THE CRITICAL STATE STABILITY

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ABSTRACT

A review of the theory of flux jump nucleation in hard and composite superconductors is presented. Stability criteria for some special cases are established. The basic physical ideas and quantitative methods of stability investigation are outlined.

1. INTRODUCTION

The critical state stability against flux jumps in hard and composite superconductors has been discussed in a number of theoretical and experimental papers. In this report we intend to review the theory of magnetic instabilities, in particular the recent progress on this question.

Before the quantitative theory will be given an outline of the basic physical ideas is helpful. The flux jump may be thought of as a coupled perturbation of a temperature T and a magnetic field H. This process is accompanied by the release of heat and by the enlargement or re-distribution of the magnetic flux captured in the superconductor. The heat and magnetic flux propagation may be described in terms of appropriate diffusivity coefficients, $\mathscr{D}_m = c^2/4\pi\sigma_f$, the magnetic diffusivity, and $\mathscr{D}_t = \kappa/\nu$ the heat diffusivity. Here σ_f is the flux flow conductivity, ν the specific heat and κ the thermal conductivity of the superconductor.

It is convenient to denote the ratio $\mathscr{D}_t/\mathscr{D}_m$ as τ. The values σ_f and κ are not too large in the case of hard superconductors, and it may be assumed that $\tau \ll 1$. Therefore, the magnetic diffusion occurs in a time short compared to that for the thermal diffusion, so that to the first order in $\tau \ll 1$ the flux jump is an adiabatic process. The experiments give a reliable confirmation of this fact. (See for example Refs. 1-3.)

The opposite situation where $\tau \gg 1$ usually takes place in superconducting composites. In this case when $\mathscr{D}_t \gg \mathscr{D}_m$ or $\tau \to \infty$ the magnetic flux in the sample is almost frozen in during the heating of the material. The explanation of this fact is evident. The normal current excited in the composite by the change of magnetic field compensates in part for the decrease in the critical current density caused by the temperature rise and prevents magnetic flux propagation in the sample.

On the basis of this intuitive speculation one may obtain the qualitative stability criteria.

First consider the case of a hard superconductor ($\tau \ll 1$). Let us assume that some fluctuation of the field, temperature, etc., results in the initial temperature rise and the corresponding energy dissipation is $Q_0 = \nu\theta_0$. The magnetic flux and the current re-distribution gives rise to the additional energy dissipation Q_1. Assuming $j_c \gg \sigma_f E$ we have:

$$Q_1 = \int j_c E \, dt \tag{1}$$

where j_c is the critical current density, E is the electric field. If the flux jump does not occur, the sample passes into a new equilibrium state with the temperature $T = T_0 + \theta$; where T_0 is the initial temperature. Since $\mathscr{D}_t \ll \mathscr{D}_m$ one may suppose that:

$$Q = \nu\theta = Q_0 + Q_1 = \nu\theta_0 + Q_1 \tag{2}$$

To evaluate Q_1 we use Eq. (1) and Maxwell's equation:

$$\nabla^2 \vec{E} = \frac{4\pi}{c^2} \frac{\partial \vec{j}}{\partial t} \tag{3}$$

Assuming for simplicity $\partial j_c/\partial H = 0$ (Bean's critical state model[4]) it follows that $\partial j_c/\partial t = -(\partial j_c/\partial T)\theta'$. Since $|\nabla^2 \vec{E}| \sim E/b^2$, where b is some typical length for each sample we obtain:

$$E \sim \frac{4\pi b^2}{c^2} \theta' \left|\frac{\partial j_c}{\partial T}\right|$$

and

$$Q_1 = \int j_c E \, dt = \frac{1}{\gamma^2} \frac{4\pi b^2}{c^2} j_c \left|\frac{\partial j_c}{\partial T}\right|$$

Here the constant $\gamma^2 \sim 1$ depends on the sample geometry, the field and current distribution. Inserting the expression for Q_1 into (2) we have:

$$\theta = \frac{\theta_0}{1 - \beta/\gamma^2} \tag{4}$$

where

$$\beta = \frac{4\pi b^2 j_c}{c^2 \nu} \left|\frac{\partial j_c}{\partial T}\right| \tag{5}$$

It follows from (4) that the temperature rise θ becomes infinite, or critical state unstable at $\beta/\gamma^2 \to 1$. Thus the stability criterion may be written as follows:

$$\beta < \gamma^2 \tag{6}$$

The formula similar to (6) for a semi-infinite flat superconductor was first given in Ref. 5 where analogous qualitative speculations were made. In that paper the length b corresponds to the depth ℓ of the shielding layer of external magnetic field. The value ℓ is defined by the equation $H(\ell) = H_\ell - 4\pi\ell j_c/c = 0$, † and inequality (6) becomes:

$$\frac{H_\ell^2}{4\pi\nu j_c} \left|\frac{\partial j_c}{\partial T}\right| < \gamma^2 \tag{7}$$

Let us determine the analogous stability criterion for the case of $\tau \gg 1$ (superconducting composites). Now the heating occurs while the magnetic flux is frozen in as discussed before. This implies $\partial j/\partial t = 0$, and as $j = j_c + \sigma E$, one gets:

$$\sigma E' + \frac{\partial j_c}{\partial T} \theta' = 0$$

hence:

$$E = \frac{\theta}{\sigma} \left|\frac{\partial j_c}{\partial T}\right|$$

and the power dissipated per unit volume Q' is:

$$Q' = j_c E = \frac{j_c \theta}{\sigma} \left|\frac{\partial j_c}{\partial T}\right|$$

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†We assume here that there is no internal flux in the sample, i.e., $H_i = 0$ in Fig. 1.

The critical state is obviously stable if the value Q' does not exceed $q = \kappa\nabla^2 T$, or:

$$q = \kappa\nabla^2 T > \frac{j_c\theta}{\sigma}\left|\frac{\partial j_c}{\partial T}\right| = Q'$$

since $|\nabla^2 T| \sim \theta/b^2$ one can put the stability criterion in the form

$$\frac{1}{\gamma_1^2}\frac{b^2 j_c}{\kappa\sigma}\left|\frac{\partial j_c}{\partial T}\right| < 1 \tag{8}$$

Here the constant $\gamma_1^2 \sim 1$ depends mainly upon sample geometry. The expression (8) may be rewritten as follows:

$$\beta/\tau < \gamma_1^2 \tag{9}$$

Note, the formulas (8) and (9) were derived assuming ideal external cooling (or isothermal boundaries). The criterion similar to (8) was first found by Hart.[6, 7]

2. HARD SUPERCONDUCTORS

In a rigorous treatment, one has to investigate the stability of Maxwell's equations combined with the heat equation, as is clear from the qualitative considerations above. The electric field E and the temperature perturbation θ will have the form

$$\theta(t,\vec{z}) = \theta_0\left(\frac{\vec{r}}{b}\right)\exp\left\{\frac{\lambda\kappa t}{\nu b^2}\right\} \tag{10}$$

$$\vec{E}(t,\vec{z}) = \vec{E}_0\left(\frac{\vec{r}}{b}\right)\exp\left\{\frac{\lambda\kappa t}{\nu b^2}\right\} \tag{11}$$

where λ is the eigenvalue to be determined. Let us introduce the notation:

$$\alpha = \alpha(x) = -\frac{4\pi b}{c}\frac{\partial j_c}{\partial H}; \qquad \beta = \beta(x) = -\frac{4\pi b^2}{c^2}\frac{j_c(x)}{\nu}\frac{\partial j_c}{\partial T}$$

Using expressions (10) and (11) one may find a set of equations for θ_0 and E_0 for the flat sample of Fig. 1 as follows:

$$\begin{cases} E_0'' + \alpha E_0' + (\beta - \lambda\tau)E_0 = \dfrac{\beta\kappa}{b^2 j_c}\theta_0'' \\ \\ \lambda\theta_0 - \theta_0'' = \dfrac{j_c b^2}{\kappa}E_0 \end{cases} \tag{12}$$

The equations (12) require four boundary conditions. We shall consider the external magnetic field H_ℓ to remain constant during the fluctuation*:

$$\frac{\partial H(\pm b)}{\partial t} = 0, \quad E_0'(\pm 1) = 0 \tag{13}$$

The two other boundary conditions can be found from the surface cooling equations:

$$w_0\theta(\pm b) + \kappa\frac{\partial\theta(\pm b)}{\partial x} = 0$$

or

$$w\theta_0(\pm 1) \pm \theta_0'(\pm 1) = 0 \tag{14}$$

*This implies that $(\partial H_\ell/\partial t)t_j \ll H_\ell$ where t_j is the rise time of the flux jump. Since $t_j \sim 10^{-4} \div 10^{-5}$ sec the assumption is usually valid.

where $w = w_0 b/\kappa$ is the heat transfer coefficient from the conductor to the coolant.

In practice, the solution E_0 and θ_0 is found in two regions which differ by the direction of the current (see Fig. 1). And the usual matching conditions must be formulated.

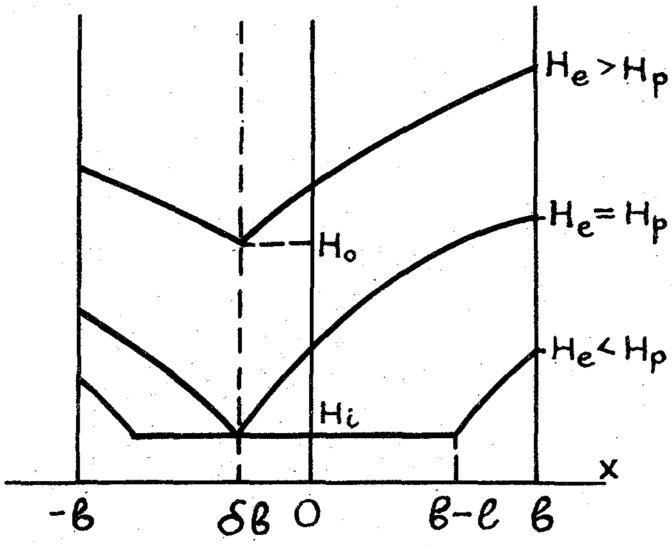

Fig. 1. The flat sample.

It is expedient to take the current density on their boundary as equal zero. Denoting this boundary $\vec{\delta}$, we have for the electric field:

$$\vec{E}(\vec{\delta}) = 0 \tag{15}$$

Also the temperature θ and heat flux $\kappa\nabla\theta$ must be continuous at $\vec{\delta}$:

$$\begin{cases} \theta_0(\vec{\delta}+0) = \theta_0(\vec{\delta}-0) \\ \nabla\theta_0(\vec{\delta}+0) = \nabla\theta_0(\vec{\delta}-0) \end{cases} \tag{16}$$

The spectrum of eigenvalues $\lambda = \lambda(\beta, \tau, \delta, w, \ldots)$ may easily be determined for the equations (12) with boundary conditions (13) to (16). The instability corresponds to $\lambda > 0$.

The equations (12) contain constant coefficients at $\alpha(x) \equiv 0$ (Bean's critical state) and may be readily solved. The spectrum of eigenvalues $\lambda = \lambda(\beta, \tau)$ for $\tau = 0$ and $\tau \ll 1$ has been investigated[8-10] and as was outlined in these papers the disturbances with $\lambda \ll 1$ (or adiabatic one) are responsible for the magnetic instabilities in hard superconductors ($\tau \ll 1$).† This fact may be proved for any critical state model (or $\alpha > 0$) by use of WKBJ method.[11] However, it seems to us evident from the qualitative speculations.

Since the stability is determined by the perturbations with $\lambda >> 1$ it allows us to simplify the calculations to the first approximation in $\tau \ll 1$.[10, 12-14] It follows from (12) that $\theta_0 \sim (1/\lambda)(j_c b^2 E_0/\kappa)$, and if the term on the right in the first of the equations (12) is small ($\sim 1/\lambda$), it can be neglected. The value $\lambda\tau \ll 1$ (see the last footnote), and to the first approximation instead of (12) we have:

$$E_0'' + \alpha(x)E_0' + \beta(x)E_0 = 0 \tag{17}$$

†Using the method of Ref. 10 one may find the values of at which the flux jump occur: $\lambda = \lambda_c = (\pi^2/4)\tau^{-1/2}$ (w=0) and $\lambda = \lambda_c = (\pi/2)^{4/3}\tau^{-1/3}$ (w>>1).

This equation must be supplied with electrodynamical boundary conditions (13) and (15).[10] The stability is violated if a nontrivial solution exists for the equation (17) with boundary conditions (13) and (15).

It is convenient to introduce a new variable y:

$$y = \frac{H_\ell - H(x)}{H_\ell - H_i}$$

and Eq. (17) becomes:

$$\frac{d^2E_0}{dy^2} + \tilde{\beta}E_0 = 0 \tag{18}$$

here

$$\tilde{\beta} = \frac{(H_\ell - H_i)^2}{4\pi\nu T_0(H)}\ ; \qquad T_0(H) = \frac{j_c}{|\partial j_c/\partial H|}$$

The boundary conditions (13) and (15) transform to:

$$E_0(1) = \frac{dE_0}{dy}\bigg|_{y=0} = 0 \tag{19}$$

The value T_0 is independent of H if $j_c(H,T) = j_0(T)\phi(H)$. The equation (18) may be solved analytically in this case and the stability criterion has the form:

$$\frac{(H_\ell - H_i)^2}{4\pi\nu\ j_c}\left|\frac{\partial j_c}{\partial T}\right| \leq \frac{\pi^2}{4} \tag{20}$$

To solve (18) in a more general case one may apply the WKBJ approximation.[11] Using the standard WKBJ procedure and boundary conditions (19) one gets the stability criterion:

$$\int_0^1 \sqrt{\tilde{\beta}}\, dy = \frac{1}{\sqrt{4\pi\nu}}\int_{H_i}^{H_\ell} \frac{dH}{\sqrt{T_0(H)}} = \int_{\delta b}^{b} \sqrt{\beta(x)}\, dx < \frac{\pi}{2} \tag{21}$$

The accuracy of the formula obtained may be evaluated as follows:

$$\frac{1}{\pi^2}(H_\ell - H_i)^2\left(\frac{1}{T_0}\frac{\partial T_0}{\partial H}\right)^2 \ll 1 \tag{22}$$

The inequality (22) is usually valid throughout the total range of external fields except for the fields very close to H_{c_2}. This results from the usual relations between the parameters in (22). Thus, the stability criterion (22) is applicable for any critical state model.

The WKBJ approximation may be used for stability investigations of samples having different geometrical configurations. For example, the case of the cylindrical symmetry is studied in the Ref. 15.

3. SUPERCONDUCTIVE COMPOSITES

Now we shall discuss the stability of the superconductive composites, i.e., the conductors consisting of normal (matrix) and superconducting (filaments) metals.

The presence of a high conductivity metal in a composite leads to the damping of perturbations with $\lambda >> 1$ and as a result to an improvement in the stability.

The methods described in the previous section allow us to investigate the stability of an individual filament covered with a normal metal layer of an arbitrary thickness d.[8,16] As shown before, the increase in d leads to a considerable stability improvement unless d is smaller than some critical value d_c. For example, under isothermal boundary conditions d_c may be evaluated as $d_c = (3c^2\nu b)/4\pi^3\kappa\sigma_n$ where σ_n is the normal metal conductivity. As an illustration, the dependence of the maximum transport current on the radius R of the wire is shown in Fig. 2 both at d=0 and $d \geq d_c$.

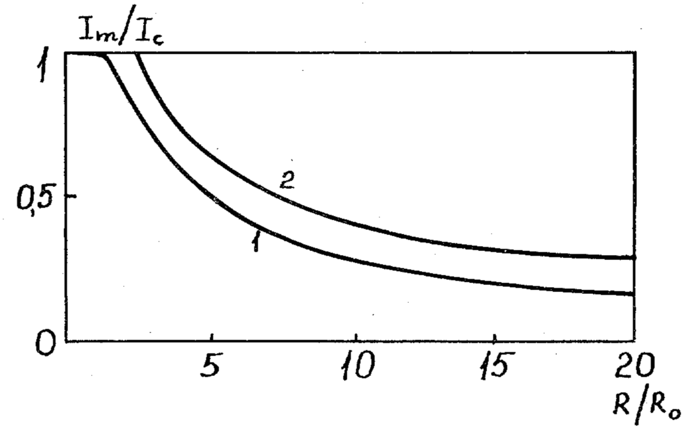

Fig. 2. The dependence of I_m on R; $I_c = \pi R^2 j_c$,

$$R_0 = \sqrt{\frac{c^2\nu}{4\pi j_c |dj_c/dT|}}\ ;$$

curve 1: d=0; curve 2: $d \geq d_c$.

Next, we discuss the stability of a whole superconducting composite containing a large number, $N >> 1$, of filaments. In such samples the instability may be initiated by collective effects or a connection between filaments even in the case when every separate filament is stable.

To derive the quantitative results one has to examine the equations describing a small perturbation in a composite. We shall treat the composite as a continuum with properties at a point given by the average over a volume in the neighborhood of the point, the volume element being large enough to contain a large number $n(1 \ll n \ll N)$ of filaments.[17,18] After averaging one has for θ_0 and E_0:

$$\begin{cases} E_0'' + \bar{\alpha}E_0' + (\bar{\beta} - \lambda\bar{\tau})E_0 = -\dfrac{\bar{\beta}\bar{\kappa}\,\theta_0''}{b^2 j_0} \\[2ex] \theta_0 - \theta_0'' = \dfrac{j_0 b^2}{\bar{\kappa}}E_0 \end{cases} \tag{23}$$

and the relation between current and electric field becomes:

$$j = j_0 + \sigma E \tag{24}$$

The values $\bar{\nu}$, j_0, $\bar{\sigma}$, $\bar{\kappa}$ are the specific heat, superconducting current density, electric and heat conductivities averaged over the composite and it is assumed that the current j does not change its direction within the volume of averaging. Denoting the fraction of superconducting material as x_s and the normal one as x_n such that $x_n + x_s = 1$, we have:

$$\bar{\nu} = x_n\nu_n + x_s\nu_s\ ; \quad j_0 = x_s j_c\ ; \quad \bar{\sigma} = x_n\sigma_n + x_s\sigma_f\ .$$

The heat conductivity transverse with respect to the filaments may be evaluated as $\bar{\kappa} \sim (1-\sqrt{x_s})\kappa_n$ (here κ_n is the normal metal heat conductivity). The accurate value of $\bar{\kappa}$ is determined by the detailed structure of the composite.

To find $\lambda = \lambda(\bar{\beta}, \bar{\tau}, \ldots)$ the equations (23) must be furnished with usual thermal and electrodynamical boundary conditions (see (13) - (16)). The value of the parameter $\bar{\tau}$ for the composites can vary from $\bar{\tau} \sim 1$ up to $10^2 \div 10^4$. For $\bar{\alpha} \equiv 0$ (Bean's model) the system of differential equations (23) contains only the constant coefficients. In Ref. 18 the spectrum of eigenvalues $\lambda = \lambda(\bar{\beta}, \bar{\tau})$ and the stability criterion in the form (flat plate) were determined:

$$\bar{\beta} < \frac{\pi^2}{4}\bar{\tau} + 3\left(\frac{\pi}{2}\right)^{4/3} \bar{\tau}^{2/3} \quad (w >> 1) \tag{25}$$

Now the stability is violated by perturbations with $\lambda = \lambda_c \ll 1$.* This fact is independent of the critical state model as can be proved by WKBJ method. Since $\lambda_c \ll 1$ and $\lambda_c \bar{\tau} >> 1$,* the equations (23) are reduced to a second order differential equation to the first approximation in $\bar{\tau} >> 1$. To show this let us rewrite the first of the equations (23) in the form:

$$\theta_0'' + \frac{\bar{\beta}}{\bar{\tau}}\theta_0 - \lambda\theta_0 = -\frac{b^2 j_0}{\lambda\bar{\tau}\bar{\kappa}}(E_0'' + \bar{\alpha}E_0') \tag{26}$$

for $\lambda\bar{\tau} >> 1$ and $\lambda \ll 1$ it becomes

$$\theta_0'' + \frac{\bar{\beta}}{\bar{\tau}}\theta_0 = 0 \tag{27}$$

And the stability breaks down if a nontrivial solution exists for the equation (27) with boundary conditions (14) and (16) as is clear from the preceding arguments. An equation similar to (27) has been found by Hart[6] by making use of qualitative considerations. In the case $j_0 = j_0(H)$ to solve equation (27) one can apply the WKBJ method. Thus, for the plate of Fig. 1 the stability criterion may be easily found:

$$\mathrm{tg}\left[\int_0^1 \sqrt{\frac{\bar{\beta}}{\bar{\tau}}}\, dx'\right] < \frac{w}{\left(\sqrt{\frac{\bar{\beta}}{\bar{\tau}}}\right)_{x=1}} \tag{28}$$

This inequality is valid for $w > 1/\bar{\tau}$ and allows us to analyze the stability with respect to external cooling.

Comparing the stability criteria for the filament and for the whole composite one may obtain the critical number N_c of filaments, that is, when $N > N_c$, the superconducting composite consisting of individually stable filaments is unstable. For $w \to \infty$ we have

$$N < N_c = \bar{\tau}\,\frac{\bar{\nu}}{\nu x_s} \tag{29}$$

and for $\bar{\tau}^{-1} \ll w \ll 1$ †

$$N < N_c = \frac{4w\bar{\tau}}{\pi^2}\,\frac{\bar{\nu}}{\nu x_s} \tag{30}$$

*For example for the case of a flat sample $\lambda_c = (w/2)^{2/3} \cdot \bar{\tau}^{-1/3}$ ($w \ll 1$, $w\bar{\tau} >> 1$) and $\lambda_c = (\pi/2)^{4/3}\bar{\tau}^{-1/3}$ ($w >> 1$).

†In practice real composites with the copper matrix cooled by liquid helium have the value $w \lesssim 1$. In principle the case $w >> 1$ may be realized for composites clad with a sufficiently thick layer of high purity copper.

4. CONCLUSIONS

The methods developed in the references 5-18 allow us to carry out a quantitative investigation of the stability of superconducting materials against flux jumps, with respect to the critical current density dependence on temperature and magnetic field, sample geometry, thermal and magnetic diffusion, external cooling, etc.

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