Flux jumps and oscillations in type-II superconductors

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Finite thermomagnetic instabilities are studied in the critical state of type-II superconductors under various heat input conditions. A thin plane-parallel plate in an external magnetic field parallel to the plate surface is considered. A study is made of how the contact of the superconductor with a normal metal influences the effect. Finally, we find the conditions under which a series of successive finite magnetic-flux jumps, i.e., oscillations of the electromagnetic field and temperature, can be observed in the specimen and we estimate their number and period.

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INTRODUCTION

Hard and composite type-II superconductors in the critical state are known to display characteristic instabilities, i.e., magnetic-flux jumps. Pertinent stability criteria have frequently been discussed in the literature (see, e.g., Refs. 1-5). In most papers on this subject a study was made of the conditions under which, notwithstanding the presence of pinning centers, there is an avalanchelike motion by the vortex structure, with an attendant high heat release and ending with the total penetration of the magnetic field into the specimen. When considered from the macroscopic point of view, this problem reduces to one of studying the character of the perturbations which develop in the electromagnetic field and temperature in a superconductor in the critical state. The stability criteria so obtained provide an extremely satisfactory description of the body of the pertinent experimental data.[5-7] A much less-investigated question is that of the conditions under which finite instabilities occur, leading to the partial penetration of magnetic flux into the specimen. Processes of this kind have been observed by many authors (see, e.g., Refs. 8-10).

In the present paper, in the linear approximation we find the range in which finite magnetic-flux jumps exist and we determine the conditions under which it is possible to observe a series of successive instabilities of this kind, i.e., oscillations of the electromagnetic field and temperature in a specimen.[1)] A plane-parallel plate is considered in an external magnetic field parallel to the surface of the specimen. A detailed study is made of the role of external cooling and the influence of the contact of a superconductor with a normal metal and an estimate is obtained for the maximum possible number of oscillations.

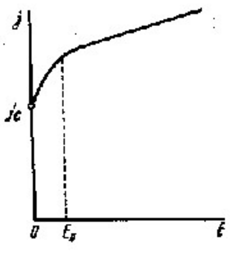

FIG. 1.

1. FORMULATION OF THE PROBLEM AND THE FUNDAMENTAL EQUATIONS

The development of perturbations of the temperature Θ and the electric (E) and magnetic (H) fields is described by the heat transfer equation and Maxwell's equations

$$\Delta \mathbf{E} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t};$$
$$\nu_s\dot{\Theta} = \varkappa_s \Delta\Theta + \mathbf{jE}, \qquad (1)$$

where ν_S and $\varkappa_S$ are the heat capacity and thermal conductivity, respectively, of the superconductor and j is the current density.

Let us consider the dependence of j on E, H, and Θ. Since we are interested in the development of fluctuations such that the vortex structure goes into motion (entry or exit of magnetic flux), the value of E is different from zero. In this case, j can be written as[6]

$$|\mathbf{j}| = j = j_c(T, H) + j_N(E), \qquad (2)$$

where j_c is the density of the critical current and j_N is the density of normal currents. As the equation of the critical state (the dependence of j_c on H, T) we choose the model given by Bean,[12] i.e., we set $\partial j_c/\partial H = 0$. This condition substantially simplifies calculations, while not qualitatively changing the results.[5] The characteristic dependence of j on E is plotted in Fig. 1. To begin with, let us consider the development of perturbations in which everywhere $E \geq E_0(T)$, where $E_0(T)$ is the limit of the

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linear segment on the curve $j_N = j_N(E)$. Then, $j_N = \sigma_f E$ (σ_f is the conductivity of the superconductor in the resistive mode). Expanding j_c in the small quantity Θ ($\Theta \ll T_c - T_0$, where T_c is the critical temperature, T_0 is the initial temperature of the specimen), we get the following expression for j:

$$j = j_c(T_0) + \sigma_f E - \left|\frac{\partial j_c}{\partial T}\right|\Theta. \tag{3}$$

Let us note that the magnetic-flux jumps, as is known, are possible only when $(\partial j_c/\partial T) < 0$ (Ref. 13). The solution of Eq. (1) will be sought in the form

$$E = E(r/b)\exp\left\{\frac{\lambda t}{t_x}\right\};$$
$$\Theta = \Theta(r/b)\exp\left\{\frac{\lambda t}{t_x}\right\},$$

where $t_x = \nu_s b^2/\varkappa_s$ is the characteristic thermodiffusion time of the specimen, 2b is the plate thickness, and λ is the eigenvalue to be determined in the problem.

We choose the coordinate system in the following manner: The external magnetic field $H_0 \parallel z$ and the xy plane is parallel to the surface of the specimen. In this case the distribution of the magnetic field and current in the plate is of the form shown in Fig. 2. Of interest to us is the one-dimensional perturbation $\Theta(r/b) = \Theta(x)$, $E(r/b) = E(x)$, with $\mathbf{E} = \{0, E_y(x), 0\}$. Let us emphasize that the very concept of the critical state implies that the current density j and the electric field density are parallel; therefore, since $\mathbf{j}_N \parallel \mathbf{E}$ only solutions with $E > 0$ are allowable solutions.[5]

Now, with Eqs. (1) and (3) it is easy to obtain an equation for determining Θ (Refs. 14 and 15):

$$\Theta^{IV} - \lambda(1+\tau)\Theta'' - \lambda(\beta - \lambda\tau)\Theta = 0, \tag{4}$$

where $\beta = \frac{4\pi}{c^2}\left|\frac{\partial j_c}{\partial T}\right|\frac{j_c b^2}{\nu_s}$, $\tau = \frac{4\pi}{c^2}\frac{\sigma_f \varkappa_s}{\nu} = \frac{D_t}{D_m}$, and D_t and D_m are the coefficients of thermal and magnetic diffusion, respectively. Here and in what follows the differentiation is performed with respect to the dimensionless variable x ($-1 \le x \le 1$). The relation between the electric field and Θ is of the form

$$E = \frac{\varkappa_s}{j_c b^2}(\lambda\Theta - \Theta''). \tag{5}$$

Naturally, we are interested in the solution which is symmetric with respect to the x axis: $\Theta(x) = \Theta(-x)$. If we are to determine $\Theta = \Theta(x)$ and the dependence of λ on (β, τ, ...), we must supplement Eq. (4) with boundary conditions. Since the electric field vanishes when $x = 1 - L/b$ ($L = cH_0/4\pi j_c$ is the depth of penetration of the magnetic field), we have $E(1 - L/b) = 0$ and, therefore, by virtue of Eq. (5),

$$\lambda\Theta - \Theta''|_{1-L/b} = 0. \tag{6}$$

At the boundary of the superconductor the magnetic field can be assumed to be constant during development of instability, i.e., $\dot{H}(1) = 0$, which is equivalent (from Maxwell's equations) to the condition $E'(1) = 0$ or

$$\lambda\Theta' - \Theta'''|_1 = 0. \tag{7}$$

Since the temperature and heat flux are continuous when $x = 1 - L/b$, it is not difficult to find that

$$\Theta' = \tilde{W}\Theta|_{1-L/b}, \tag{8}$$

where

$$\tilde{W} = \sqrt{\lambda}\,\mathrm{th}\left[\sqrt{\lambda}\left(1 - \frac{L}{b}\right)\right]. \tag{9}$$

At the outer boundary of the superconductor the corresponding relation is of the form

$$\Theta' = -W\Theta, \tag{10}$$

where $W = W_0 b/\varkappa_s$ and W_0 is the coefficient of heat transfer from the superconductor to the coolant. If the specimen is covered with a layer of normal metal with a thickness d (Fig. 3), then conditions (6) and (8) obviously do not change. Since the magnetic field on the surface $x = (b + d)/b$ can now be considered constant during development of an instability, it is easy to use Maxwell's equations to get

$$\frac{E'}{E}\Big|_{x=1} = -\sqrt{\lambda\tau_1}\,\mathrm{th}\left(\sqrt{\lambda\tau_1}\,\frac{d}{b}\right), \tag{11}$$

where $\tau_1 = \tau\sigma_n/\sigma_s$ and σ_n is the electrical conductivity of the normal metal. Such a calculation, if one proceeds from the heat transfer equation, leads to the missing condition of the form

$$\frac{\Theta'}{\Theta}\Big|_{x=1} = -W_d, \tag{12}$$

where

$$W_d = \sqrt{\lambda\bar{\gamma}}\,\frac{W + \sqrt{\lambda\bar{\gamma}}\,\mathrm{th}(\sqrt{\lambda\bar{\gamma}}\,d/b)}{W\,\mathrm{th}(\sqrt{\lambda\bar{\gamma}}\,d/b) + \sqrt{\lambda\bar{\gamma}}},$$
$$\gamma = \frac{\nu_n\varkappa_s}{\nu_s\varkappa_n}, \quad \bar{\gamma} = \frac{\nu_n\varkappa_n}{\nu_s\varkappa_s}, \tag{13}$$

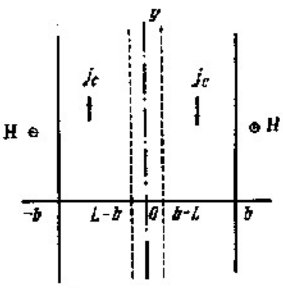

FIG. 2.

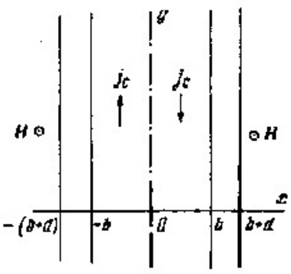

FIG. 3.

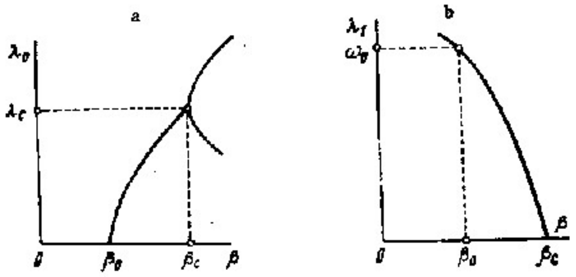

FIG. 4.

and ν_n and $\varkappa_n$ are the heat capacity and thermal conductivity, respectively, of the normal metal.

In the present paper we consider the case of a thin plate, $b - L \ll b$. In this case condition (8) is replaced by

$$\Theta'\Big|_{x=1-\frac{L}{b}} = 0. \tag{14}$$

The opposite limiting case will be considered in a separate paper.

2. THE PHYSICAL PICTURE OF INSTABILITY DEVELOPMENT

The development of instability is in many ways determined by the ratio τ of the coefficients of thermal and magnetic diffusion. Hard superconductors are characterized by small values of τ ($\tau \ll 1$). In composite superconducting materials (matrix of a normal metal with one configuration or other of superconducting strands interspersed in it) $\tau \gg 1$ because of the high conductivity of the normal metal.[15] Boundary conditions play a significant role along with τ.

Let us consider a qualitatively simpler case: $\tau \ll 1$, $L = b$. In the range of parameters of interest to us (immediately before a magnetic-flux jump) perturbations whose development in the fundamental approximation proceeds adiabatically,[1,14] i.e., $|\lambda| \gg 1$, prove to be characteristic. With the same accuracy it follows from Eq. (5) that

$$E = \frac{\lambda \varkappa_s}{j_c b^2}\Theta.$$

Substituting the last relation into Maxwell's equation, we get for E the equation

$$E'' + \beta E = 0. \tag{15}$$

It follows from Eq. (15) and the boundary conditions that "fast" perturbations ($|\lambda| \gg 1$) first arise at $\beta = \frac{\pi^2}{4}\frac{b^2}{L^2} \cong \frac{\pi^2}{4}$ (a half-wave fits into a plate thickness).

The quantity λ is defined, as usual, as the ratio between relaxation processes. In the given case these are the normal currents and thermal diffusion. In the linear approximation, the normal currents do not affect thermal diffusion and in proportion to the term $\sigma E \propto \lambda\tau\Theta$ compensate the perturbation of the current density, $\delta j \propto \beta\Theta$. As a result, $\delta j \propto (\beta - \lambda\tau)\Theta$. In a characteristic time $t \sim t_\varkappa/\lambda$ the temperature at the given point decreases by a value proportional to Θ/λ because of thermal diffusion. When this is taken into account the estimate for δj becomes

$$\delta j \propto \left(\beta - \frac{\beta^2}{\lambda} - \lambda\tau\right)\Theta. \tag{16}$$

Similar to the condition above is the following condition for the onset of "fast" perturbations:

$$\beta - \lambda\tau - \frac{\beta^2}{\lambda} = \frac{\pi^2}{4}. \tag{17}$$

Equation (17) serves to determine the function $\lambda = \lambda(\beta, \tau)$ (in the range $|\lambda| \gg 1$) which, obviously, is of the form

$$\lambda = \frac{\beta - \frac{\pi^2}{4} \pm \sqrt{\left(\beta - \frac{\pi^2}{4}\right)^2 - \frac{\pi^4\tau}{4}}}{2\tau}. \tag{18}$$

It follows from Eq. (18) that in the range of β values

$$\frac{\pi^2}{4} = \beta_0 < \beta < \beta_c = \frac{\pi^2}{4}(1 + 2\sqrt{\tau}),$$

the quantity λ turns out to be complex: $\lambda = \lambda_0 + i\lambda_1$, with $\lambda_0 > 0$. Let us note that when $\beta = \beta_0$, we have $\lambda = i\pi^2/(4\cdot\sqrt{\tau}) \equiv i\lambda_c$, $\lambda_c = \lambda(\beta_c)$. A real positive value $\lambda = \lambda_c$ first appears at $\beta = \beta_c$ which thus determines the stability criterion for magnetic-flux jumps.[9,14] The plot of $\lambda = \lambda(\beta, \tau)$ is shown in Fig. 4.

Let us now consider qualitatively the role of thermal boundary conditions. The parameter β contains the characteristic spatial scale l of the perturbation.[2)] In the case of adiabatic boundary conditions $l = L \cong b$. In the case of isothermal boundary conditions, the perturbation developing with the characteristic time $t = t_\varkappa/\lambda$ is practically absent in a layer with a thickness of the order of $b/\sqrt{\lambda}$ near the surface. The characteristic dimension l which determines the parameter β also changes correspondingly. As a result, the estimate for δj now is of the form $\delta j \propto (\beta - \lambda\tau - 2\beta/\sqrt{\lambda})\Theta$ and the equation of interest to us for determining the function $\lambda = \lambda(\beta, \tau)$ is:

$$\beta - \lambda\tau - \frac{2\beta}{\sqrt{\lambda}} = \frac{\pi^2}{4}. \tag{19}$$

Let us note that here the main term (with respect to $|\lambda| \gg 1$) is the term corresponding to the change in scale (in proportion to $1/\sqrt{\lambda}$). The correction of the order of $1/\lambda$ which is associated with the redistribution of heat proves to be of a higher order of smallness and we thus discarded it. The solution of Eq. (19) obviously is similar in form to that pictured in Fig. 4, differing only in the values of the parameters $\lambda = 2.31(1/\tau)^{2/3}$ for $\beta = \beta_0 = (\pi^2/4)(1 + 0.9\tau^{1/3})$;

$$\lambda = \lambda_c = \left(\frac{\pi^2}{4\tau}\right)^{2/3} \quad \text{for} \quad \beta = \beta_c = \frac{\pi^2}{4}(1 + 2.2\tau^{1/3}).$$

Thus, before a magnetic-flux jump there may be oscillations of the electric field and the temperature (let us recall that $\beta = \beta_c$ corresponds to a magnetic-flux jump). Let us consider this situation in greater detail. All of the relations which we have obtained are valid only when there is a linear relation between the current density j and the electric field **E**. Since the condition $E \geq E_0(T)$ must be

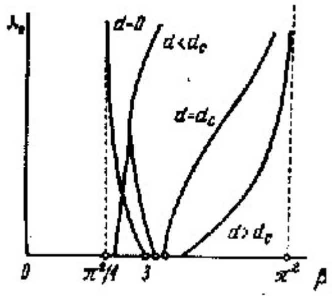

FIG. 5.

satisfied if this is to be the case, the magnitude of the electric field in the specimen has a significant influence on the character of the effects observed. Indeed, if from the outset there is no electric field in the specimen, then even a quite large fluctuation in the time $t \propto t_\kappa/\lambda_\kappa$ will decrease to such an extent that the condition $E < E_0(T)$ is satisfied and the perturbation will certainly be extinguished. Thus, in this case the initial growth of fluctuation with an increment λ_0/t_κ will lead to a change in the magnetic flux in the specimen only by a finite value. Consequently, finite magnetic-flux jumps with an amplitude which is proportional to the initial perturbation will be observed experimentally.

Suppose now that the stability of the critical state is studied in an alternating magnetic field. Then, from the very outset an electric field of the order of $E \sim (b/c)\dot{H}$ ($\dot{H}$ is the rate of change of the external field) exists in the specimen. If $\dot{H} > (c/b)E_0$, then electric-field oscillations with an amplitude of up to $(b/c)\dot{H}$ can in fact be observed in the specimen. The characteristic rates of change of the external field, as a rule, are always low in comparison with the rate of change of the magnetic field during a flux jump. This makes it possible to estimate the number of oscillations N observed at a given value of $\dot{H}$. Let $\lambda_1 \neq 0$ in the range $\Delta\beta = \beta_c - \beta_0 \ll \beta_c$ of values of the parameter β.[3] The corresponding range of values of the magnetic field ΔH is

$$\Delta H = \frac{\Delta\beta}{\partial\beta/\partial H} = 2\frac{\Delta\beta}{\beta_c}H_j,$$

where H_j is the field of the magnetic-flux jump [$\beta(H_j) = \beta_c$]. The value of the external magnetic field lies in this range for a time $\Delta t \sim \Delta H/\dot{H}$, whence N is of the order

$$N \sim \frac{\Delta H \lambda_1}{\dot{H} t_\kappa}.$$

It is seen that $N \gg 1$ if the rate of change of the external magnetic field is in the range

$$\frac{cE_0}{b} < \dot{H} < \frac{\Delta H \lambda_1}{t_\kappa}.$$

Next we shall make a detailed analysis of the range of the spectrum of eigenvalues λ ($\lambda_0 > 0$, $\lambda_1 \neq 0$) for arbitrary τ, thickness of the layer of metal of the coating, and thermal boundary conditions. In the general case, complex values of λ with $\lambda_0 \geq 0$ do obviously exist if $\lambda_c \neq 0$. Indeed, in this case in the vicinity of $\beta = \beta_c$ the eigenvalue λ can be rewritten as

$$\lambda = \lambda_c\left\{1 \pm a\sqrt{\frac{\beta - \beta_c}{\beta_c}}\right\},$$

where a is some real number. It is seen that when $\beta < \beta_c$, the value $\lambda_1 \neq 0$.

3. EXISTENCE DOMAIN OF FINITE INSTABILITIES

The range of values of the parameters β, τ, W, and d in which finite instabilities exist is determined, as is clear from the preceding section, by

$$\lambda_0(\beta, \tau, W, d) \geqslant 0, \quad \lambda_1(\beta, \tau, W, d) \neq 0$$

or

$$\beta_0(\tau, W, d) \leqslant \beta < \beta_c(\tau, W, d).$$

In order to find the function $\lambda = \lambda(\beta, \tau, W, d)$ it is necessary to write the condition of the existence of a nontrivial solution of Eq. (4) with the necessary boundary conditions. When the corresponding determinant is equated to zero the result is an involved equation which, in the general case, can be solved only by computer. Here we consider two limiting cases admitting analytic solution: $\tau \ll 1$ (hard superconductors) and $\tau \gg 1$ (composite superconductors). In each limiting case we derived (with the required accuracy) a simple algebraic equation for the function $\lambda = \lambda(\beta, \tau, W, d)$, the values of the parameters, β_0, β_c, λ_c, $\omega_0 = \lambda_1(\beta_0)$, and the maximum possible number of oscillations N. The order of exposition is specified by the character of the external heat transfer.

<u>a. Adiabatic boundary conditions (W = 0).</u> The spectrum of eigenvalues $\lambda = \lambda(\beta, \tau)$ is given in this case[14] by the equation

$$k_2(k_1^2 - \lambda)\,\mathrm{tg}\,k_2 = k_1(k_2^2 + \lambda)\,\mathrm{th}\,k_1, \tag{20}$$

where

$$k_{1,2}^2 = \sqrt{\lambda(\beta - \lambda\tau) + \frac{\lambda^2(1+\tau)^2}{4}} \pm \frac{\lambda(1+\tau)}{2}.$$

In hard superconductors ($\tau \ll 1$) flux jumps occur rapidly ($\lambda_c \gg 1$), which allows Eq. (20) to be solved analytically. The relation $\lambda = \lambda(\beta, \tau)$ (in the range $|\lambda| \gg 1$) is of the form of Eq. (18) and for the value N we have the estimate

$$N \sim \frac{H_j}{\dot{H}t_\kappa}\frac{\Delta\beta}{\beta_c}\lambda_1 \sim \frac{H_j}{\dot{H}t_\kappa}.$$

When $\tau = \tau_c = 1/21$ the quantity λ_c vanishes[16] at the point $\beta_c = \beta_1 = 3b/L$. Therefore, in the range $(\tau_c - \tau)/\tau_c \ll 1$ we have $\lambda_c \ll 1$ (the flux jump proceeds slowly). Then, to within τ inclusively, it is not difficult to get an equation from Eq. (20) for determining the spectrum $\lambda = \lambda(\beta, \tau)$ ($|\lambda| \ll 1$):

$$\lambda^2 - 10\left[\frac{\tau_c - \tau}{\tau_c} + 3(\beta - \beta_1)\right]\lambda - 175(\beta - \beta_1) = 0. \tag{21}$$

The obvious solution of Eq. (21) leads to the following estimates:

$$\lambda_c = 5\frac{\tau_c - \tau}{\tau_c};\ \beta_c = \beta_1;$$
$$\omega_0 = 7.6\sqrt{\frac{\tau_c - \tau}{\tau_c}};\ \beta_0 = \beta_1 - 0.33\frac{\tau_c - \tau}{\tau_c};$$
$$N \sim \left(\frac{\tau_c - \tau}{\tau_c}\right)^{3/2} \frac{H_f}{\dot{H}t_\kappa}.$$

Let us note that under these conditions the characteristic period $2\pi t_\kappa/\omega_0$ of the oscillations is shorter than the time t_κ/λ_c of the flux jump.

When $\tau > \tau_c$ the quantity $\lambda_c = 0$ and, therefore, oscillations as well as finite instabilities of the nature under study are absent.

Let us consider a superconductor covered with a layer of normal metal (see Fig. 3), with adiabatic boundary conditions on its outer boundary. Of greatest interest here is the case of hard superconductors since the effect of the normal metal with a high conductivity is significant only with respect to "fast" perturbations.[4] The equation for determining $\lambda = \lambda(\beta, \tau_1, d)$ when $\tau \ll 1$ is of the form

$$\begin{aligned} &W_d\sqrt{\lambda\tau_1}\,\mathrm{th}\left(\sqrt{\lambda\tau_1}\frac{d}{b}\right)(k_1^2 + k_2^2)\,[k_1^3\,\mathrm{tg}\,k_2 + k_2^3\,\mathrm{th}\,k_1] \\ &+ \sqrt{\lambda\tau_1}\,\mathrm{th}\left(\sqrt{\lambda\tau_1}\frac{d}{b}\right) \times \left[(k_1^4 - k_2^4)\,\mathrm{th}\,k_1\,\mathrm{tg}\,k_2 + 2\left(1 - \frac{1}{\mathrm{ch}\,k_1 \cos k_2}\right)k_1^3 k_2^3\right] \\ &+ k_1 k_2 (k_1^2 + k_2^2)\,[k_1^3\,\mathrm{th}\,k_1 - k_2^3\,\mathrm{tg}\,k_2] + W_d\left[2\frac{k_1^3 k_2^3}{\mathrm{ch}\,k_1 \cos k_2}\right. \\ &\left. + k_1 k_2 (k_1^4 + k_2^4) + k_1^2 k_2^2 (k_1^2 - k_2^2)\,\mathrm{tg}\,k_2\,\mathrm{th}\,k_1\right] = 0. \end{aligned} \tag{22}$$

Here, W_d is the heat transfer in the coating and it can be found from Eq. (13) since $W = 0$:

$$W_d = \sqrt{\lambda\gamma}\,\mathrm{th}\left(\sqrt{\lambda\gamma}\frac{d}{b}\right).$$

As the thickness of the coating increases, the eigenvalues λ decrease and the curve $\lambda = \lambda(\beta) > 0$ shifts to the region of higher β. Beginning from some value $d = d_c$ (see below for definition) λ_c vanishes. The evolution of $\lambda = \lambda(\beta)$ as d varies is shown in Fig. 5. The behavior of the curve $\lambda = \lambda(\beta, d)$ as described here is obvious since when the thickness of the layer of normal metal is quite large (greater than the thickness of the skin at the appropriate frequency) "fast" perturbations are suppressed by the normal currents. Consequently, "slow" perturbations become characteristic; their development is affected by the contact with the normal metal only in proportion to the heat transfer inside the coating, i.e., very weakly.

In the case $d \ll d_c$ the values $\lambda \sim \lambda_c$ of interest to us satisfy the condition $|\lambda| \gg 1$ and, from Eq. (22) we can easily find, with the necessary accuracy, an equation for determining $\lambda = \lambda(\beta, d, \tau)$:

$$\lambda^2\tau\left(1 + 2\frac{\sigma_n d}{\sigma_s b}\right) - \left(\beta - \frac{\pi^2}{4}\right)\lambda + \frac{\pi^4}{16} = 0. \tag{23}$$

The corresponding eigenvalues of the parameters are

$$\lambda_c = \frac{\pi^2}{4}\frac{1}{\sqrt{\tau\left(1 + 2\frac{\sigma_n d}{\sigma_s b}\right)}};\ \beta_c = \frac{\pi^2}{4}\left(1 + 2\sqrt{\tau\left(1 + 2\frac{\sigma_n d}{\sigma_s b}\right)}\right);$$
$$\omega_0 = \lambda_c;\ \beta_0 = \frac{\pi^2}{4};\ N \sim \frac{H_f}{\dot{H}t_\kappa}.$$

When $(d_c - d)/d_c \ll 1$, we have $\lambda_c \ll 1$ and from Eq. (22) we can easily find a simple equation for determining the spectrum $\lambda = \lambda(\beta, \tau, d)$ in the vicinity of the point $\beta = \beta_2 = 3[1 + d/b(\nu_n/\nu_s)]$ [note that $\lambda(\beta_2) = 0$ for any thickness of the coating]:

$$\lambda^2 - 14\lambda\left[3(\beta - \beta_2) + \frac{d_c - d}{d_c}\right] - 295(\beta - \beta_2) = 0.$$

Here[5]

$$d_c = \frac{8}{3}\frac{b}{11\nu_n/\nu_s + 105\tau_1}.$$

Accordingly, the parameter values of interest to us are:

$$\lambda_c = 7\frac{d_c - d}{d_c};\ \beta_c = \beta_2 = 3\left(1 + \frac{d}{b}\frac{\nu_n}{\nu_s}\right);$$
$$\omega_0 = 10\sqrt{\frac{d_c - d}{d_c}};\ \beta_0 = \beta_2 - 0.33\frac{d_c - d}{d_c};$$
$$N \sim \left(\frac{d_c - d}{d_c}\right)^{3/2}\frac{H_f}{\dot{H}t_\kappa}.$$

Note that here, as in the case $\tau \to \tau_c$, the characteristic period of the oscillations is greater than that of the flux-jump time and, therefore, the maximum number of oscillations is relatively small.

b. Isothermal boundary conditions ($W = \infty$). With cooling, the value of λ_c is always different from zero[5]; thus, there always exists a range of parameters in which finite instabilities or oscillations can appear.

The equation for determining the function $\lambda = \lambda(\beta, \tau)$ is of the form[5]

$$\begin{aligned} &2k_1k_2(\lambda + k_2^2)(k_1^2 - \lambda) + k_1k_2[(k_2^2 + \lambda)^2 + (k_1^2 - \lambda)^2] \\ &\times \mathrm{ch}\,k_1 \cos k_2 + (k_1^2 - k_2^2)(k_1^2 - \lambda)(k_2^2 + \lambda)\,\mathrm{sh}\,k_1 \sin k_2 = 0. \end{aligned} \tag{25}$$

An analytic solution of Eq. (25) can be obtained in two limiting cases: $\tau \ll 1$ and $\tau \gg 1$. In the first case, the eigenvalues $|\lambda| \gg 1$ and the function $\lambda = \lambda(\beta, \tau)$ are described according to the $\tau \ll 1$ approximation by Eq. (19). For the maximum number of oscillations we can easily find the expression

$$N \sim \frac{1}{\tau^{1/3}}\frac{H_f}{\dot{H}t_\kappa}.$$

In the second case, the eigenvalues $|\lambda| \ll 1$ which, essentially, according to the $\tau \gg 1$ approximation, leads to the equation

$$\lambda^{3/2}\tau - \left(\beta - \frac{\pi^2}{4}\tau\right)\lambda^{1/2} + \frac{\pi^2}{2}\tau^{1/2} = 0. \tag{26}$$

Accordingly, the values of the parameters are

$$\lambda_c = \left(\frac{\pi^2}{4}\right)^{2/3}\left(\frac{1}{\tau}\right)^{1/3};\ \beta_c = \frac{\pi^2}{4}\tau(1 + 2.2\tau^{-1/3});$$
$$\omega_0 = 1.26\lambda_c;\ \beta_0 = \frac{\pi^2}{4}\tau(1 + 0.9\tau^{-1/3});$$
$$N \sim \frac{1}{\tau^{2/3}}\frac{H_f}{\dot{H}t_\kappa}.$$

Let us also note that for $\tau \sim 1$ the domain of existence of finite instabilities, $\Delta\beta/\beta_c = (\beta_c - \beta_0)/\beta_c$, is comparatively large (of the order of unity). In particular, when $\tau = 1$ numerical computations yield

$$\lambda_c = 3;\ \beta_c = 17.6;\ \omega_0 = \lambda_c;\ \beta_0 = \frac{\beta_c}{2} = 8.8.$$

Let us consider a superconductor coated with a layer of normal metal which has isothermal conditions on its boundary (see Fig. 3). In this case, the function $\lambda = \lambda(d, \beta, \tau)$ in the range $|\lambda| \gg 1$ undergoes considerable changes as the thickness of the coating increases from $d \sim d_1 = b\sigma_s/\sigma_n \ll b$ to $d \sim d'_c = 3b/\pi^2\tau_1$ (Ref. 5). When the value of W_d specifying the heat transfer in the coating, $W_d = \sqrt{\lambda\gamma}\cot(\sqrt{\lambda\gamma}\, d/b)$, is inserted in Eq. (22), we get an equation for determining $\lambda = \lambda(\beta, \tau, d)$. In the range $d_1 < d < d_c$ in the main approximation with respect to $|\lambda| \gg 1$, it reduces to a simple relation of the form of Eq. (19) with the change $\tau \to 2\tau_1 d/b$, which makes it easy to calculate all the characteristic parameters of the problem, especially the maximum number of oscillations:

$$N \sim \left(\frac{b}{2\tau_1 d}\right)^{1/2} \frac{H_j}{\dot{H} t_\varkappa}. \tag{27}$$

When $d > d'_c$ an approximate algebraic equation can also be obtained for determining the function $\lambda = \lambda(\beta, \tau, d)$ since here, too, finite instabilities develop "rapidly" ($|\lambda| \gg 1$):

$$\lambda(\pi^2 - \beta) - 2\pi^2\left(\frac{\lambda}{\tau_1}\right)^{1/2} + \pi^4 = 0. \tag{28}$$

It follows from Eq. (28) that

$$\lambda_c = \pi^4\tau_1;\ \beta_c = \pi^2\left(1 - \frac{1}{\pi^2\tau_1}\right);$$
$$\omega_0 = \frac{\lambda_c}{2};\ \ \beta_0 = \pi^2\left(1 - \frac{2}{\pi^2\tau_1}\right);$$
$$N \sim \frac{H_j}{\dot{H} t_\varkappa}.$$

c. Arbitrary thermal boundary conditions. Calculation of $\lambda = \lambda(\beta, \tau, W)$ for arbitrary values of W is an extremely complicated undertaking which, in the general case, can be accomplished only with a computer. Note that in the case $W \gg 1$, the character of the curve $\lambda = \lambda(\beta, \tau, W)$ is qualitatively the same as for $W = \infty$. If $W < 1$, then for $\tau > \tau_c$ the eigenvalues λ satisfy the condition $|\lambda| \ll 1$ and the situation can be considered analytically. It is not difficult to find that for hard superconductors with $\tau_c < \tau < 1$ the relation $\lambda = \lambda(\beta, \tau, W)$ in the main approximation with respect to $|\lambda| \ll 1$ is described by

$$1.2(\tau - \tau_c)\lambda^2 + (\beta_1 - \beta)\lambda + 3W = 0. \tag{29}$$

It follows from Eq. (29) that

$$\lambda_c = 1.6\left(\frac{W}{\tau - \tau_c}\right)^{1/2};\ \beta_c = \beta_1 + 3.8[W(\tau - \tau_c)]^{1/2};$$
$$\omega_0 = \lambda_c;\ \beta_0 = \beta_1;$$
$$N \sim W\frac{H_j}{\dot{H} t_\varkappa}.$$

As is seen from the formulas given above, the characteristic flux-jump time $t_j \sim t_\varkappa/\lambda_c$ (and therefore, the oscillation period) can be relatively large (within the bounds of $\lambda_c^{-1} \gg 1$), as was apparently observed in Ref. 8.

For composite superconductors under the conditions of weak cooling, the function $\lambda = \lambda(\beta, \tau, W)$ can be found from

$$\lambda^{4/3}\tau + (W\tau - \beta)\lambda^{1/2} + W\tau^{1/2} = 0. \tag{30}$$

Equation (30) was obtained with the condition that $1/\tau < W < 1$. It is now easy to find the values of the parameters:

$$\lambda_c = \left(\frac{W}{2}\right)^{2/3}\tau^{-1/3};\ \beta_c = W\tau\left[1 + 1.9\frac{1}{(W\tau)^{1/3}}\right];$$
$$\omega_0 = 1.26\lambda_c;\ \beta_0 = W\tau\left[1 + 0.8\frac{1}{(W\tau)^{1/3}}\right];$$
$$N \sim \frac{W^{1/3}}{\tau^{2/3}}\frac{H_j}{\dot{H} t_\varkappa}.$$

CONCLUSION

A study was made of the domain of existence of finite instabilities or oscillations in type-II superconductors. Various conditions of heat input and contact of the superconductor with a normal metal are considered. It is shown that in the case of an adiabatically thermal-insulated specimen finite instabilities are observed in a narrow range of values of an external magnetic field [in proportion to $(\tau_c - \tau)$ or $(d_c - d)$ only when $\tau < \tau_c$ and $d < d_c$. Oscillations can always be observed when there is external heat transfer. If the heat transfer is low, however, then the range of external magnetic field in which oscillations are observed is correspondingly small. It is shown that the oscillation period (apart from the range $(\tau_c - \tau)/\tau_c \ll 1$ or $(d_c - d)/d_c \ll 1$) is of the order of the characteristic magnetic-flux-jump time. This is accordance with the experimental data obtained in Refs. 8 and 9 (the results of Refs. 8 and 9 will be analyzed in detail and compared with theory separately when the case $L \ll b$ is considered).

NOTATION

Here, 2b denotes the film thickness; Θ is the temperature perturbation; T_c is the critical temperature of the superconductor; H is the magnetic field; $\dot{H}$ is the rate of change of the external magnetic field; E is the electric field; j_c is the critical current density; W_0 is the heat transfer coefficient; ν_s, $\varkappa_s$, σ_s and ν_n, $\varkappa_n$, σ_n are the heat capacity, thermal conductivity, and electrical conductivity of the superconducting and normal metals, respectively.

1)A preliminary report on this work was published in Ref. 11.

2)When $L \neq b$, the parameter β should be defined as[5] $\beta = \frac{4\pi}{c^2}\left|\frac{\partial j_c}{\partial T}\right|\frac{j_c L^2}{\nu_s}$.

3)Since $b \approx L$, we have $\beta \propto H^2$.

4)The coating has practically no effect on the character of the development of flux jump in combined superconductors ($\tau \gg 1$), regardless of the thermal boundary conditions.[5]

5)In a similar calculation given in Ref. 14 no account was taken of the heat transfer in the coating and this led to an incorrect result.

[5]R. Hancox, Phys. Lett. 16, 208 (1965).