Computational Learning Theory	Spring Semester, 2005/6

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Lecturer:Yishay Mansour *Scribe:Hila Pochter, Yaakov Hoch, Omry Tuval* [1]

4.1 Lecture overview

In this lecture we will concern ourselves with the existence and price of Nash equilibrium in several game classes. We will:

- Define a class of games called *congestion games* and we'll show the existence of a pure Nash Equilibirum in any congestion game.
- Define a class called *potential games* and we'll study the existence of pure equilibrium in those games. (Actually the two classed are equivalent)
- Study two variants of a *Network Creation* game (unfair and fair), and study the price of anarchy (when a Nash equilibrium exists)
- Define the Price of Stability (PoS) and analyze the PoS in a Network Creation game
- Define a *Bandwidth Sharing* game and discuss the equilibria of this game.

4.2 Congestion Games

4.2.1 Example

Let us start with an illustrative example of a congestion game. Players A,B and C have to go from point S to T using road segments SX,XY,...etc. (See Figure 4.1) Numbers on edges denote the cost for a single user for using the corresponding road segment, where the actual cost is a function of the actual number of players using that road segment(i.e. a *discrete delay* function). For example: if segment SX is used by a 1,2, or 3 users, the cost on that segment would be 2,3, or 5, respectively. The total cost for a player is the sum of the costs on all segments he uses.

[1]Partially based on 2004 scribe notes by Nir Yosef and Ami Koren

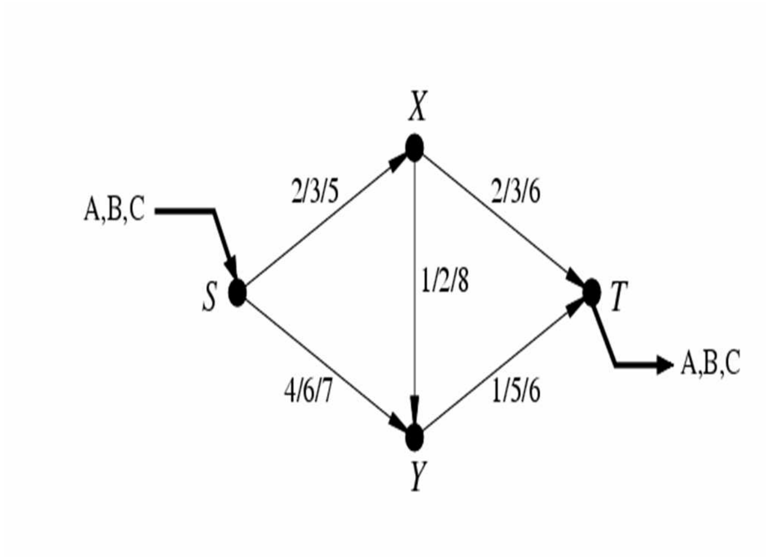

Figure 4.1: Example of a congestion game

4.2.2 Congestion game - Definition

A congestion model (N,M,$(A_i)_{i \in N}$,$(c_j)_{j \in M}$) is defined as follows:

- $N = \{1..n\}$ denotes the set of n players.
- $M = \{1..m\}$ denotes the set of m facilities.
- For $i \in N$, let A_i denotes the set of strategies of player i, where each $a \in A_i$ is a non empty subset of the facilities.
- For $j \in M$, $c_j \in R^n$ denotes the vector of costs, where $c_j(k)$ is the cost related to facility j, if there are exactly k players using that facility.

Let $A = \times_{i \in N} A_i$ be the set of all possible joint actions. For any $\vec{a} \in A$ and for any $j \in M$, let $n_j(\vec{a})$ be the number of players using facility j, assuming $\vec{a}$ is the current joint action, i.e. $n_j(\vec{a}) = \mid \{i \mid \ M_j \in a_i\} \mid$. The cost function for player i is $u_i(\vec{a}) = \sum_{j \in a_i} c_j(n_j(\vec{a}))$.

Remark 4.1 *How can* Routing *with unspllitable flow be modeled as a congestion game? The facilities are the edges $M = E$, the possible strategies are the possible routes for player i: $A_i = P_i$, and the cost of the edge is the latency, i.e. $c_e(k) = l_e(k)$*

4.2.3 Deterministic equilibrium

Theorem 4.2 *Every finite congestion game has a pure Nash equilibrium.*

Proof: Let $\vec{a} \in A$ be a joint action.
Let Φ: $A \to R$ be a potential function defined as follows: $\Phi(\vec{a}) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(\vec{a})} c_j(k)$
Consider the case where a single player changes its strategy from a_i to b_i (where $a_i, b_i \in A_i$).
Let Δu_i be the change in its cost caused by the the change in strategy:
$\Delta u_i = u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}}) = \sum_{j \in b_i - a_i} c_j(n_j(\vec{a}) + 1) - \sum_{j \in a_i - b_i} c_j(n_j(\vec{a}))$.
(explanation: change in cost = cost related to the use of new facilities minus cost related to use of those facilities which are not in use anymore due to strategy change)
Let $\Delta\Phi$ be the change in the potential caused by the change in strategy:
$\Delta\Phi = \Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) = \sum_{j \in b_i - a_i} c_j(n_j(\vec{a}) + 1) - \sum_{j \in a_i - b_i} c_j(n_j(\vec{a}))$
(explanation: immediate from potential function's definition).
Thus we can conclude that for a single player's strategy change we get $\Delta\Phi = \Delta u_i$.

That's an interesting result: We can start from an arbitrary joint action $\vec{a}$, and at each step let one player reduce it's cost. That means, that at each step Φ is reduced (identically). Since Φ can accept a finite amount of values, it will eventually reach a local minima. At this point, no player can achieve any improvement, therefore we reach a Nash equilibrium. □

Remark 4.3 Φ *is actually an* exact potential function *as we will define shortly.*

4.2.4 Weighted Congestion Game

The previous theorem showed that a congestion game always has a pure equilibrium.
What about *Weighted Congestion games*, where the load on each facility caused by different players is different?
In this case each player is assigned a non negative weight w_i and the cost of a facility j is $c_j(\sum_{i|M_j \in a_i} w_i)$. We'll see that a pure equilibrium does not necessarily exists. Let's consider the following example:

Two Players wish to choose a route $s - t$, each has a weight $w_1 = 1, w_2 = 2$. The edge's discrete delay functions are as shown in the figure. A necessary condition for a pure equilibrium is that each player chooses a route that is in his *BestResponse* given the other player's chosen route. That is, $a_1 \in BR_1(a_2)$ and $a_2 \in BR_2(a_1)$
In this example there are only four $s - t$ routes, and by going over all 4 options for a_i it is easy to see that the two necessary conditions can not hold at the same time, and therefore in this example there is no pure equilibrium.

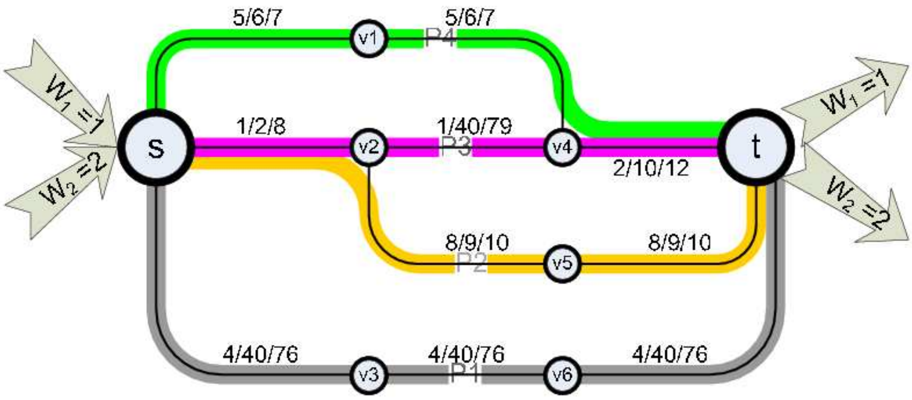

Figure 4.2: Weighted Congestion Game - no PNE. Taken from [1]

4.3 Potential games

4.3.1 Potential functions

Let $G =< N, (A_i), (u_i) >$ be a game where $A = \times_{i\in N} A_i$ is the collection of all deterministic strategy vectors in G.

Definition A function Φ: $A \rightarrow R$ is an *exact potential* for game G if
$\forall_{\vec{a}\in A}\forall_{a_i,b_i\in A_i}\ \Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) = u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}})$

Definition A function Φ: $A \rightarrow R$ is a *weighted potential* for game G if
$\forall_{\vec{a}\in A}\forall_{a_i,b_i\in A_i}\ \Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) = \omega_i(u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}})) = \omega_i\Delta u_i$
Where $(\omega_i)_{i\in N}$ is a vector of positive numbers (weight vector).

Definition A function Φ: $A \rightarrow R$ is an *ordinal potential* for a minimum game G if
$\forall_{\vec{a}\in A}\forall_{a_i,b_i\in A_i}\ (u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}}) < 0) \Rightarrow (\Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) < 0)$ (Intuition: when a player decreases his cost, the potential function also decreases.).

Remark 4.4 *Considering the above definitions, it can be seen that the first two definitions are special cases of the third.*

4.3.2 Potential games

Definition A game G is called an *ordinal potential game* if it has an an ordinal potential function.

Theorem 4.5 *Every finite ordinal potential game has a pure equilibrium.*

Proof: Analogous to the proof of Theorem 4.2: Given an initial strategy vector, each time a player changes strategy and reduces his cost, the potential function also decreases. since this is a finite game, the potential function can have a finite set of values and therefore the process of successive improvements by players must reach a local minima of the potential function. No improvements (by any player) are possible at this point, and therefore this is a pure equilibrium. □

4.3.3 Examples

Exact potential game

Consider an undirected graph $G = (V, E)$ with a weight function $\vec{\omega}$ on its edges. In this game the players are the vertices and the goal is to partition the vertices set V into two distinct subsets D_1, D_2 (where $D_1 \cup D_2 = V$):
For every player i, choose $s_i \in \{-1, 1\}$ where choosing $s_i = 1$ means that $i \in D_1$ and $s_i = -1$ means that $i \in D_2$. The weight on each edge denotes how much the corresponding vertices 'want' to be on the same set. Thus, define the value function of player i as $u_i(\vec{s}) = \sum_{j \neq i} \omega_{i,j} s_i s_j$. (A player 'gains' $\omega_{i,j}$ for players that are in the same set with him, and 'loses' for player in the other set. Note that $\omega_{i,j}$ can be negative.) Each player tries to maximize its utility function.

On the example given in Figure 4.3 it can be seen that players 1,2 and 4 have no interest in changing their strategies, However, player 3 is not satisfied, it can increase his profit by changing his set to D_1.
Using $\Phi(\vec{s}) = \sum_{j<i} \omega_{i,j} s_i s_j$ as our potential function, let us consider the case where a single player i changes its strategy (shifts from one set to another):
$\Delta u_i = \sum_{j \neq i} \omega_{i,j} s_i s_j - \sum_{j \neq i} \omega_{i,j} (-s_i) s_j = 2 \sum_{j \neq i} \omega_{i,j} s_i s_j = \Delta(\Phi)$
Which means that Φ is an exact potential function, therefore we conclude that the above game is an exact potential game.

Remark 4.6 *Any congestion game (as defined earlier) is an exact potential game. The proof of Theorem 4.2 is based on this property of congestion games.*

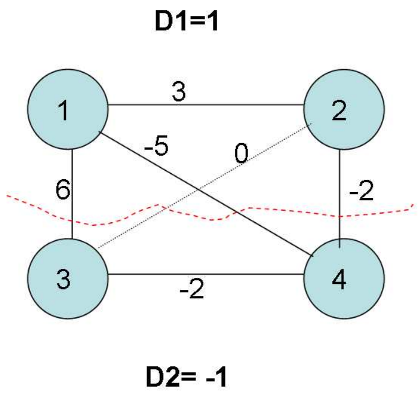

Figure 4.3: Example for an exact potential game

Weighted potential game

Consider the following load balancing congestion model $(N, M, (\omega_i)_{i\in N})$ with M identical machines, N jobs and $(\omega_i)_{i\in N}$ weight vector($\omega_i \in R^+$). The load on a machine is defined as the sum of weights of the jobs which use it: $L_j(\vec{a}) = \sum_{i:\ a_i=j} \omega_i$ where $\vec{a} \in [1..M]^N$ is a joint action.
Let $u_i(\vec{a}) = L_{a_i}(\vec{a})$ denote the cost function of player i. We would like to define a potential function whose change in response to a single player's strategy change will be correlated with the change in the player's cost function.
The potential function is defined as follows: $\Phi(\vec{a}) = \sum_{j=1}^{M} \frac{1}{2} L_j^2$, Consider the case where a single job shifts from its selected machine M_1 to another machine M_2 (where M_1 and M_2 are two arbitrary machines):
Let Δu_i be the change in its cost caused by the strategy change:
$\Delta u_i = u_i(M_2, \vec{a_{-i}}) - u_i(M_1, \vec{a_{-i}}) = L_2(\vec{a}) + \omega_i - L_1(\vec{a})$.
(Explanation: change in job's load = load on new machine minus load on old machine)
Let $\Delta\Phi$ be the change in the potential caused by the strategy change:
$\Delta\Phi = \Phi(M_2, \vec{a_{-i}}) - \Phi(M_1, \vec{a_{-i}}) = \frac{1}{2}[(L_1(\vec{a}) - \omega_i)^2 + (L_2(\vec{a}) + \omega_i)^2 - L_1^2(\vec{a}) - L_2^2(\vec{a})] =$
$= \omega_i(L_2(\vec{a}) - L_1(\vec{a})) + \omega_i^2 = \omega_i(L_2(\vec{a}) + \omega_i - L_1(\vec{a})) = \omega_i \Delta u_i$
Therefore, we can conclude that load balancing on identical machines is a weighted potential game.

4.3.4 Finite Improvement path

Let's consider a finite game G as a directed graph, where the vertices are strategy vectors, $V = A$ and there is an edge between vertices when it is an improvement step.
An improvement step is a change from $\vec{a} \in A$ to $\vec{b} \in A$, where $\vec{a}$ and $\vec{b}$ differ only in the

strategy of a single player i and $\Delta u_i < 0$

Remark 4.7 *In such a graph a pure equilibrium is a sink.*

Lemma 4.8 *For every game G such that for every joint action $\vec{a} \in A$ there exists an improvement path ending in a pure equilibrium, there exists an* ordinal potential function *Φ.*

Proof: Let $\Phi(\vec{a})$ be the length of the longest possible improvement path in the game G starting from $\vec{a}$. The function Φ is well defined because of the property of G assumed in the lemma.
Consider an improvement step from $\vec{a_1} \in A$ to $\vec{a_2} \in A$. For contradiction assume that $\Phi(\vec{a_2}) \geq \Phi(\vec{a_1})$. Therefore from $\vec{a_2}$ there exists an improvement path of length $1 + \Phi(\vec{a_2})$ which is a contradiction to $\Phi(\vec{a_1})$ being the *longest* improvement path starting from $\vec{a_1}$. This shows that $\Phi(\vec{a_2}) < \Phi(\vec{a_1})$, and that means Φ is an *ordinal potential function.* □

4.4 Network Creation Game

We have a graph G=(V,E). Each edge e has a price $C(e)$. Each player i has two nodes s_i and t_i that he wants to connect. Each player i offers $p_i(e)$ for the edge e. Let's denote by p joint action of the players, and $G(p) = (V, E_p)$ is the graph resulting from the players' strategies, where $e \in E_p$ iff $\sum_i p_i(e) \geq C(e)$. Player i's cost function $C_i(p)$ is equal to ∞ if s_i and t_i are not connected and otherwise it is $\sum_{e \in E_p} p_i(e)$. The player's aim is to minimize this cost (yet to have s_i connected to t_i) We define the social cost to be $C(p) = \sum_i c_i(p)$.

Remark 4.9 *Notice that in a Nash equilibrium the players will pay exactly the cost of each edge bought in G(p) and every one of them will have a path from s_i to t_i in G(p).*

Theorem 4.10 *A pure Nash equilibrium does not always exist for the creation game*

Proof: Let's look at the following game in Figure 4.4.

- In every NE we will buy exactly 3 edges.
- Without loss of generality assume that the edges bought are $(s_1, s_2), (s_1, t_2), (t_1, s_2)$.
- Only player 1 pays for (s_2, t_1) (he's the only player who needs it).
- Only player 2 pays for (s_1, t_2) (he's the only player who needs it).
- Without loss of generality, suppose player 1 pays (at least ϵ) for (s_1, s_2).
  Player 1 can change his strategy and

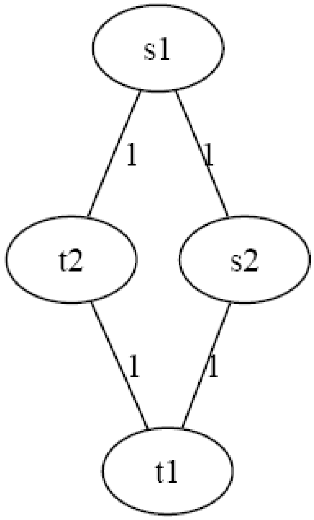

Figure 4.4: Creation Game with no Nash equilibrium (taken from [2])

  - not pay for (s_1, t_1) and (s_1, s_2) gaining $(1 + \epsilon)$.
  - buy (t_1, t_2) paying 1.

- Thus there is no pure Nash equilibrium

□

Remark 4.11 *In the above proof, the problem is that player 1 ignores the fact that in the resulting network player 2 has no motivation to continue paying for (s_1, t_2). This is a serious weakness of the Nash equilibrium concept: it ignores the fact that other players can and might react to a certain player changing his strategy.*

We now define a social cost function $C(p) = \sum c_i(p)$ and assume we are given a game in which there exists a Nash equilibrium.

1. $PoA \leq N$: Every player i, given the other players actions, the cost of connecting s_i to t_i is at most the cost of connecting them regardless of the other players. Which in turn is at most the total cost of the optimal solution. So every player pays at most OPT and the total cost in a NE is at most $N \cdot OPT$.

2. $PoA \geq N$: Consider Figure 4.5. This is a network of a single source single sink network creation game with N players. Assume that all the players wants to connect from s to t. In the social optimum solution all the players buy together the edge from s to t. Each player pays only $\frac{1}{N}$. The social cost in this case is 1. Now look at the worst

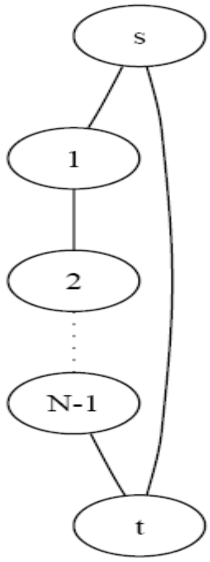

Figure 4.5: POA = N (taken from [2])

case Nash equilibrium. In this case each player buys one edge in the leftmost path , each player pays 1. (Pay attention that none of the players can gain by not buying an edge since then s and t won't be connected).The social cost in this case is N. Therefore $PoA \geq N$.

Definition [Price of Stability]

$$PoS = \min_{p \in PNE} \frac{C(p)}{OPT(p)}$$

In the previous example the PoS is 1 since the optimum is a Nash equilibrium . Now we will show a case in which the PoS is high. Consider Figure 4.6.

The social optimal cost is $1 + 3\epsilon$, when the players buy the leftmost path and 3 of the ϵ edges in the square on the right. However the lowest cost achieved in an equilibrium is $N - 2 + \epsilon$, when the players buy the two edges with cost $\frac{N}{2} - 1 - \epsilon$ and three ϵ edges in the right square. Note that there is no other Nash equilibrium due to the square on the right which we have shown that does not admit a Nash equilibrium.

4.5 Network Creation Game with fair cost

We will consider a modification of the previous game. Instead of allowing the players to directly set the cost, players will choose the edges and the cost will be divided equally

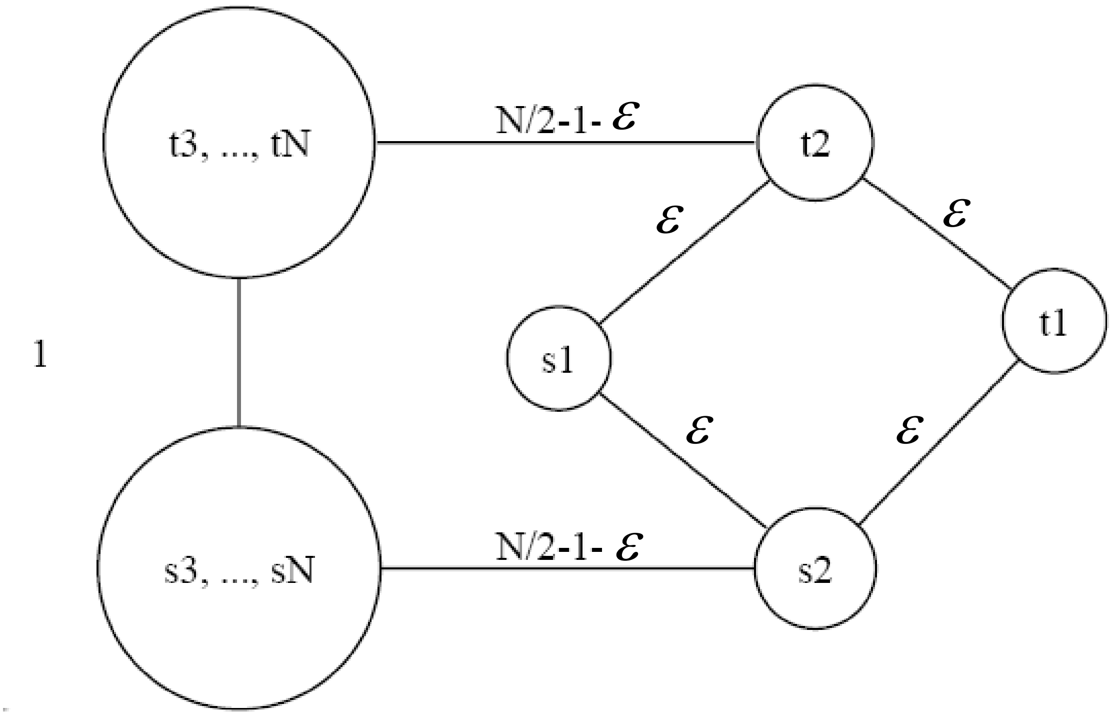

Figure 4.6: $PoS \approx N - 2$ (taken from [2])

between all players participating in an edge. The strategies for player i are $a_i \in A_i$ where $a_i \subseteq E$. Define

$$n_e(a) = |\{i \ : \ e \in a_i\}|$$

The cost of edge e to the player choosing it is $c_i(a) = \sum_{e \in a_i} \frac{c(e)}{n_e(a)}$
The social cost is $C(a) = \sum_i c_i(a)$
We can define the game as a congestion game with $c_e(k) = \frac{c(e)}{k}$.

$$u_i(a) = \sum_{e \in a} c_e(n_e(a))$$

since this is a congestion game there exists a pure Nash equilibrium !

Theorem 4.12

$$PoS \leq H(N) = \sum_{l=1}^{N} \frac{1}{l}$$

Proof:

$$\Phi(a) = \sum_{e \in E} \sum_{l=1}^{n_e(a)} \frac{c(e)}{l} = \sum_{e} c(e) \cdot H(n_e(a))$$

Consider $a^* = argmin_a \Phi(a)$ which is obviously a Nash equilibrium. Let a_{opt} be the optimal solution.

$$H(N) \cdot C(a_{opt}) \geq \sum_{e \in E_{opt}} c(e) \cdot H(n_e(a_{opt})) \geq \Phi(a_{opt}) \geq \Phi(a^*) \geq C(a^*)$$

□

Theorem 4.13

$$PoS \geq H(N) = \sum_{l=1}^{N} \frac{1}{l}$$

Proof:

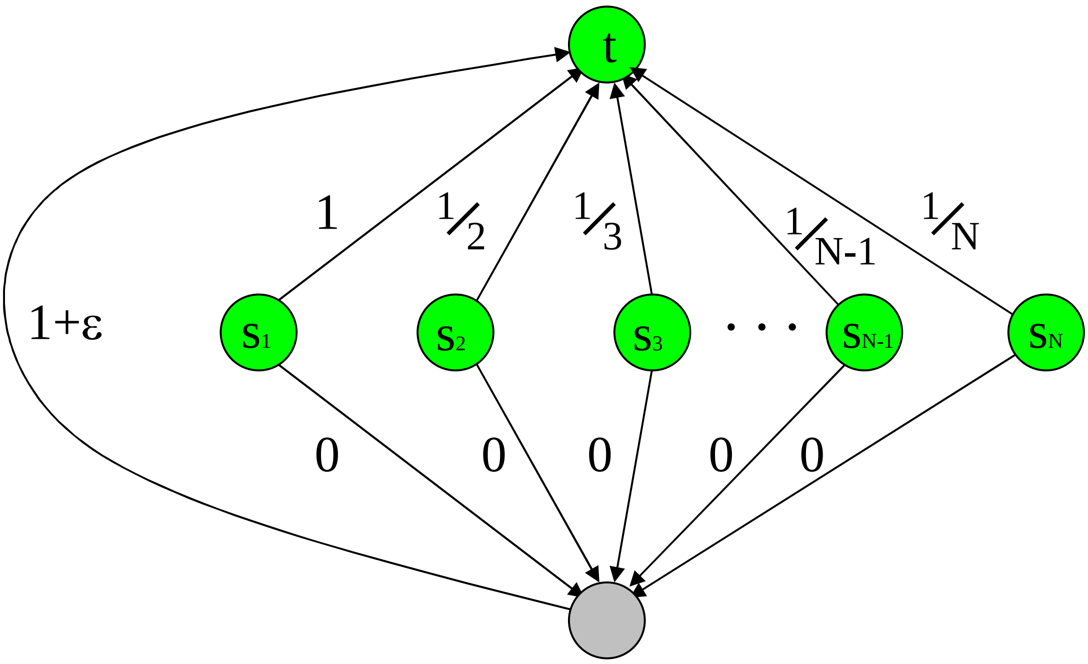

Figure 4.7: $PoS \geq H(N)$ (taken from [3])

Consider at the following game (Figure 4.7): This is a single target game with N players. In the social optimal solution each player would buy the 0 cost edge from him to the bottom node and the $1 + \epsilon$ edge (which its cost be shared equally between all the players).
The social cost in this case is $1 + \epsilon$. The only Nash equilibrium that exists in this case is the one in which each player i buys the $\frac{1}{i}$ edge from him to t, which gives us a social cost of $H(N)$. We will show that this is the only Nash equilibrium: Each player i has only 2 ways to connect s_i to t. Let's assume that a group Γ is connecting using the $1 + \epsilon$ edge. Let's i

be the player with the highest index in the group Γ. Player i would pay in this case $\frac{1+\epsilon}{|\Gamma|}$ and if he chooses to use the $\frac{1}{i}$ edge from him to t he would pay $\frac{1}{i}$. Since $\|\Gamma\| \leq i$, player i would rather use the $\frac{1}{i}$ edge. Therefore this is not an equilibrium. In this case $PoS \geq \frac{H(N)}{1+\epsilon}$ □

4.6 Bandwidth Sharing

We have a link of limited capacity C and N players who want a share of the bandwidth. Each user r has a specific utility function $U_r(d)$, which represents his satisfaction when he receives a bandwidth d. All utility functions $U_r(d)$ are assumed to be strictly monotone increasing, continuously differentiable non-negative and strictly concave.

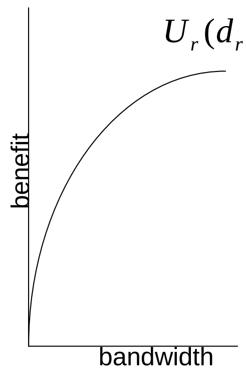

Figure 4.8: concave utility function

The optimal solution is

$$max \sum_{r \in N} U_r(d_r)$$
$$s.t. \sum_r d_r \leq C, d_r \geq 0$$

where d_r is the bandwidth allocated for the user r. In the social optimal solution we'll have exactly $\sum_r d_r = C$ because the utilities are strictly increasing, and so the optimal solution cannot have any leftover bandwidth. Furthermore, in the optimal solution we have

$$U'_s(d_s) = U'_r(d_r)$$

for all $d_r, d_s > 0$. Otherwise we could transfer bandwidth from one player to the other and increase the total utility.(For the sake of simplicity we will assume $U'_r(0) = \infty$ which ensures that $\forall r \ d_r > 0$).

4.6.1 Description of the Game

Player r pays w_r and receives $d_r = C\frac{w_r}{W}$,where $W = \sum w_i$. The utility function for each player is

$$Q_r(w^r, w^{-r}) = U_r(d_r) - w_r$$

In equilibrium we have

$$Q'_r = 0$$

since if $Q'_r > 0$ the player has an incentive to increase his payment w_r. Now we have

$$Q'_r = (U_r(\frac{w_r}{W} \cdot C) - w_r)' = U'_r(\frac{w_r}{W} \cdot C) \cdot C\frac{W - w_r}{W^2} - 1 = 0$$

Since $d_r = \frac{w_r}{W} \cdot C$ we have,

$$\begin{aligned} U'_r(dr) \cdot C \cdot (\frac{1}{W} - \frac{w_r}{W^2}) &= 1 \\ U'_r(dr)(1 - \frac{w_r}{W}) &= \frac{W}{C} \end{aligned}$$

Now by rearranging we have,

$$U'_r(dr)(1 - \frac{d_r}{C}) = \frac{W}{C}$$

We now define a new utility function

$$\hat{U}_r(d_r) = (1 - \frac{d_r}{C})U_r(d_r) + \frac{d_r}{C}\left[\frac{1}{d_r}\int_0^{d_r} U_r(z)dz\right]$$

Its derivative is,

$$\hat{U}'_r(d_r) = (1 - \frac{d_r}{C})U'_r(d_r) - \frac{1}{C}U_r(d_r) + \frac{1}{C}U_r(d_r) = (1 - \frac{d_r}{C})U'_r(d_r) = \frac{W}{C}.$$

Notice that an optimal solution to the problem with the utility function $\hat{U}_r(d_r)$ will be a Nash equilibrium with the original utility function (in the optimal solution all the derivatives are equal). Since the utility functions (and their derivatives) are concave, there is a single optimum. Since the derivative of $\hat{U}_r(d_r)$ is equal to

$$U'_r(dr)(1 - \frac{d_r}{C})$$

we have that the $U'_s(d_s)$ are equal for all players, which happens exactly when we are in the Nash equilibrium (remember that we have a unique NE)
Now let's try to find our the Price of Anarchy. (See Figure 4.9.)

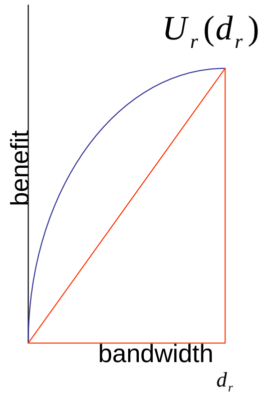

Figure 4.9: Notice the triangular area compared to the integrated area

It is easy to see that

$$\frac{1}{d_r}\int_0^{d_r} U_r(z)dz \geq \frac{1}{2}u_r(d_r)$$

Then we get

$$\hat{U}_r(d_r) \geq \frac{1-d_r}{C}U_r(d_r) + \frac{d_r}{C}\frac{1}{2}U_r(d_r) \geq \frac{1}{2}u_r(d_r)$$

which means that $PoA \leq 2$.
A better analysis shows that $PoA = \frac{4}{3}$.

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