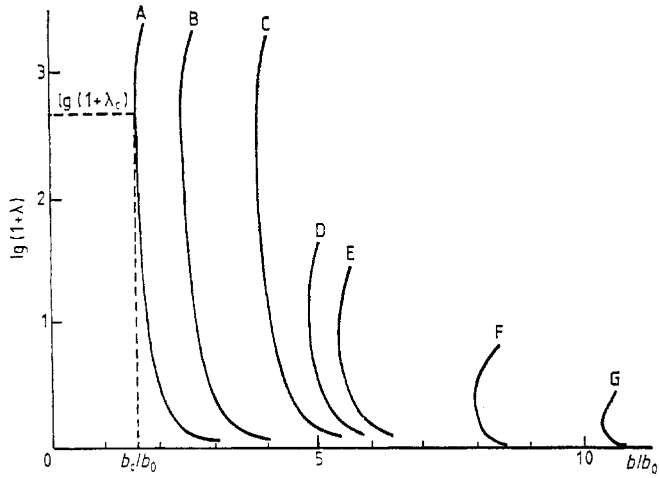

Figure 2. Dispersion curves for the plate at $W=10^{10}$ and $\mu_0=$(A) 0, (B) 0.6, (C) 0.8, (D) 0.9, (E) 1, (F) 1.5, and (G) 2.

The heat flux $\boldsymbol{q}_1$ causes the additional heat release Q_1 (besides $j_c E$) in equation (6) due to the variation in length of the flux lines during the motion. Indeed, consider a wire with the current ($\boldsymbol{j}\|z$) where the flux lines annihilate at the wire axis. Then the heat $2\pi r Q_1\, \mathrm{d}r = s^*T\, \mathrm{d}l$ will be released per unit time in a volume $\mathrm{d}V = 2\pi r\, \mathrm{d}r$ (the dimension along the z axis is equal to unity and $\mathrm{d}l = n_f v 2\pi\, \mathrm{d}r$ is the total length variation of all flux lines passing through the volume $\mathrm{d}V$ per unit time). Hence $Q_1 = sTE/r$ is nothing other than Q_1 from equation (6) written for cylindrical coordinates. In the planar case, the flux lines are rectilinear and $Q_1=0$. Then the effects associated with heat transfer by the fluxoids manifest themselves only on the superconductor–coolant interface, because the heat s^*T (per unit length of the fluxoid) is absorbed when the fluxoid penetrates into the superconductor.

3. The plate in the magnetic field

Let us consider the influence of the thermomagnetic effects on the stability of the critical state for the case of the plate (figure 1) in a constant magnetic field $H_0\|y$. In this case equations (5) and (6) can be solved easily and the dispersion equation $\lambda=\lambda(b)$ may be found in the general form. However, this equation is very cumbersome and we shall obtain the stability criterion using a more simple and obvious method.

For hard superconductors $\tau \ll 1$ and one can use the adiabatic approximation $\lambda_c \to \infty$, $\lambda_c\tau \to 0$. In this case we may neglect the diffusion of heat (Mints and Rakhmanov 1977). Then using equations (5) and (6) one finds $\varepsilon=\lambda\theta$ and the equation for ε may be written in the form:

$$\varepsilon'' + 2\mu\varepsilon' + \beta\varepsilon = 0. \tag{11}$$

The boundary condition for this equation at $x_1=1$ is not $\varepsilon'(1)=0$ (see equation (9)) since the heat diffusion is important near the surface ($x_1 > x_0 = 1 - l_\kappa$, where $l_\kappa \sim \lambda^{-1/2}$ is the heat length) due to external cooling. To obtain the boundary condition for equation (11) we integrate equations (5) and (6) with respect to x_1 from x_0 to 1.

$$\varepsilon'(x_0)=\lambda\left\{\beta\int_{x_0}^{1}\theta\,\mathrm{d}x_1+2\mu[\theta(1)-\theta(x_0)]+\tau\int_{x_0}^{1}\varepsilon\,\mathrm{d}x_1\right\} \tag{12}$$

$$\lambda\int_{x_0}^{1}\theta\,\mathrm{d}x_1=\theta'(1)-\theta'(x_0)+\int_{x_0}^{1}\varepsilon\,\mathrm{d}x_1. \tag{13}$$

$\theta(1)$ can be found from the thermal boundary condition (9). Using equations (9), (12) and (13) we obtain the boundary condition for equation (11) (in the final expressions $x_0=1$). For isothermal cooling ($W\to\infty$, $\theta(1)\to 0$):

$$\varepsilon'(1)=-2\mu\varepsilon(1). \tag{14}$$

In this case the stability criterion is given by ($W\to\infty$, $\lambda\gg\beta$):

$$b<b_0[\pi-\tan^{-1}(\mu_0^{-2}-1)^{1/2}]/(1-\mu_0^2)^{1/2} \tag{15}$$

where $b_0^2=\nu c^2/4\pi j_c j_t$ and $\mu_0^2=\mu^2/\beta$.

The boundary condition for equation (11) under adiabatic cooling ($W=0$) at $\alpha\ll 1$, $\mu\ll 1$ is:

$$\varepsilon'(1)=-\alpha\beta\varepsilon(1). \tag{16}$$

The stability criterion may be written as†

$$b<b_0(\pi/2+\alpha_0-\mu_0) \tag{17}$$

where $\alpha_0=sT/j_c b_0$. The stability criteria (15) and (17) are found to the lowest order in $\lambda_c^{-1/2}$.

Thus the influence of the thermomagnetic effects on the stability of the critical state depends on the dimensionless parameters α_0 and μ_0 connected with Nernst and Ettingshausen effects respectively. The temperature dependence and the order of magnitudes of α_0 and μ_0 may be obtained assuming, for instance, that $\nu=\nu_0T^3/T_c^3$, $j_c=j_0[1-(T/T_c)]$, $s=s_0T[1-(T/T_c)]/T_c$ (T_c is the superconducting transition temperature). One finds

$$\left.\begin{aligned}\alpha_0&=2a\left[\frac{T}{T_c}\left(1-\frac{T}{T_c}\right)\right]^{1/2}\\ \mu_0^2&=a^2\frac{T_c}{T}\left(1-\frac{T}{T_c}\right)^3\\ a^2&=\frac{\pi T_c s_0^2}{\nu_0 c^2}\end{aligned}\right\}. \tag{18}$$

The thermomagnetic effects become essential at low temperatures when $\mu_0\sim 1$ or $T/T_c\gtrsim a^2$. The estimation of a is possible for alloys with $H_c\sim 10^3$–10^4 Oe because the experimental data for $s(T, B)$ are known for such materials (Fiory and Serin 1966, Otter and Solomon 1966, 1967, Lowell *et al* 1969). (When $B\to 0$ $s_0\simeq 1.2\times 10^{11}$ Oe cm s^{-1} K^{-1}.) If $T_c=10$ K, $\nu\sim 10^4$–10^5 erg cm^{-3} K^{-1} one obtains $a\simeq 0.2$–0.07 or $\mu_0\lesssim 1$ when $T\gtrsim 5\times(10^{-2}$–$10^{-3})T_c$. Thus for the temperatures $T\gtrsim 1$ K we have α_0, $\mu_0\ll 1$.

Let us consider the situation qualitatively. When $W\to\infty$ the large temperature gradient $\theta'(1)\sim-\theta_m\lambda^{1/2}$ (the 'heat barrier') arises near the surface (figure 1). It leads to an increase of the surface current $I_s\sim -sT_0\theta_m$ which prevents the penetration of the

† In equations (15) and (17) we have neglected the small terms $\tau^{1/3}$ and $\tau^{1/2}$ respectively (Mints and Rakhmanov 1977).

fluxoids into the superconductor. Meanwhile there are positive bulk gradients of θ (figure 1) that decrease the stability. When $W \rightarrow \infty$ the competition between these effects leads to an increase in stability. When $W \rightarrow 0$ the 'heat barrier' arises because of cooling of the surface by the Nernst effect. In this case the thermomagnetic effects may increase or decrease the stability depending on the ratio of α_0 (the 'heat barrier') and μ_0 (the bulk gradients of θ) (see equation (17)).

Let us consider the dynamics of the critical state in the plate. The value of λ_c (see figure 2) is determined by the relaxation processes connected with the heat conductivity and the normal currents which have been neglected above. When $\lambda \gg \beta$, α, $\mu \ll 1$ the dispersion equation may be written in the form (Gurevich and Mints 1979):

$$\tan p = -p \frac{W(\lambda^{1/2} - \mu) + \lambda + \lambda^{1/2}(2\mu - \alpha\beta)}{W(\mu\lambda^{1/2} + \beta) + 2\alpha\mu\lambda^{3/2} + \lambda(\alpha\beta - \mu)} \tag{19}$$

where $p^2 = \beta - \lambda\tau - \beta^2/\lambda$. The equation for λ_c may be obtained from equation (19) by differentiation with respect to λ under the condition $\partial b/\partial\lambda = 0$. When $W \gg \lambda_c^{1/2}$, one finds

$$\beta W + (2\mu - \alpha\beta)\lambda_c - (W\tau + 4\alpha\mu)\lambda_c^{3/2} = 0. \tag{20}$$

Hence λ_c is determined by the thermomagnetic effects when $W \ll W_c$ where

$$W_c \sim \alpha_0\mu_0/\tau. \tag{21}$$

From the above estimates for α_0 and μ_0 (see equation (18)) we find $W_c \sim 10$–10^2 which by the order of magnitude corresponds to the cooling by liquid helium. If $W \gg W_c$, λ_c does not depend on α_0 and μ_0 up to $\mu_0 \gtrsim 1$ (figure 2). When $W \ll 1$, one obtains from equation (19)

$$\lambda_c = \tfrac{1}{4}\pi^2\tau^{-1/2} \qquad \alpha_0\mu_0 \ll \tau^{3/4} \tag{22}$$

$$\lambda_c = (\tfrac{1}{2}\pi)^{8/3}(2\alpha_0\mu_0)^{-2/3} \qquad \alpha_0\mu_0 \gg \tau^{3/4}. \tag{23}$$

Thus λ_c is defined by the normal currents, the heat conductivity and the thermomagnetic effects. The influence of the thermomagnetic effects on λ_c is essential due to the small values of τ in hard superconductors.

4. Cylindrical geometry

In this section we shall consider: (i) a wire with radius R and with transport current $I = \pi R^2 j_c$ in a constant magnetic field $H_\perp \perp z$ at $W \gg 1$; (ii) a tube in the time-dependent magnetic field $H_\parallel(t) = \dot{h}_0 t + h_1 \sin \omega t$ parallel to the axis of the tube (z axis). In these cases $\beta_c \ll 1$ when $s = 0$ and the thermomagnetic effects influence both the stability and the dynamics of the critical state. In this section we shall use the following parameters:

$$\xi = \mu_0 - \alpha_0/2 \qquad \eta = 2\alpha_0\mu_0. \tag{24}$$

4.1. Wire with transport current

The general solution of equations (5) and (6) cannot be obtained for arbitrary λ with cylindrical coordinates. However it will be shown below that the perturbations with $\lambda \ll 1$ are the most 'dangerous' in some cases. Therefore we may write the solution of equations (5) and (6) in the form of a power series in $\lambda \ll 1$. Analysis of this solution shows that two kinds of dispersion curves are possible. They are illustrated in figure 3

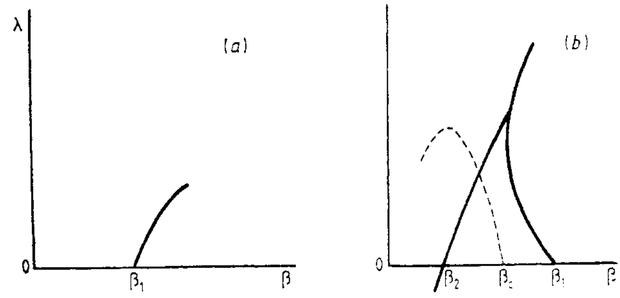

Figure 3. Dispersion curves for (*a*) the wire or (*b*) the tube. Full curves, Re λ; broken curve, Im λ.

schematically. The full and the broken curves show Re λ and Im λ respectively. If $\beta > \beta_2$ (figure 3) Re $\lambda > 0$, so that the evolution of the instability is accompanied by oscillations of θ and ε when $\beta_2 < \beta < \beta_c$. (The oscillations in the critical state when $s=0$ have been considered by Mints (1978), Maksimov and Mints (1979).) The oscillation frequencies will be found in the next section and here we shall investigate case (*a*) in figure 3. This case is realised when $\partial\lambda/\partial\beta|_{\beta_1} > 0$. Neglecting the normal currents ($\tau \ll \max(\mu_0{}^2, \eta)$) one obtains $\partial\lambda/\partial\beta|_{\beta_1} > 0$ if $\xi < 0$.

Let us find the magnitude of β_1. Equation (5) implies that the electric field is uniform $\varepsilon(r_1, \varphi) = \varepsilon_0 + 0(\lambda)$ when $\lambda \to 0$ (r_1 and φ are the polar coordinates). In this case the criterion of the critical state instability may be formulated as the condition that the I–E characteristic is negative when $\varepsilon_0 \to 0$ ($\partial I/\partial\varepsilon_0 < 0$). The full current I depends on the temperature distribution $\theta(r_1, \varphi)$ which satisfies the static thermal diffusion equation with the boundary condition $\theta(1, \varphi) = 0$.

$$\nabla^2\theta + [1 + \alpha\rho(1 + \rho^2 r_1{}^2 - 2\rho r_1 \sin\varphi)^{-1/2}]\,\varepsilon_0 = 0 \tag{25}$$

where $\rho = \Delta B/H_\perp$, $\Delta B = 2\pi j_c R/c$ and r_1 is the dimensionless coordinate ($0 \leqslant r_1 \leqslant 1$). The second term in the brackets in equation (25) arises due to the flux line length variations during the motion. The solution of equation (25) may be found in the general form. We shall consider here two simple limits: $\rho \ll 1$ (the self-magnetic field of the current is much smaller that $H_\perp$) and $\rho \gg 1$. In the former case the induction B is almost uniform throughout the wire cross-section ($H_\perp \| x$) and the magnetic field of the current induces a weak curving of the flux lines. Calculating $I(\varepsilon_0)$ up to terms of order $\rho \ll 1$, one finds:

$$\beta_1 = 8\tau(1 + 2\rho_0\xi) \tag{26}$$

where $\rho_0 = \Delta B_0/H_\perp$, $\Delta B_0 = 2\pi j_c b_0/c$. All the thermomagnetic parameters appearing in equation (26) are multiplied by a small value ρ_0. This is due to the fact that the thermomagnetic effects do not influence β_1 when $\rho \to 0$ because in this case the distribution of B is symmetrical and all the flux lines are parallel to the x axis. For fields $H_\perp \sim 10$–10^2 kOe and $\nu \sim 10^4$ erg cm^{-3} K^{-1} one can find $\rho_0 \sim (\pi\nu T_c)^{1/2}/H_\perp \sim 10^{-1}$–$10^{-2}$. Therefore if $\rho \ll 1$ the thermomagnetic effects influence the stability only weakly even at α_0, $\mu_0 \sim 1$.

In the limit of $\rho \gg 1$ the induction dependence of $\sigma(B) = \sigma_H H_{c_2}/B$ (see for example Gor'kov and Kopnin 1975) is important because $B(0, \varphi) \sim H_\perp \ll \Delta B$. If $\rho \to \infty$ equation (26) may be solved easily and the condition $\partial I/\partial\varepsilon_0 = 0$ leads to the equation for β_1:

$$\beta_1{}^{3/2} - (16/3)\,\xi\beta_1 - 8\eta\beta_1{}^{1/2} - 16\tau_1 = 0 \tag{27}$$

where $\tau_1=\tau\sigma_H H_{c_2}/\sigma\Delta B_0$ is the magnitude of $\tau(B)$ when $B=\Delta B_0$. If $s=0$, one finds

$$\beta_1=(16\tau_1)^{2/3}. \tag{28}$$

When $s\neq 0$ and $\mu_0{}^3\gg\tau_1$ the stability is controlled by the thermomagnetic effects only. The results of the numerical solution of equation (27) are shown in figure 4.

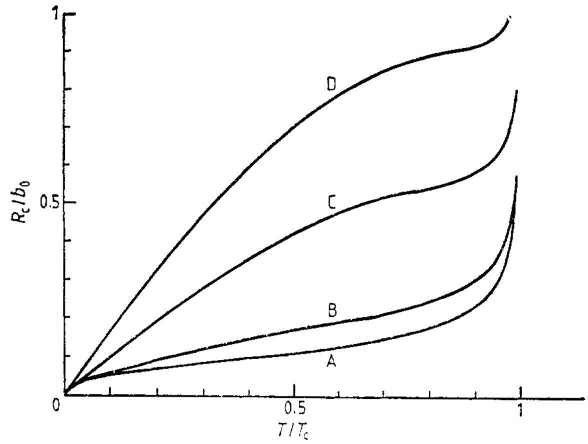

Figure 4. Temperature dependence of R_c for a wire with the transport current in a self-field for $a=$(A) 0, (B) 0.1, (C) 0.3, (D) 0.5.

4.2. Tube in a time-dependent magnetic field

Let us consider a tube, inside radius r_0 and outside radius r_0+2b. The applied magnetic field $H_\parallel=\dot{h}_0 t+h_1\sin\omega t$ is parallel to the axis of the tube and $l_0=cH_\parallel/4\pi j_c>2b$.

Let us denote the heat transfer coefficients on the tube interior and exterior as $W_0{}^+$ and $W_0{}^-$ respectively. If $W_0{}^+>W_0{}^-$ then $\nabla_r\theta<0$ and the thermomagnetic effects stabilise the critical state. If $W_0{}^->W_0{}^+$ then the situation is opposite.

The solution of equations (5) and (6) can be obtained in the planar limit $r_0\gg b$ (which is equivalent to the case of two parallel plates). The dispersion curves are shown qualitatively in figure 3.

First let us consider the region where $\lambda\ll 1$. One may easily find the equation for β_1 using the condition $\partial I/\partial\varepsilon_0=0$ or the direct expansion of ε and θ with respect to λ:

$$\beta_1{}^{3/2}W_+W_-+2\beta_1(W_++W_-)+3\beta_1{}^{1/2}[1-\tau W_+W_--\xi(W_+-W_-)]$$
$$-\tfrac{3}{4}(\tau+\eta)(W_++W_-)=0 \tag{29}$$

where $W_\pm=W_0{}^\pm b_0/\kappa$. When $W_+=W_-\gg 1$ the thermomagnetic effects do not influence β_1, ($\beta_1=3\tau$). If $W_+\neq W_-$ the magnitude of β_1 depends on the relation between W_+ and W_-. For example

$$\beta_1{}^{1/2}=\tfrac{3}{4}\{[\xi^2+\tfrac{4}{3}(\tau+\eta)]^{1/2}+\xi\} \qquad W_+\gg 1,\ W_-=0 \tag{30}$$

$$\beta_1{}^{1/2}=\tfrac{3}{4}\{[\xi^2+\tfrac{4}{3}(\tau+\eta)]^{1/2}-\xi\} \qquad W_-\gg 1,\ W_+=0. \tag{31}$$

The values of ξ and η at $T\sim 4$ K are of the order of 10^{-1} and 10^{-2} respectively. Therefore in hard superconductors with $\tau\gtrsim 10^{-3}$ the critical state stability is controlled mainly by the thermomagnetic effects. Next we shall treat the case $W_+\gg 1$, $W_-=0$ in more detail assuming that the magnetic flux enters the superconductor ($\dot{h}_0>0$), $\xi,\eta\ll 1$ and $\tau=0$.

Let us consider the complex solutions for $\lambda = \lambda(b)$. Calculating $b = b(\lambda)$ up to terms of the order of λ we find that $\partial\lambda/\partial\beta|_{\beta_1} < 0$ when $\mu_0 > 3.05\alpha_0$ (see figure 3). Thus the oscillations of θ and ε may occur at low temperatures $T \gtrsim T_c/6$. The analytical expressions for $\lambda_0 = \mathrm{Re}\,\lambda$ and $\omega_0 = \mathrm{Im}\,\lambda$ may be obtained in the limit $|\lambda_c| \gg 1$. In this case the dispersion equation can be expanded into a power series in $\omega_0^{-1/2}$ by analogy with the case of the plate (Gurevich and Mints 1979). The results have the form:

$$\lambda_0 = \beta[2(\beta^{1/2} - \xi)^2 - \eta]/\eta^2 \tag{32}$$

$$\omega_0 = 2\beta(\beta^{1/2} - \xi)[\eta - (\beta^{1/2} - \xi)^2]^{1/2}/\eta^2 \tag{33}$$

$$\beta_2^{1/2} = \xi + (\eta/2)^{1/2} \qquad \beta_c^{1/2} = \xi + \eta^{1/2} \qquad \mathrm{Re}\,\lambda^{1/2} \lesssim 1. \tag{34}$$

When $\beta = \beta_c$, one obtains

$$\lambda_c = \beta_c/\eta \simeq \mu_0/2\alpha_0 \simeq T_c/4T. \tag{35}$$

The critical state is unstable when $\beta > \beta_2$ ($\lambda_0 > 0$), so that at $\beta_2 < \beta < \beta_c$ the evolution of the instability is accompanied by the oscillations of θ and ε with the frequency ω_0. Note that we are considering the oscillations of θ and ε superimposed on the constant background electric field $E_0 = \dot{h}_0 r_0/2c$ arising due to the term $\dot{h}_0 t$ in $H_\parallel(t)$. It is necessary that $j_c E > 0$ during the instability where E is the total electric field in the superconductor. The condition of the applicability of equations (32)–(35) is $|\lambda_c| \gg 1$ or $T_c/4T \gg 1$. However the condition $\mu_0 \ll 1$ is not satisfied at low temperatures (see equation (18)). When $\mu_0 \sim 1$ we may put $\alpha_0 = \tau = 0$. It may be shown that the situation illustrated in figure 3 (*a*) is realised if $\mu_0 > 0.62$. Thus the oscillations of θ and ε occur when

$$3.05\,\alpha_0 < \mu_0 < 0.62. \tag{36}$$

Let us consider the response of the system to the external magnetic field $h_1 \sin \omega t$ when $\Delta\omega = \omega - \omega_0 \ll \omega_0$, $\lambda_0 < 0$, $|\lambda_0| \ll 1$.

Using the Laplace transformation of equations (5) and (6) we find that the steady-state solution of equations (5) and (6) near resonance has the form:

$$T = T_0 + \frac{j_c b^2 E_0}{2\kappa}(4 - x_1^2) + T_1 \{\cos(\omega t + \varphi) - \exp[(x_1 - 2)(\omega/2)^{1/2}]$$

$$\times \cos[\omega t + \varphi + (x_1 - 2)(\omega/2)^{1/2}]\} \tag{37}$$

$$E = E_0 + E_1 \cos(\omega t + \varphi) \tag{38}$$

$$E_1 = \frac{h_1 \kappa}{b c \nu} \frac{\omega_0^{3/2}\, 2^{1/2}}{\eta(\lambda_0^2 + \Delta\omega^2)^{1/2}} \qquad T_1 = \frac{j_c b^2}{\kappa} \frac{E_1}{\omega_0} \qquad \tan\varphi = \frac{|\lambda_0|}{\Delta\omega} \tag{39}$$

where $x_1 = (r - r_0)/b$ $(0 \leq x_1 \leq 2)$.

Taking $\nu = 1.5 \times 10^4$ erg cm^{-3} K^{-1}, $\kappa = 4 \times 10^3$ erg cm^{-3} K^{-1}, $\eta = 0.02$, $j_c = 1.5 \times 10^5$ A cm^{-2}, $T_c = 10$ K (the data for Nb–25% Zr alloy at 4 K) one obtains $b_c \simeq 2 \times 10^{-3}$ cm. The amplitude T_1 may be evaluated using equations (39). For $h_1 = 0.1$ Oe, $\omega_0 = 3$, $\lambda_0 = -0.1$ we have $T_1 \simeq 0.3$ K. The frequency of the oscillations is of the order of $\omega_0\kappa/\nu b_c^2 \sim 2 \times 10^5$ s^{-1}. Taking $r_0 = 3$ cm we find that the condition of the applicability of the above expressions ($E_0 > E_1$) yields the lower boundary of $\dot{h}_0 \simeq 30$ kOe s^{-1} that corresponds to $E_1 \simeq 5 \times 10^{-4}$ V cm^{-1}.

The terms for $W_- \gg 1$, $W_+ = 0$ (or for the case $W_+ \gg 1$, $W_- = 0$ and $\dot{h}_0 < 0$) can be obtained from the above equations by the substituting $\xi \to -\xi$.

5. Conclusions

We have examined the influence of thermomagnetic effects on the stability of the critical state for some simple systems. It has been shown that these effects influence the stability criterion weakly when $\beta_c \sim 1$. The exceptions may be: (i) the critical state stability at low temperatures ($T \gtrsim 1$ K) when $\mu_0 \sim 1$; (ii) systems with $\beta_c \ll 1$, for example a wire with a transport current $I = \pi R^2 j_c$, a tube and so on; (iii) alloys with high H_{c_2}. If $B \ll H_{c_2}$ then $s_0 \propto \ln (H_{c_2}/H_0)$ (Kopnin 1975). Therefore we expect an increase of a for alloys with $H_{c_2} \sim 10^5$–10^6 Oe. In this case the interval of T where $\mu_0 \gtrsim 1$ may increase to several degrees.

Note that for nonsymmetrical cooling $W_+ \neq W_-$ (§4.2) the stability of the critical state depends on the sign of $W_+ - W_-$. This effect gives a chance to separate the contribution of the thermomagnetic effects if the direction of the magnetic field is changed ($H \to -H$).

The thermomagnetic effects are essential when one considers the dynamics of the critical state. For example these effects lead to oscillations of θ and ε during the flux jump in the tube at low temperatures $T \gtrsim T_c/6$. It is important to note that the radical change of the dispersion curves $\lambda = \lambda(L)$ may arise already when α_0, $\mu_0 \ll 1$ because of the small values of τ in hard superconductors.

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