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On the theory of flux jumps in hard superconductors

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Abstract. The critical state stability against flux jumps in hard superconductors is considered by means of the WKBJ method. The universal stability criterion for an arbitrary critical state model is obtained.

1. Introduction

The inherent magnetic instabilities (flux jumps) in hard superconductors have been discussed by many authors (e.g. Hancox 1965, Wipf 1967, Swartz and Bean 1968, Wilson *et al* 1970, Kremlev 1973, 1974). Stability criteria were found for different examples of approximation by a simple model (such as a constant critical current density or a critical current density that depends on the magnetic field as $j_c \sim 1/(H+B_0)$). The flux jumping is usually treated as adiabatic process (Swartz and Bean 1968) but this was conclusively proved only for $\partial j_c/\partial H=0$ (Mints and Rakhmanov 1975).

In the present paper the critical-state stability criterion is determined for an arbitrary function $j_c=j_c(H)$ and general evidence is given of the adiabatic nature of flux jumping in hard superconductors.

2. The general equation (WKBJ solution)

Flux jumping in superconductors is accompanied by the change of electric (E) and magnetic (H) fields and by an increase in the temperature T. The expression for any of these values may be found from the heat and Maxwell equations. The relation between the current j and the electric field E may be written as follows:

$$j=j_c(T, H)+\sigma_f E$$

where σ_f is the flux-flow conductivity.

For example, for a flat sample (figure 1) the electric field E may be expressed in the form:

$$E=E_0(x/b) \exp(\lambda t\kappa/\nu b^2)$$

where ν is the specific heat, κ is the thermal conductivity and $2b$ is the thickness of the sample. The general equation for E_0 mentioned by Kremlev *et al* (1976) is as follows:

$$E_0^{\mathrm{iv}}+(\alpha+2\gamma'/\gamma)\,E_0'''+[\gamma''/\gamma+2(\alpha\gamma)'/\gamma-\lambda(1+\tau)]\,E_0''+[(\alpha\gamma)''/\gamma-\alpha\lambda-2\lambda\tau\gamma'/\gamma]\,E_0' \\ +\lambda\,(\lambda\tau-\beta-\tau\gamma''/\gamma)\,E_0=0 \qquad (1)$$

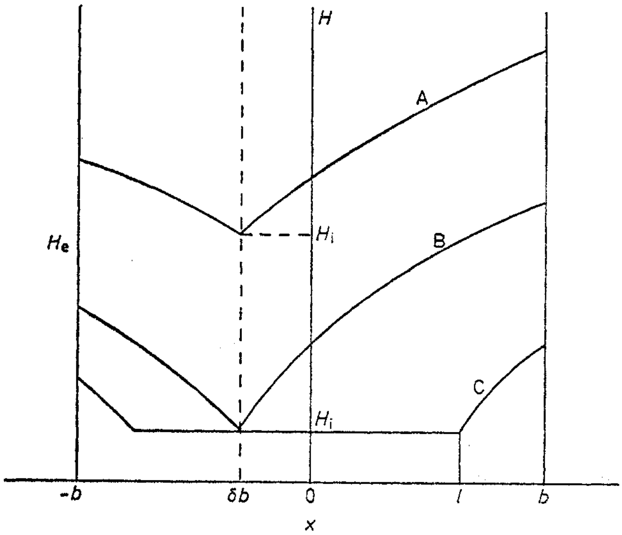

Figure 1. The slab of thickness $2b$ in the external field. A, $H_e > H_p$; B, $H_e = H_p$; C, $H_e < H_p$.

where

$$\alpha = \alpha(x) = -\frac{4\pi b}{c} \frac{\partial j_c}{\partial H}$$

$$\beta = \beta(x) = -\frac{4\pi b^2}{c^2} \frac{j_c(x)}{\nu} \frac{\partial j_c}{\partial T}$$

$$\gamma = \gamma(x) = -(\partial j_c/\partial T)^{-1}$$

$$\tau = \frac{D_t}{D_m} = \frac{4\pi}{c^2} \frac{\kappa \delta_f}{\nu}.$$

D_m and D_t are the magnetic and thermal diffusivities respectively. Equation (1) is derived by a procedure similar to the one used by Kremlev (1974).

Equation (1) must have four boundary conditions. If we assume that the external magnetic field is constant during the jump, we have $\partial H(\pm b)/\partial t = cE'(\pm b) = 0$. There are two more boundary conditions provided by the surface cooling equations $\kappa T'(\pm b) \pm hT(\pm b) = 0$ (h is the heat transfer coefficient). The function $j_c(x)$ changes its sign at $x = \delta b (-1 \leqslant \delta \leqslant 1$, see figure 1) and equation (1) is solved in the two independent regions $x < \delta b$ and $x > \delta b$. Then the matching conditions at $x = \delta b$ have to be formulated (Mints and Rakhmanov 1975) as

$$T(\delta b + 0) = T(\delta b - 0) \qquad T'(\delta b + 0) = T'(\delta b - 0) \qquad E_0(\delta \pm 0) = 0.$$

If E_0 is substituted into the boundary and matching relations the dependence $\lambda = \lambda(\beta, \tau, \delta, \ldots)$ may be found from the requirement that non-trivial solutions of equation (1) should exist. The critical state is unstable if $\lambda(\beta, \tau, \delta, \ldots) > 0$.

The dependence of the coefficients of equation (1) on x arises because the dependence of the critical current on the magnetic field is not neglected. Therefore, these coefficients vary by a considerable amount over the length l which may be defined as

$$l^{-1} = \frac{1}{j_c} \left| \frac{\mathrm{d}j_c}{\mathrm{d}x} \right| = \frac{1}{j_c} \left| \frac{\partial j_c}{\partial H} \cdot \frac{\mathrm{d}H}{\mathrm{d}x} \right| = \frac{4\pi}{c} \left| \frac{\partial j_c}{\partial H} \right| = \frac{\alpha}{b}.$$

If $l \gg b$ (or $\alpha \ll 1$) equation (1) contains slowly varying coefficients and may be solved by the WKBJ method.† As usual we shall use E in the form $E_0 = \exp[\mathrm{i}f(x/b)]$, where $f(x/b)$ is a smoothly varying function. By substitution of this expression into equation (1) one can find (to the first approximation with respect to $\alpha(x) \ll 1$)

$$f(x/b) = \pm 1/b \int [\![\{-\tfrac{1}{2}\lambda(1+\tau) \pm [\tfrac{1}{4}\lambda^2(1-\tau)^2 + \lambda\beta]^{1/2}\}^{1/2} + \ldots]\!]\, \mathrm{d}x. \qquad (2)$$

Note that the WKBJ solution obviously coincides with the accurate one for case of $\alpha = 0$ (Bean's critical state model, Bean 1964).

We can use solution (2) and the boundary and matching conditions to find the equation for $\lambda(\beta, \tau, \delta, \ldots)$ from the requirement of the existence of the non-trivial solution E_0; e.g. for $h=0$ (adiabatic insulation) it is easy to obtain

$$K_2{}^3(0)\tan\left(\frac{1}{b}\int_0^b K_2(x)\,\mathrm{d}x\right) = K_1{}^3(0)\tanh\left(\frac{1}{b}\int_0^b K_1(x)\,\mathrm{d}x\right) \qquad (3)$$

where

$$K_{1,2}(x) = [\pm\tfrac{1}{2}\lambda(1+\tau) + (\tfrac{1}{4}\lambda^2(1-\tau)^2 + \lambda\beta(x))^{1/2}]^{1/2}.$$

(For simplicity $\delta = 0$.) Equation (3) is a generalization of the expression obtained by Kremlev for $\alpha = 0$.

The curves $\lambda = \lambda(\beta_e)$ where $\beta_e = \beta(H_e)$ for different τ have the form shown in figure 2; this can be demonstrated by equation (3). In particular, if we have determined the value λ_c (see figure 2), we may evaluate the building-up time of the jump as $t_j \sim \nu b^2/\lambda_c\kappa$ (Mints and Rakhmanov 1975). The stability is broken down by 'fast' (or adiabatic) perturbations ($\lambda_c \gg 1$) in any critical state model if $\tau \ll 1$ (which applies to hard superconductors).

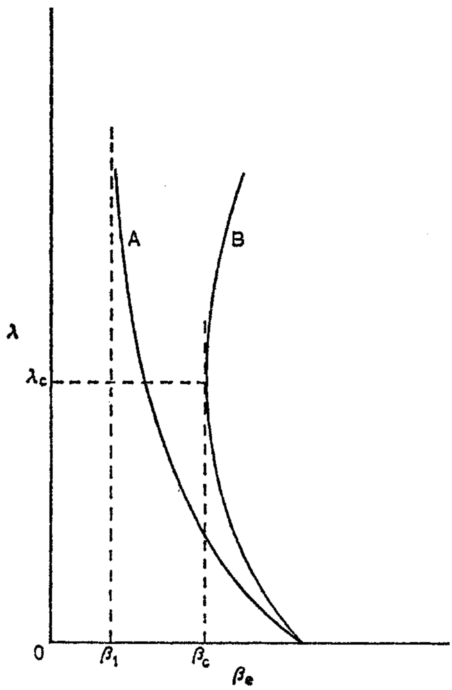

Figure 2. The functions λ (β_e) for different τ. Adiabatic boundary conditions. A, $\tau = 0$; B, $\tau > 0$.

† The WKBJ solution for nth-order linear differential equations had been obtained by Heading (1961) and the proof of its validity has been demonstrated by Wasow (1962).

3. Simplified scheme

As has been shown, the flux jumping is an adiabatic process for $\tau \ll 1$, and the general equation (1) may be reduced to a second-order one (Kremlev *et al* 1976)

$$E_0'' + \alpha(x)\, E_0' + \beta(x)\, E_0 = 0. \tag{4a}$$

This equation requires only electrodynamical boundary conditions. The equilibrium will be violated if a non-trivial solution of equation (4*a*) exists with boundary conditions $E_0'(\pm 1) = E_0(\delta) = 0$.

Let us introduce a new variable y (see figure 1):

$$y = \frac{H_e - H(x)}{H_e - H_i}$$

and equation (4*a*) becomes

$$\frac{d^2 E_0}{dy^2} + \tilde{\beta} E_0 = 0 \tag{4b}$$

where

$$\tilde{\beta} = \frac{(H_e - H_i)^2}{4\pi\nu T_0(H)} \qquad T_0(H) = \frac{j_c}{|\partial j_c/\partial T|}.$$

The boundary conditions may be rewritten as $dE_0(0)/dy = E_0(1) = 0$.

The value T_0 does not depend on H if the critical current density may be represented as $j_c(T, H) = j_0(T)\,\phi(H)$ and $\tilde{\beta}$ = constant. Equation (4*b*) may be solved explicitly and the corresponding stability criterion easily found:

$$H_e - H_i \leq H_j = \left(\frac{\pi^3 \nu j_0(T)}{|\partial j_0/\partial T|} \right)^{1/2}.$$

This expression coincides with the criterion obtained for the Bean model ($\alpha = 0$, Swartz and Bean 1968).

In a more general case, equation (4*b*) may be solved by means of the usual WKBJ method (Heading 1962) provided $d\tilde{\beta}^{1/2}/dy < \tilde{\beta}^{1/2}$ or

$$\frac{H_e - H_i}{T_0(H)} \left| \frac{dT_0(H)}{dH} \right| < 1.$$

The WKBJ solution and the boundary conditions can be used to find the subsequent stability criterion:

$$\int_0^1 \tilde{\beta}^{1/2}\, dy = \frac{1}{(4\pi\nu)^{1/2}} \int_{H_i}^{H_e} \left(\frac{1}{j_c} \left| \frac{\partial j_c}{\partial T} \right| \right)^{1/2} dH = \int_{\delta b}^{b} (\beta(x))^{1/2}\, dx < \frac{\pi}{2} + \epsilon. \tag{5}$$

The theory of the WKBJ method allows us to evaluate the accuracy ϵ of the latter expression as

$$|\epsilon| \sim \frac{1}{\pi^2} (H_e - H_i)^2 \left(\frac{1}{T_0(H)} \frac{dT_0}{dH} \right)^2.$$

As an illustration let us consider the critical-state model where the dependency of j_c on H has the following form:

$$j_c = j_0(T)\,(1 - h)^{\mu} \tag{6}$$

where

$$h = H/H_{c2}(T).$$

Substitution of expression (6) into the inequality (5) gives

$$\frac{H_{c2}}{(4\pi\nu T_1)^{1/2}} \int_{h_0}^{h_e} \left(1 + t\,\frac{h}{1-h}\right)^{1/2} \mathrm{d}h < \frac{\pi}{2}. \tag{7}$$

Here

$$\mathrm{d}j_0/\mathrm{d}T = -j_0/T_1 \qquad \mathrm{d}H_{c2}/\mathrm{d}T = -H_{c2}/T_2$$

$$t = \mu T_1/T_2 \qquad h_e = h(H_e) \qquad h_i = h\,(H_i).$$

The integral on the left-hand side of inequality (7)—$\Psi(h_e)$ is shown in figure 3 for different values of h_0 and t. Bean's critical-state model corresponds to $t=0$. The stability may be significantly affected by the dependency of j_c on H in the high-fields regions; this follows from the results shown in figure 3.

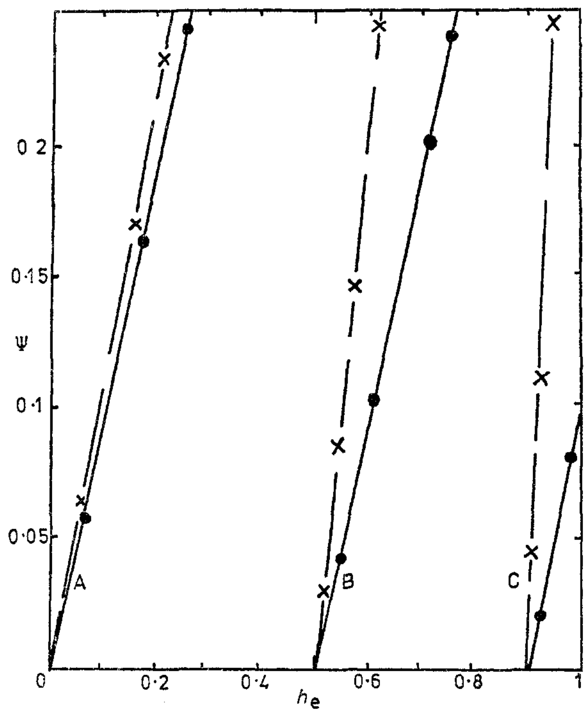

Figure 3. The function ψ (h_e) for different t and h_0. ● $t=0$; × $t=1$. A, $h_0=0$; B, $h_0=0{\cdot}5$; C, $h_0=0{\cdot}9$.

The criterion (5) is evidently valid for non-uniform samples with smoothly varying properties; in particular, it follows from (5) that the use of such materials does not lead to considerable improvement in stability. The WKBJ method gives us an opportunity to find out the stability criteria for samples of different geometrical configurations. For example, a cylindrically symmetrical sample is considered in the paper by Mints and Rakhmanov (1975b). The qualitative dependence of the stability criterion upon sample geometry is found to be just the same as for the Bean model.

The disturbances with $\lambda \gg 1$ can not be caused by external cooling; hence, the instability at $\lambda = \lambda_c \gg 1$ (if it can occur) is absolute from this point of view. For the superconductor clad with a normal metal the equilibrium may be violated by the perturbations with $\lambda = \lambda_c \lesssim 1$. But the instability region may be shifted up to $\lambda_c \gg 1$ (absolute instability) by effects such as external cooling and increasing the normal layer thickness. The instability will pass through two stages in this case. Firstly, the flux redistributes rapidly inside the superconductor (the jump, $\lambda = \lambda_c \gg 1$). Secondly the external magnetic flux sweeps into the sample (the rate of the process is determined by the normal metal parameters). This was experimentally observed by Onishi (1974). For the situation in question, the boundary conditions are $E(\pm b) = 0$ (during the jump), as at $\lambda \gg 1$ the electric field E decays inside the normal layer (skin effect) and the term $\frac{1}{2}\pi$ on the right-hand side of inequality (5) has to be changed by π as can be seen readily from the preceding calculations.

The instability is always absolute in the absence of the normal cladding.

4. Conclusions

In this paper the equations describing the initial stage of magnetic instability in hard superconductors are solved by the WKBJ method. In the process, the stability criterion valid for a wide class of functions $j_c = j_c(H)$ is obtained and general evidence is given of the adiabatic nature of flux jumping in hard superconductors.

This method allows us to carry out the stability investigations taking into account effects such as the transport current, sample geometry, magnetic and thermal diffusivities, cladding of the superconductor with a normal metal and surface cooling in the same manner as has been done for the Bean model (i.e. at $\partial j_c / \partial H = 0$). The method may be applied to the investigation of superconducting composites and materials with smoothly varying properties.

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