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ON THE ENERGY SPECTRUM OF EXCITATIONS IN TYPE-II SUPERCONDUCTORS

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The quasi-classical approximation has been applied in order to study the energy spectrum of the bound excitations with the radial quantum number $n \neq 0$ in the isolated vortex line core in a pure type-II superconductor. The qualitative estimations of the number of levels and the numeric calculations of the spectrum for the special form of potentials $A_\theta(r)$ and $\Delta(r)$ have been made.

THE PURPOSE of this work is to study the energy spectrum of excitations in the core of vortex in a pure type-II superconductor. The external magnetic field is considered to be relatively weak (the case of isolated vortex). It is convenient for this aim to make use of Bogoliubov–de Gennes equations[1] for the excitation wave function $\Psi = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{pmatrix} \mathcal{H}_e & \Delta \\ \Delta^* & -\mathcal{H}_e^* \end{pmatrix} \Psi = \epsilon \Psi$$

where the $\mathcal{H}_e = (1/2m)[-i\hbar\nabla - (e/c)\mathbf{A}]^2 - \epsilon_F$ is a one-electron Hamiltonian in the presence of magnetic field, described by a vector potential $\mathbf{A}$ and $\Delta(r)$ is a pair potential. In the cylindrical coordinates r, θ and z and in a gauge for which Δ is real, the magnetic field is described by a vector potential $A_z = A_r = 0$, $A_\theta = A_\theta(r)$ and $A_\theta(r) \to 0$ as r goes to infinity ($\Delta(r) \to \Delta_0$ as $r \to \infty$).

We may express the wave function in the form:

$$\Psi = \mathrm{f} \exp \{ik_F z \cos\alpha + i\mu\theta - i\sigma_z \theta/2\} \qquad (1)$$

where $k_F \cos\alpha$ is the component of wave vector in the z direction coinciding with the vortex line axis, μ (the magnetic quantum number) is half an odd integer,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The equation for f is the following:

$$\frac{\hbar^2}{2m}\sigma_z \left\{-\frac{d^2 \mathrm{f}}{dr^2} - \frac{1}{r}\frac{d\mathrm{f}}{dr} + \left[\mu - \frac{\sigma_z e r}{\hbar c}A_\theta(r)\right]^2 \frac{\mathrm{f}}{r^2} - k_F^2 \sin^2\alpha\, \mathrm{f}\right\} + \sigma_x \Delta(r)\mathrm{f} = \epsilon \mathrm{f} \qquad (2)$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The states with $\epsilon < \Delta_0$ are bound. Each of them is defined by the radial quantum number n, the magnetic quantum number μ and the angle α. The level with $n = 0$ was first obtained in reference 2. The states with $n \neq 0$ will be considered in our work.

One may see qualitatively the existence of states with $n \neq 0$ and the appropriate region of angles α already from the results of work in reference 2. The level with $n = 0$ was calculated in reference 2 by means of matching the phase of wave function. This phase contains an arbitrary item πn (where n is an integer). Using the last circumstance and following reference 2 we have

$$\epsilon_n(\mu, \alpha) = \Delta_0 \left\{\frac{\pi\mu d}{2\sin\alpha}\frac{\Delta_0}{\epsilon_F} + \pi n\sqrt{d \sin\alpha}\right\} \qquad (3)$$

$n = 0, 1, 2, 3, \ldots, d = (\zeta/\Delta_0)(d\Delta/dr)_{r=0} \sim 1$ (ζ is a coherence distance).

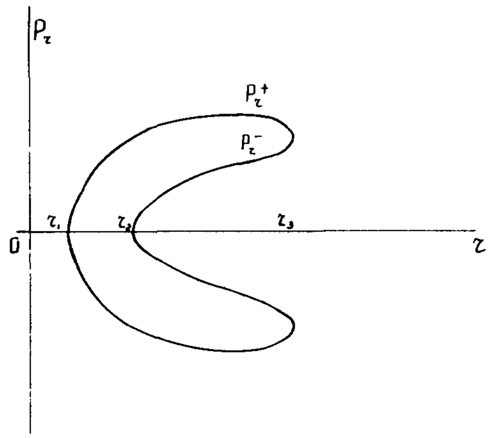

FIG. 1. The inherent form of the phase path of the excitation with $\epsilon < \Delta_0$.

From the expression (3) one can easily derive that $\epsilon_n > \Delta_0$ for $\sin\alpha \sim 1$, but for a small α the function $\epsilon_n(\mu, \alpha)$ has a minimum if

$$\alpha_m = \left(\frac{\mu\Delta_0}{\epsilon_F}\frac{\sqrt{d}}{n}\right)^{2/3}$$

with corresponding value of ϵ_n

$$\epsilon_{\min}(n) = \frac{3\pi}{2}\left[\mu\frac{\Delta_0}{\epsilon_F}(nd)^2\right]^{1/3}\Delta_0. \tag{4}$$

The quantity $\epsilon_{\min}(n)$ may be less than Δ_0 when μ and n are not too large. Nevertheless, the expression (3) represents only a qualitative relation for it has been derived as the first term in the development as a series in ϵ/Δ.[2] Generally, the assumption $\epsilon/\Delta \ll 1$ is not correct if $n \neq 0$.

In order to investigate the energy spectrum for $n \neq 0$ the quasi-classical approximation may be used.[4] For the problem in question the rule of quantization has the usual form:

$$\oint P_r(r)\mathrm{d}r = 2\pi\hbar(n + \gamma) \tag{5}$$

where $\gamma \sim 1$ is determined by the number of turning points,

$$P_r^{\pm} = \sqrt{-\frac{\hbar^2\mu^2}{r^2} + \hbar^2 k_F^2 \sin^2\alpha \pm 2m\sqrt{(\epsilon+\rho)^2 - \Delta^2(r)}} \tag{6}$$

is a classical momentum of excitation and

$$\rho = \frac{\mu e\hbar}{mcr}A_\theta(r).$$

Inherent phase path $P_r = P_r(r)$ for $\epsilon < \Delta_0$ is drawn in Fig. 1. According to reference 4 $\gamma = 1/2$ if the number of the turning points is even. This approximation is applicable for $n \neq 0$, $\mu^2 \gg 1$ and the accuracy of the method is known to be $1/\pi^2 n$.

To evaluate the number of levels let ϵ in (6) be equal to Δ_0. Then $r_3 \to \infty$ but $A_\theta(r) \to 0$ and $\Delta(r) \to \Delta_0$ as r increases, the integral in (5) converges rapidly and the contribution from the region $r > \zeta$ to (5) is negligible.

Suppose that $\mu \ll \epsilon_F \sin\alpha/\Delta_0$. Then the region $r_1, r_2 < r < \zeta$ mainly contributes to the integral in (5) and $n\hbar \sim (P_r^+ - P_r^-)\zeta$. One may develop the square root (6) as a series in $(\hbar^2 k_F^2 \sin^2\alpha)^{-1}$ if $\sin^2\alpha \gg \Delta_0/\epsilon_F$. And for n we have:

$$n \sim 1/\alpha. \tag{7}$$

For $\alpha^2 \lesssim \Delta_0/\epsilon_F$ the value of difference $P_r^+ - P_r^-$ is of the order of each P_r and therefore $\sim \hbar k_F \alpha$. One may obtain in this case:

$$n \sim \frac{\epsilon_F}{\Delta_0}\alpha. \tag{8}$$

For $\mu \gg \epsilon_F \sin\alpha/\Delta_0 (r_1 \sim r_2 \sim \mu/k_F \sin\alpha > \zeta)$ the integral in (5) vanishes exponentially and the levels with $n \neq 0$ disappear. From the preceding discussion it is clear that the number of levels has a maximum

$$n_{\max} \sim \sqrt{\frac{\epsilon_F}{\Delta_0}} \gg 1 \tag{9}$$

for

$$\alpha \sim (\Delta_0/\epsilon_F)^{1/2}.$$

As the illustration of presented above speculations the computation of energy spectrum was carried out by means of quasi-classical method with the potentials being chosen in the form[3]

$$A_\theta(r) = \frac{\hbar c}{2er}\frac{1}{\cosh\left(\frac{ar}{\zeta}\right)}; \quad \Delta(r) = \Delta_0 \tanh\left(\frac{dr}{\zeta}\right) \tag{10}$$

for various a and d. The results of calculation for $a = 0.4$ and $d = 1.2$ are shown in Figs. 2 and 3. They are in agreement with qualitative estimations described above and practically do not change their values if to omit the factor $[\cosh(ar/\zeta)]^{-1}$ in the expression (10).

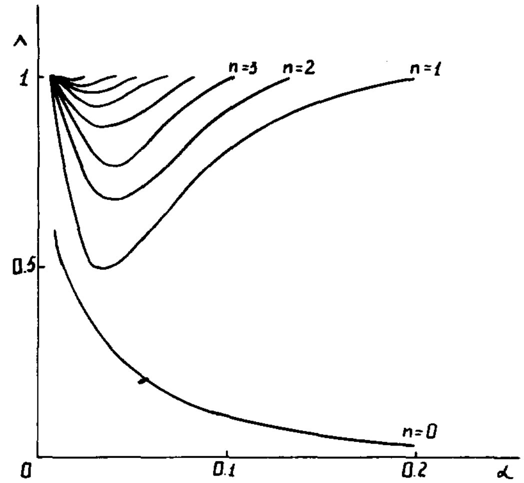

FIG. 2. The energy of the excitations as a function of α for various $n - \epsilon_n(\alpha)$. $d = 1.2, a = 0.4, \mu\Delta_0/\epsilon_F \ll 1$, $\epsilon_F/\Delta_0 = 10^3$.

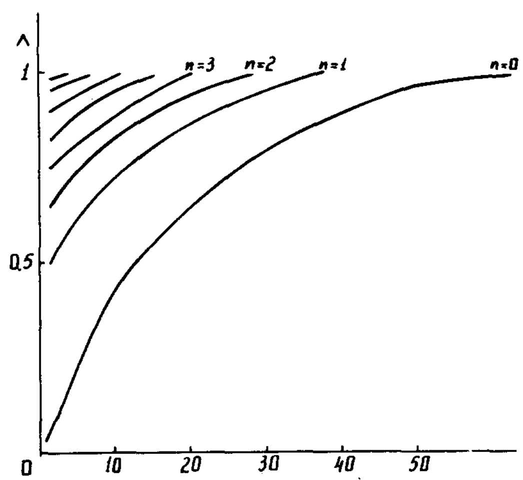

FIG. 3. The dependence of the energy of excitations upon $\mu - \epsilon_n(\mu)$. $d = 1.2, a = 0.4, \alpha \gg \Delta_0/\epsilon_F, \epsilon_F/\Delta_0 = 10^3$.

Now consider the effect of value d on the results in question. With increased Ginsburg–Landau parameter $\mathcal{K}$ d decreases,[3,5] which obviously leads to the growth of the number of levels and to the decrease of minimum $\epsilon_n(\alpha)$ (see Fig. 2). With diminishing temperature, d increases,[5] so the number of levels reduces. However the qualitative picture and numerical results do not change significantly up to $T/T_c \sim 0.05$, for low temperatures the quasi-classical consideration seems to be not applicable.

The procedure analogous to reference 2 can be carried out for the comparison with the method WKBJ

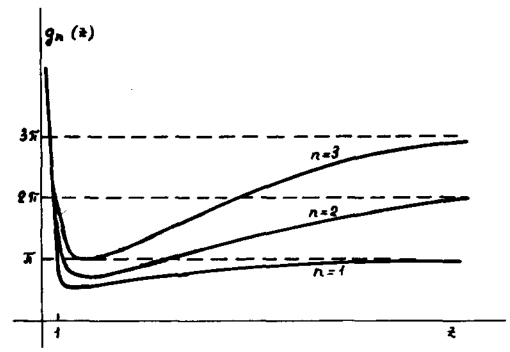

FIG. 4. The appearance of functions $g_n(z)$.

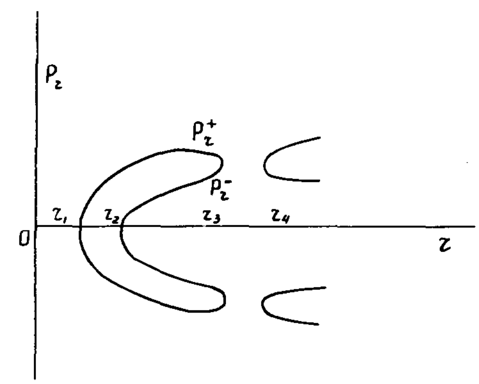

FIG. 5. The inherent phase path of excitation with $\epsilon > \Delta_0$ and $\mu < 0$.

if it is possible ($n \sim 1, \epsilon < \Delta$). To avoid the limitation $\epsilon/\Delta \ll 1$ one may develop the method of work[2] up to the higher order of ϵ/Δ. The procedure appears to be convergent if

$$\frac{\alpha^{3/2}}{\mu\sqrt{d}}\frac{\epsilon_F}{\Delta_0} > 1.$$

The calculations using the approximation mentioned above can be performed explicitly to give

$$\epsilon_n(\mu, \alpha) = \Delta_0\left\{\frac{\pi\mu d\Delta_0}{2\alpha\epsilon_F} + g_n\left(\frac{\alpha^{3/2}\epsilon_F}{\mu\sqrt{d}\,\Delta_0}\right)\sqrt{d\alpha}\right\}. \quad (11)$$

The qualitative behavior of $g_n(z)$ is shown in Fig. 4.

The iterative procedure converges rapidly in the vicinity of minimum of $\epsilon_n(\mu, \alpha)$ for little n. In this region we have calculated the spectrum by means of matching using the potentials in the form (10). The difference between results obtained with the help of quasi-classical approximation and latter on is not more than 10 per cent given the accuracy of both methods.

At last a few words about excitations with $\epsilon > \Delta_0$ and $\mu < 0$. The inherent path of $P_r(r)$ is shown in Fig. 5. The existence of a closed part of the curve $P_r(r)$ means the possibility of arising quasi-discrete states (if $\oint P_r(r)\,dr = 2\pi\hbar(n + \frac{1}{2})$ for the closed part of curve in Fig. 5). The quasi-discrete levels result in the peaks in the density of states. Unfortunately, the quantitative assessment using the potentials (10) has pointed out the quasi-classical approximation not to be applicable in this case. To this time we have no definite answer on the question.

The states with $n \neq 0$, for example, are responsible for arising some additional peaks in the resonance electromagnetic radiation and ultrasonic absorption spectrums of appropriate frequencies.

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