Homogeneous Quasi-Continuous Sliding Mode Control

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1 Introduction

Control under heavy uncertainty conditions is one of the central topics of the modern control theory. The sliding-mode control approach [51, 53, 9] to the problem is based on the exact keeping of a properly chosen constraint by means of high-frequency control switching. Although very robust and accurate, the approach also features certain drawbacks. The standard sliding mode may be implemented only if the relative degree of the constraint is 1, i.e. control has to appear explicitly already in the first total time derivative of the constraint function. Another problem is that the high-frequency control switching may cause dangerous vibrations (the so-called chattering effect) [15, 16, 17].

The issues can be settled in a few ways. High-gain control with saturation is used to overcome the chattering effect approximating the sign-function in a narrow boundary layer around the switching manifold [46], the sliding-sector method [20] is suitable to control disturbed linear time-invariant systems. The sliding-mode order approach [10, 26] considered in this chapter is capable to treat successfully both the chattering and the relative-degree restrictions preserving the sliding-mode features and improving its accuracy.

High order sliding mode (HOSM) [10, 26, 31] is actually a movement on a discontinuity set of a dynamic system understood in Filippov's sense [13]. The sliding order characterizes the dynamics smoothness degree in the vicinity of the mode. Let the task be to provide for keeping a constraint given by equality of a smooth function σ to zero. While successively differentiating σ along trajectories, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes $\sigma = 0$ may be classified by the number r of the first successive total derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist due to some reason like trajectory nonuniqueness. That number is called sliding order. The words "rth order sliding" are often abridged to "r-sliding".

The standard sliding mode, on which most variable structure systems (VSS) are based, is of the first order ($\dot{\sigma}$ is discontinuous). While the standard

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modes feature finite time convergence, convergence to HOSMs may be asymptotic as well. While the standard sliding mode precision is proportional to the time interval between the measurements or to the switching delay, r-sliding mode realization may provide for up to the rth order of sliding precision with respect to the measurement interval (Levant 1993). Properly used, HOSM practically removes the chattering effect and the rth order sliding mode is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{(r-1)} = 0. \quad (1)$$

forming an r-dimensional condition on the state of the dynamic system.

Asymptotically stable HOSMs appear in many systems with traditional sliding-mode control [18]. In particular, if the relative degree of the constraint is higher than 1, an auxiliary constraint is usually built, being a linear combination of the original constraint and its successive total time derivatives, so that it has the first relative degree [46]. As a result, an asymptotically stable HOSM with respect to the original constraint arises. HOSMs are also deliberately introduced in systems with dynamical sliding modes [44, 37]. The limit sliding accuracy in the asymptotic representation is the same in that case, as that of the standard 1-sliding mode [46]. The asymptotic convergence to the constraint inevitably complicates the overall system performance analysis.

While finite-time-convergent arbitrary-order sliding-mode controllers are still theoretically studied [30, 31, 14], 2-sliding controllers are already successfully implemented for the solution of practical problems [5, 7, 12, 29, 45, 40, 22, 47, 48, 49, 50]. The term "r-sliding controller" replaces here and further the longer expression "finite-time-convergent r-sliding-mode controller".

Construction of r-sliding controllers, $r \geq 3$, is rather difficult due to the high dimension of the problem. Thus, only one family of such controllers [30] was known until recently [33, 34, 36]. Almost all known HOSM controllers possess specific homogeneity called the r-sliding homogeneity [34]. Thus, new finite-time convergent HOSM controllers are naturally constructed based on the homogeneity-based approach. The homogeneity makes the convergence proofs of the HOSM controllers standard and provides for the highest possible asymptotic accuracy in the presence of measurement noises, delays and discrete measurements [34]. With τ being the sampling interval, the accuracy $\sigma = O(\tau^r)$ is attained. An output-feedback controller with the same asymptotical features is obtained, when a recently developed robust exact homogeneous differentiator of the order $r-1$ [27, 31] is included as a standard part of the homogeneous r-sliding controller.

The r-sliding controllers [30, 31] actually require only the knowledge of the system relative degree r. The produced control is a discontinuous function of the tracking error σ and of its real-time-calculated successive derivatives σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$. The chattering effect is successfully treated, provided the control derivative is used as a new control input [26, 4, 30]. Unfortunately, the discontinuity set of controllers [30, 31] is a complicated stratified union of manifolds with codimension varying in the range from 1 to r, which causes certain transient chattering. To avoid it one needs to increase artificially the relative degree r, inevitably complicating the controller implementation [30, 31]. The finite-time-stable exact tracking is lost with alternative controllers developed in [50] and [3] for $r = 2$ and $r = 3$ respectively.

A sliding-mode controller of a new type is developed in this chapter, being a feedback function of σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$, continuous everywhere except the manifold (1) of the r-sliding mode. The mode $\sigma \equiv 0$ is established after a finite-time transient. In the presence of errors in evaluation of the output σ and its derivatives, a motion in some vicinity of (1) takes place. Therefore, control is practically a continuous function of time, since actually the trajectory never hits the manifold (1) with $r > 1$. The controller design is based on the homogeneity reasoning.

Provided the robust exact finite-time-convergent differentiator [27, 31] is implemented, an output-feedback controller is obtained. It provides for exact tracking $\sigma \equiv 0$ if the measurements of the tracking error σ are exact, and for σ proportional to the maximal measurement error otherwise. Its transient features are much better than those of the known r-sliding controllers [30, 31] (Sections 4, 7). Simulation demonstrates the practical applicability of the new controller.

2 A "Black-box" control problem and its sliding-mode solution

Discontinuous differential equations.

Definition 1. *A differential inclusion $\dot{x} \in F(x)$ is further called a Filippov differential inclusion if the vector set $F(x)$ is non-empty, closed, convex, locally bounded and upper-semicontinuous [13]. The latter condition means that the maximal distance of the points of $F(x)$ from the set $F(y)$ vanishes when $x \to y$. Solutions are defined as absolutely-continuous functions of time satisfying the inclusion almost everywhere.*

Solutions of Filippov inclusions exist for any initial conditions and have most of the well-known standard properties except uniqueness [13].

Definition 2. *It is said that a differential equation $\dot{x} = f(x)$ with a locally-bounded Lebesgue-measurable right-hand side is understood in the Filippov sense, if it is replaced by a special Filippov differential inclusion $\dot{x} \in F(x)$. In the most usual case, when f is continuous almost everywhere, the procedure is to take $F(x)$ being the convex closure of the set of all possible limit values of f at a given point x, obtained when its continuity point y tends to x. In the general case approximate-continuity [42] points y are taken (one of the equivalent definitions by Filippov [13]). A solution of $\dot{x} = f(x)$ is defined as a solution of $\dot{x} \in F(x)$.*

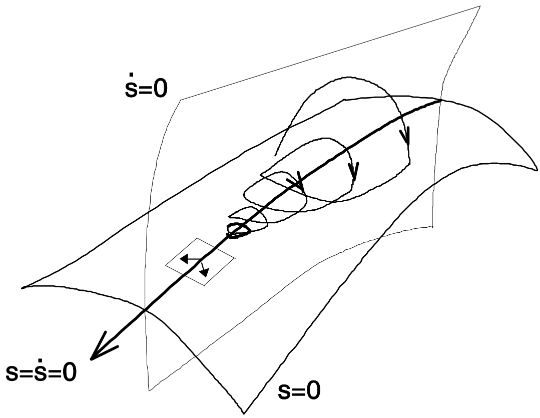

Fig. 1. Convergence to 2-sliding mode

Values of f on any set of the measure 0 do not influence the Filippov solutions. Note that with continuous f the standard definition is obtained. Following is the formal definition of the high order sliding mode. For simplicity we restrict ourselves to sliding modes with respect to scalar constraint functions.

Consider a discontinuous differential equation $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f(\cdot)$ is a locally bounded measurable vector function. Let it be understood in the Filippov sense and let a constraint be given by an equation $s(x) = 0$, where $s : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth function.

Definition 3. *Suppose that total time derivatives $s, \dot{s}, \ddot{s}, ..., s^{(r-1)}$ along the system trajectories exist and are continuous (single-valued) functions of x. Then, the r-th order sliding set is determined by the equalities*

$$s = \dot{s} = \ddot{s} = ... = s^{(r-1)} = 0 \tag{2}$$

forming an r-dimensional condition on the state of the dynamic system.

Let the r-sliding set (2) be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories). Then the corresponding motion on the set (2) is said to be in the r-sliding mode with respect to the constraint function s.

The rth derivative $s^{(r)}$ is mostly supposed to be discontinuous, non-existent or not a single-valued function. The definition is extended to the non-autonomous case adding the fictitious equation $\dot{t} = 1$. A sliding mode is called stable if the corresponding integral sliding set (2) is stable. A typical possible trajectory when approaching a 2-sliding mode is shown in Fig. 1. Note that if the sliding order is higher than 1, the entrance into the sliding mode can require finite or infinite time as well.

Black-Box control problem.

Let a Single-Input-Single-Output (SISO) system to be controlled have the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \qquad (3)$$
$$\sigma \;:\; (t,x) \longmapsto \sigma(t,x) \in \mathbb{R},$$

where σ is the measured output of the system, u is control. Smooth functions a, b, σ are assumed unknown, the dimension n can also be uncertain. The task is to make σ vanish in finite time by means of a possibly discontinuous feedback and to keep $\sigma \equiv 0$. The solutions are understood in the Filippov sense [13], and the system trajectories are supposed to be infinitely extendable in time for any bounded Lebesgue-measurable input. Although it is formally not needed, the weakly minimum-phase property is often required in practice.

Extend the system by means of a fictitious equation $\dot{t} = 1$. Let $\tilde{x} = (x,t)^t$, $\tilde{a}(\tilde{x}) = (a(t,x),1)^t$, $\tilde{b}(\tilde{x}) = (b(t,x),0)^t$. Then the system takes on the form

$$\dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{b}(\tilde{x})u, \; \sigma = \sigma(\tilde{x}). \qquad (4)$$

According to [21] the equality of the relative degree of system (4) to r means that the Lie derivatives $L_{\tilde{b}}\sigma, L_{\tilde{b}}L_{\tilde{a}}\sigma, \ldots, L_{\tilde{b}}L_{\tilde{a}}^{r-2}\sigma$ equal zero identically in a vicinity of a given point and $L_{\tilde{b}}L_{\tilde{a}}^{r-1}\sigma$ is not zero at the point. It is assumed that the relative degree r [21] of the system is constant and known.

The equality of the relative degree to r means, in a simplified way, that u first appears explicitly only in the rth total time derivative of σ. As follows from [21] the equation

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad g(t,x) \neq 0, \qquad (5)$$

holds, where $h(t,x) = L_{\tilde{a}}^r\sigma = \sigma^{(r)}|_{u=0}$, $g(t,x) = L_{\tilde{b}}L_{\tilde{a}}^{r-1}\sigma = \frac{\partial}{\partial u}\sigma^{(r)}$. Thus, h, g may be defined on the basis of the input-output relations, and are unknown smooth functions. The uncertainty prevents immediate reduction of (3) to the standard form (5). Suppose that the inequalities

$$|\sigma^{(r)}|_{u=0}| \leq C, \quad 0 < K_m \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_M. \qquad (6)$$

hold for some $K_m, K_M, C > 0$. These conditions are satisfied at least locally for any smooth system (3) having a well-defined relative degree at a given point with $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$. Assume that (6) holds globally. Then (5), (6) imply the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \qquad (7)$$

The problem is solved in two steps. First a bounded feedback control

$$u = \Psi(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \tag{8}$$

is constructed, such that all trajectories of (7), (8) converge in finite time to the origin $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ of the r-sliding phase space σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$. At the next step the lacking derivatives are real-time evaluated, producing an output-feedback controller. The function Ψ is assumed to be a Borel-measurable function, which provides for the Lebesgue measurability of composite functions to be obtained further. Actually all functions used in the sliding-mode control theory satisfy this restriction. In particular, any superposition of piece-wise continuous functions is Borel-measurable.

The differential inclusion (7), (8) is understood here in the Filippov sense, which means that the right-hand side set is enlarged producing a Filippov inclusion. To this end the above Filippov procedure is applied to the function Ψ and the obtained Filippov set is substituted for u in (7), producing a Filippov inclusion to replace (7), (8). Any solution of (7), (8) is defined as a solution of the constructed Filippov inclusion. Every time when a differential inclusion is considered in this chapter, an appropriate Filippov inclusion replaces it.

Note that the function Ψ has to be discontinuous at the origin. Indeed, otherwise u is close to the constant $\Psi(0, 0, ..., 0)$ in a small vicinity of the origin, and, taking $c \in [-C, C]$ and $k \in [K_m, K_M]$ so that $c + k\Psi(0, 0, ..., 0) \neq 0$, achieve that (8) cannot stabilize the dynamic system $\sigma^{(r)} = c + ku$. Thus, $\sigma^{(r-1)}$ is to be discontinuous along the trajectories of the original system (3), (8), which means that the r-sliding mode $\sigma \equiv 0$ is to be established. All known high-order sliding controllers [4, 7, 26, 31, 36] may be considered as controllers for (7) steering σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ to 0 in finite time. Inclusion (7) does not "remember" either the original system (3) or the way it was obtained. Thus, such controllers are obviously robust with respect to any perturbations preserving the system relative degree and (6).

3 Homogeneous sliding modes and their features

Definition 4. *A function $f : \mathbb{R}^n \to \mathbb{R}$ (respectively a set-valued function $F(x) \in \mathbb{R}$, $x \in \mathbb{R}^n$) is called homogeneous of the degree $q \in \mathbb{R}$ with the dilation $d_\kappa : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n)$ [2], where m_1, ..., m_n are some positive numbers (weights), if the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ (respectively $F(x) = \kappa^{-q} F(d_\kappa x)$) holds for any x and $\kappa > 0$.*

For example, $x^3 + y^2$ is a homogeneous function of the degree 6 with the weights 2 and 3 of x and y respectively.

Definition 5. *A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, (respectively a vector-set field $F(x) \in \mathbb{R}^n$) is called homogeneous of the degree $q \in \mathbb{R}$ with the dilation $d_\kappa : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n)$ [2], where m_1, ..., m_n are* *some positive numbers (weights), if the identity* $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$ *holds for any* x *and* $\kappa > 0$ *(respectively* $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ *).*

The non-zero homogeneity degree q of a vector field (vector-set inclusion) can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, \ldots, m_n$.

Definition 6. *A discontinuous differential equation* $\dot{x} = f(x)$ *is called homogeneous in the Filippov sense, provided it is understood in the Filippov sense and the corresponding equivalent Filippov inclusion* $\dot{x} \in F(x)$ *is homogeneous.*

Hence, the Filippov homogeneity property is not affected by the values of f on a point set of the measure 0. Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$) with respect to the combined time-coordinate transformation $G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x)$, $p = -q$, where p can be considered as the weight of t.

The natural question arises what is the difference between a function and a scalar vector field (a set-valued function and a scalar vector-set field) in this context. The answer is that the latter is associated with a scalar differential equation (inclusion), i.e. the differentiation with respect to time implicitly appears. For example, the function x^3 is a homogeneous function of the degree 3 with the dilation $d_\kappa : x \mapsto \kappa x$, i.e. with the weight 1 of x. At the same time the vector field x^3 is associated with the differential equation $\dot{x} = x^3$ and has the degree 2, which corresponds to the weight -2 of the time variable t. The corresponding weight balance equation is as follows: weight of x minus weight of t equals homogeneity degree of the right-hand side function, i.e. 1 - (-2) = 3. Thus $\begin{pmatrix} x^{1/5} y^{2/3} + y \\ x^{1/5} + y^{1/3} \end{pmatrix}$ is an example of a homogeneous vector field with the dilation $d_\kappa : (x, y) \mapsto (\kappa^5 x, \kappa^3 y)$, weights 5, 3, and the homogeneity degree -2.

Definition 7. *A differential inclusion* $\dot{x} \in F(x)$ *(equation* $\dot{x} = f(x)$*) is further called globally uniformly finite-time stable at 0, if it is Lyapunov stable and for any* $R > 0$ *there exists* $T > 0$ *such that any trajectory starting within the disk* $||x|| < R$ *stabilizes at zero in the time* T*.*

Definition 8. *A differential inclusion* $\dot{x} \in F(x)$ *(equation* $\dot{x} = f(x)$*) is further called globally uniformly asymptotically stable at 0, if it is Lyapunov stable and for any* $R > 0$, $\varepsilon > 0$ *exists* $T > 0$ *such that any trajectory starting within the disk* $||x|| < R$ *enters the disk* $||x|| < \varepsilon$ *in the time* T *to stay there forever.*

A set D is called dilation retractable if $d_\kappa D \subset D$ for any $\kappa < 1$.

Definition 9. *A homogeneous differential inclusion* $\dot{x} \in F(x)$ *(equation* $\dot{x} = f(x)$*) is further called contractive if there are 2 compact sets* D_1, D_2 *and* $T > 0$ *such that* D_2 *lies in the interior of* D_1 *and contains the origin;* D_1 *is dilation-retractable; and all trajectories starting at the time 0 within* D_1 *are localized in* D_2 *at the time moment* T*.*

Theorem 1. *[34] Let $\dot{x} \in F(x)$ be a homogeneous Filippov inclusion with a negative homogeneous degree $-p$, then definitions 7-9 are equivalent and the maximal settling time is a continuous homogeneous function of the initial conditions of the degree p.*

Equivalence of definitions 7, 8 with negative homogeneous degrees is proved also in [41]. An important consequence is that, due to the obvious robustness of the contractivity property, finite-time stability of a homogeneous differential inclusion with negative homogeneous degree is insensitive with respect to homogeneous perturbations small in some bounded region.

Let $\dot{x} \in F(x)$ be a homogeneous Filippov differential inclusion. Consider the case of "noisy measurements" of x_i with the noise magnitude $\tau^{m_i} c_i$, $i = 1, ..., n$,

$$\dot{x} \in F(x_1 + \tau^{m_1}[-c_1, c_1], ..., x_n + \tau^{m_n}[-c_n, c_n]), \;\; \tau > 0, c_i > 0.$$

Applying successively the closure of the right-hand-side graph and the convex closure at each point x, obtain some new Filippov differential inclusion $\dot{x} \in F_\tau(x)$.

Theorem 2. *[34] Let $\dot{x} \in F(x)$ be a globally uniformly finite-time-stable homogeneous Filippov inclusion with the homogeneity weights m_1, ..., m_n and the degree $-p < 0$, and let $\tau > 0$, c_1, ..., $c_n > 0$. Suppose that a continuous function $x(t)$ be defined for any $t \geq -\tau^p$ and satisfy some initial conditions $x(t) = \xi(t)$, $t \in [-\tau^p, 0]$. Then if $x(t)$ is a solution of the disturbed inclusion $\dot{x}(t) \in F_\tau(x(t + [-\tau^p, 0]))$, $0 < t < \infty$, the inequalities $|x_i| \leq \gamma_i \tau^{m_i}$ are established in finite time with some positive constants γ_i dependent only on the constants c_i and independent of τ and ξ.*

Note that Theorem 2 covers the cases of retarded or discrete noisy measurements of all or some of the coordinates and any mixed cases. In particular, infinitely extendable solutions certainly exist in the case of noisy discrete measurements of some variables or in the constant time-delay case.

4 Quasi-continuous homogeneous sliding-mode control

Suppose that feedback (8) imparts homogeneity properties to the closed-loop inclusion (7), (8). Due to the term $[-C, C]$, with $C > 0$ the right-hand side of (7) can only be a homogeneous set-valued function of the homogeneity degree 0. Indeed, with a positive degree the right hand side of (7), (8) approaches zero near the origin, which is not possible with $C > 0$. With a negative degree it is not bounded near the origin, which contradicts the local boundedness of Ψ. Thus, the homogeneity degree of the right-hand side of (7) is to be 0, and the homogeneity degree of the function $\sigma^{(r-1)}$ is to be opposite to the homogeneity degree of the whole system (i.e. equals the weight of the time variable).

Scaling the system homogeneity degree to -1, achieve that the homogeneity weights of t, σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are $1, r, r-1, ..., 1$ respectively. This homogeneity is further called *the r-sliding homogeneity.*

Definition 10. *The inclusion (7), (8) and controller (8) are called r-sliding homogeneous if for any $\kappa > 0$ and $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ the combined time-coordinate transformation*

$$G_\kappa : (t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \mapsto (\kappa t, \kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}) \tag{9}$$

preserves the closed-loop differential inclusion (7), (8).

Note that the Filippov differential inclusion corresponding to the closed-loop inclusion (7), (8) is also r-sliding homogeneous. Indeed, the convexity, the limiting process and the Lebesgue measurability are invariant with respect to the linear time-coordinate transformation (9).

Transformation (9) transfers (7), (8) into

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\Psi(\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}).$$

Hence, (8) is r-sliding homogeneous iff

$$\Psi(\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}) = \Psi(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}). \tag{10}$$

Such a homogeneous controller is inevitably discontinuous at the origin $(0, ..., 0)$, unless Ψ is a constant function. It is also uniformly bounded, since it is locally bounded and takes on all its values in any vicinity of the origin.

A controller is called r-sliding homogeneous *in the broader sense* if (9) preserves the resulting trajectories of (7). In particular, the sub-optimal 2-sliding controller [4, 7] is homogeneous, though it does not have the feedback form (8). Since the values of Ψ on any zero-measure set do not affect the corresponding Filippov inclusion and the homogeneity in the Filippov sense, such changes of Ψ also do not affect the trajectories and the controller homogeneity in the broader sense. Almost all known r-sliding controllers, $r \geq 2$, are r-sliding homogeneous. The only important exception is the terminal sliding mode controller $u = -\alpha \operatorname{sign}(\dot{\sigma} + \beta\sigma^\rho)$, where $\rho = (2k+1)/(2m+1)$, $\alpha, \beta > 0$, $k < m$, and k, m are natural numbers [38]. Indeed, the identity $\operatorname{sign}(\kappa\dot{\sigma} + \beta(\kappa^2\sigma)^\rho) = \operatorname{sign}(\dot{\sigma} + \beta\sigma^\rho)$ requires $\rho = 1/2$ and $\sigma \geq 0$. The following r-sliding homogeneous controllers solve the general problem stated in Section 2.

Nested-sliding-mode (nested-SM) controller [30, 31].

This controller is based on a complicated switching motion, which can be approximately described by a sequence of nested sliding modes (Section 6). Let $p \geq r$, $i = 1, ..., r-1$, $\beta_1, ..., \beta_{r-1}$ be some positive numbers. Denote

$$N_{1,r} = |\sigma|^{(r-1)/r},\ N_{i,r} = (|\sigma|^{p/r} + |\dot{\sigma}|^{p/(r-1)} + ... + |\sigma^{(i-1)}|^{p/(r-i+1)})^{(r-i)/p},$$

$$\psi_{0,r} = \text{sign}\sigma,\ \Psi_{i,r} = \text{sign}(\sigma^{(i)} + \beta_i N_{i,r}\Psi_{i-1,r}).$$

Obviously, $N_{i,r}$ and $\Psi_{i,r}$ are r-sliding homogeneous functions of the weights $r-i$ and 0 respectively, $N_{i,r}$ is also a positive-definite function of $\sigma, \dot{\sigma}, ..., \sigma^{(i-1)}$. Then the corresponding r-sliding homogeneous controller is defined as $u = -\alpha\Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$. The proposed controller is easily generalized: coefficients of $N_{i,r}$ may be any positive numbers, etc. Certainly, the number of choices of β_i is infinite. Following are the controllers with $r \leq 5$, and p β_i tested for $r \leq 4$. p is here the least common multiple of $1, 2, ..., r$. The first is the relay controller, the second is close to [38].

1. $u = -\alpha\, \text{sign}\, \sigma$
2. $u = -\alpha\, \text{sign}(\sigma + |\sigma|^{1/2}\text{sign}\sigma)$,
3. $u = -\alpha\, \text{sign}(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + |\sigma|^2)^{1/6}\text{sign}(\dot{\sigma} + |\sigma|^{2/3}\text{sign}\sigma)$,
4. $u = -\alpha\, \text{sign}\{\dddot{\sigma} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + |\sigma|^3)^{1/12}\text{sign}[\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^3)^{1/6}\text{sign}(\dot{\sigma} + 0.5|\sigma|^{3/4}\text{sign}\sigma)]\}$,
5. $u = -\alpha\, \text{sign}(\sigma^{(4)} + \beta_4(|\sigma|^{12} + |\dot{\sigma}|^{15} + |\ddot{\sigma}|^{20} + |\dddot{\sigma}|^{30})^{1/60}\text{sign}(\sigma^{(3)} + \beta_3(|\sigma|^{12} + |\dot{\sigma}|^{15} + |\ddot{\sigma}|^{20})^{1/30}\text{sign}(\ddot{\sigma} + \beta_2(|\sigma|^{12} + |\dot{\sigma}|^{15})^{1/20}\text{sign}(\dot{\sigma} + \beta_1|\sigma|^{4/5}\text{sign}\sigma))))$.

Quasi-continuous controller [33, 36].

As it was shown in Section 2, any controller (8) solving the stated problem of the finite-time establishment and keeping of $\sigma \equiv 0$ is inevitably discontinuous at the r-sliding mode set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$. In order to reduce the chattering, a controller is designed, which is continuous everywhere except this set. Such a controller is naturally called quasi-continuous, for in practice, in the presence of measurement noises, singular perturbations and switching delays, the motion will take place in some vicinity of the r-sliding set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ never hitting it with $r > 1$. This means that the control will actually be *a continuous function of time.* Let $i = 1, ..., r-1$. Denote

$$\varphi_{0,r} = \sigma,\ N_{0,r} = |\sigma|,\ \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \text{sign}\,\sigma,$$
$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r},\ N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)},$$
$$\Psi_{i,r} = \varphi_{i,r}/N_{i,r},$$

where $\beta_1, ..., \beta_{r-1}$ are positive numbers. The following proposition is easily proved by induction.

Proposition 1. *Let $i = 0, ..., r-1$. $N_{i,r}$ is positive definite, i.e. $N_{i,r} = 0$ iff $\sigma = \dot{\sigma} = ... = \sigma^{(i)} = 0$. The inequality $|\Psi_{i,r}| \leq 1$ holds whenever $N_{i,r} > 0$. The function $\Psi_{i,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ is continuous everywhere (i.e. it can be redefined by continuity) except the point $\sigma = \dot{\sigma} = ... = \sigma^{(i)} = 0$.*

Theorem 3. *Provided β_1,..., β_{r-1}, $\alpha > 0$ are chosen sufficiently large in the list order, both above designs result in the r-sliding homogeneous controller*

$$u = -\alpha \Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \tag{11}$$

providing for the finite-time stability of (7), (11). The finite-time stable r-sliding mode $\sigma \equiv 0$ is established in the system (3), (11).

Any time the finite-time stability is mentioned in this chapter it means that the maximal possible transient time is a locally bounded function of initial conditions [34]. The main points of the proof are given in Section 6. It follows from Proposition 1 that control (11) is continuous everywhere except the r-sliding mode $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$. Another quasi-continuous controller is constructed in [34] as a homogeneous regularization of the above nested-sliding-mode controller.

Each choice of $\beta_1, ..., \beta_{r-1}$ determines a controller family applicable to all systems (3) of relative degree r. Parameter $\alpha > 0$ is chosen specifically for any fixed C, K_m, K_M, most conveniently by computer simulation, avoiding redundantly large estimations of C, Km, KM. Obviously, α is to be negative with $\frac{\partial}{\partial u}\sigma^{(r)} < 0$. Following are quasi-continuous controllers with $r \leq 4$ and simulation-tested β_i.

1. $u = -\alpha \operatorname{sign} \sigma$,
2. $u = -\alpha(\dot{\sigma} + |\sigma|^{1/2} \operatorname{sign} \sigma)/(|\dot{\sigma}| + |\sigma|^{1/2})$,
3. $u = -\alpha[\ddot{\sigma} + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{-1/2}(\dot{\sigma} + |\sigma|^{2/3} \operatorname{sign} \sigma)]/$
   $[|\ddot{\sigma}| + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}]$,
4. $\varphi_{3,4} = \dddot{\sigma} + 3[\ddot{\sigma} + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{-1/3}(\dot{\sigma} + 0.5|\sigma|^{3/4} \operatorname{sign} \sigma)]$
   $[|\ddot{\sigma}| + (|\dot{\sigma}| + 0.5|\sigma|^{3/4}]^{-1/2}$,
   $N_{3,4} = |\dddot{\sigma}| + 3[|\ddot{\sigma}| + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2}, \quad u = -\alpha\varphi_{3,4}/N_{3,4}$.

While the control is a continuous function of time everywhere except the r-sliding set, it may have infinite derivatives when certain surfaces are crossed. All further Theorems are standard consequences [34] of the r-sliding homogeneity of controller (11) and Theorems 1, 2.

Theorem 4. *Let the control value be updated at the moments t_i, with $t_{i+1} - t_i = \tau = const > 0$, $t \in [t_i, t_{i+1})$ (the discrete sampling case). Then controller (11) provides in finite time for keeping the inequalities $|\sigma| < \mu_0\tau^r$, $|\dot{\sigma}| < \mu_1\tau^{r-1}$, ..., $|\sigma^{(r-1)}| < \mu_{r-1}\tau$ with some positive constants μ_0, μ_1, ..., μ_{r-1}.*

That is the best possible accuracy attainable with discontinuous $\sigma^{(r)}$ [26]. The following result shows robustness of controller (11) with respect to measurement errors.

Theorem 5. *Let $\sigma^{(i)}$ be measured with accuracy $\eta_i\varepsilon^{(r-i)/r}$ for some fixed $\eta_i > 0$, $i = 1, ..., r-1$. Then with some positive constants μ_i the inequalities $|\sigma^{(i)}| \leq \mu_i\varepsilon^{(r-i)/r}$, $i = 0, ..., r-1$, are established in finite time for any $\varepsilon > 0$.*

The convergence time may be reduced by changing coefficients β_j. Another way is to substitute $\lambda^{-j}\sigma^{(j)}$ for $\sigma^{(j)}$ and $\lambda^r\alpha$ for α , $\lambda > 0$, causing convergence time to be diminished approximately by λ times [35].

Implementation of r-sliding controller when the relative degree k is less than r. Introducing successive time derivatives $u, \dot{u}, ..., u^{(r-k-1)}$ as new auxiliary variables and $u^{(r-k)}$ as a new control, achieve different modifications of each r-sliding controller intended to control systems with relative degrees $k = 1, 2, ..., r$. The resulting control is $(r-k-1)$-smooth function of time with $k < r$, a Lipschitz function with $k = r-1$ and a bounded "infinite-frequency switching" function with $k = r$.

Chattering removal. The same trick removes or considerably attenuates the chattering effect. For example, substituting $u^{(r-1)}$ for u in (8), receive a local r - sliding controller to be used instead of the relay controller $u = -\text{sign}\sigma$ and attain rth order sliding precision with respect to τ by means of $(r-2)$-smooth control with Lipschitz $(r-2)$th time derivative. It needs to be modified for global usage.

Controlling systems nonlinear on control. Consider a system $\dot{x} = f(t, x, u)$ nonlinear on control. Let $\frac{\partial}{\partial u}\sigma^{(i)}(t, x, u) = 0$ for $i = 1, ..., r-1$, $\frac{\partial}{\partial u}\sigma^{(r)}(t, x, u) > 0$. It is easy to check that the problem is now reduced to that considered above with relative degree $r+1$ by introducing the new auxiliary state variable u and the new control $v = \dot{u}$.

5 Output-feedback sliding-mode control

Asymptotic features of the known high-order sliding mode controllers [26, 30, 31, 7] are easily obtained from Theorem 2. Any r-sliding homogeneous controller can be complemented by an $(r-1)$th order differentiator [1, 5, 24, 27, 31, 23, 52] producing an output-feedback controller. In order to preserve the demonstrated exactness, finite-time stability and the corresponding asymptotic properties, the natural way is to calculate $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ in real time by means of a robust finite-time convergent exact homogeneous differentiator [31]. Its application is possible due to the boundedness of $\sigma^{(r)}$ provided by the boundedness of the feedback function Ψ in (8). Following is the short description of the differentiator.

5.1 Arbitrary-order real-time exact robust differentiation.

Suppose that it is known that the input signal is compounded of a smooth signal $f_0(t)$ to be differentiated and a noise being a bounded Lebesgue-measurable function of time. Both signals are unknown and only their sum is available. It is proved that if the base signal $f_0(t)$ has pth derivative with Lipschitz's constant $L > 0$, the best possible kth order differentiation accuracy is $d_k \; L^{k/(p+1)} \; \varepsilon^{(p-k+1)/(p+1)}$, where $d_k > 1$ may be estimated (this asymptotic representation may be improved with additional restrictions on $f_0(t)$). Moreover, it is proved that such a robust exact differentiator really exists [27, 31].

Let the aim be to find real-time robust estimations of $f_0(t), f_0(t), ..., f_0^{(p)}(t)$, being exact in the absence of measurement noise and continuously depending on the noise magnitude. The differentiator is recursively constructed.

Let a $(p-1)$th-order differentiator $D_{p-1}(f(t), L)$ produce outputs D_{p-1}^i $(i = 0, 1, ..., p-1)$, which are estimates of $f_0, \dot{f}_0, , ..., f_0^{(p-1)}$ for any input signal f with $f_0^{(p-1)}$ having Lipschitz constant $L > 0$. Then, the pth order differentiator has the outputs $z_i = D_p^i$, $i = 0, 1, ..., p$, defined as follows:

$$
\begin{aligned}
&\dot{z}_0 = \nu, \quad \nu = -\lambda |z_0 - f(t)|^{\frac{p}{p+1}} \operatorname{sign}(z_0 - f(t)) + z_1, \\
&z_1 = D_{p-1}^0(\nu, L), \quad \ldots, \quad z_p = D_{p-1}^{p-1}(\nu, L)
\end{aligned}
\tag{12}
$$

Here $D_0(f(t), L)$ is a simple nonlinear filter

$$
D_0 : \qquad \dot{z} = -\lambda \operatorname{sign}(z - f(t)) \;, \qquad \lambda > L. \tag{13}
$$

In other words the pth order differentiator has the form

$$
\begin{aligned}
&\dot{z}_0 = \nu_0, \quad \nu_0 = -\lambda_0 |z_0 - f(t)|^{\frac{p}{p+1}} \operatorname{sign}(z_0 - f(t)) + z_1, \\
&\ldots \\
&\dot{z}_i = \nu_i, \quad \nu_i = -\lambda_i |z_i - \nu_{i-1}|^{\frac{p-i}{p-i+1}} \operatorname{sign}(z_i - \nu_{i-1}) + z_{i+1}, \\
&\ldots \\
&\dot{z}_p = -\lambda_p \operatorname{sign}(z_p - \nu_{p-1})
\end{aligned}
\tag{14}
$$

It is easy to check that the above differentiator can be rewritten in the non-recursive form

$$
\begin{aligned}
&\dot{z}_0 = z_1 - \kappa_0 |z_0 - f(t)|^{\frac{p}{p+1}} \operatorname{sign}(z_0 - f(t)) \\
&\dot{z}_1 = z_2 - \kappa_1 |z_0 - f(t)|^{\frac{p-1}{p+1}} \operatorname{sign}(z_0 - f(t)) \\
&\ldots \\
&\dot{z}_i = z_i - \kappa_i |z_0 - f(t)|^{\frac{p-i}{p+1}} \operatorname{sign}(z_0 - f(t)) \\
&\ldots \\
&\dot{z}_p = -\kappa_p \operatorname{sign}(z_0 - f(t))
\end{aligned}
\tag{15}
$$

for suitable positive constant coefficients κ_i. The coefficients are easier to be found for form (14), for in that case the pth order differentiator requires only one parameter to be found, if the lower-order differentiators are already built. Having been found for $L = 1$, the parameters are easily recalculated for any L. In the following Theorems [31] the performance of the proposed differentiator in the presence of bounded measurement noises, and with discrete-time implementation, is studied.

Theorem 6. *Let the input noise satisfy the inequality* $|f(t) - f_0(t)| \leq \varepsilon$. *Then the following inequalities are established in finite time for some positive constants* μ_i, ν_i *depending only on the parameters of differentiator (14)*

$$|z_i - f_0^{(i)}(t)| \leq \mu_i \, \varepsilon^{\frac{(p-i+1)}{(p+1)}}, \; i = 0, ..., p;$$
$$|v_i - f_0^{(i+1)}(t)| \leq \nu_i \, \varepsilon^{\frac{(p-i)}{(p+1)}}, \; i = 0, ..., p-1.$$

Exact differentiation is provided with $\varepsilon = 0$. Using recursive high-order differentiators the noise propagation is obviously counteracted as compared with the cascade implementation of first-order differentiators. Consider the discrete-sampling case, when $z_0(t_j) - f(t_j)$ is substituted for $z_0 - f(t)$, with $t_j \leq t < t_{j+1}$, $t_{j+1} - t_j = \tau > 0$.

Theorem 7. *Let* $\tau > 0$ *be the constant sampling interval in the absence of noises. Then the following inequalities are established in finite time for some positive constants* μ_i, ν_i *depending exclusively on the parameters of differentiator (14)*

$$|z_i - f_0^{(i)}(t)| \leq \mu_i \, \tau^{p-i+1}, \; i = 0, ..., p;$$
$$|v_i - f_0^{(i+1)}(t)| \leq \nu_i \, \tau^{p-i}, \; i = 0, ..., p-1.$$

Theorem 8. *Let parameters* λ_{0i}, $i = 0, 1, ..., p$, *of differentiator (14) provide for exact* $p-$*th order differentiation with* $L = 1$. *Then the parameters* $\lambda_i = \lambda_{0i} L^{1/(p-i+1)}$ *are valid for any* $L > 0$ *and provide for the accuracy* $|z_i - f_0(i)(t)| \leq \mu_i L^{i/(p+1)} \varepsilon^{(p-i+1)/(p+1)}$ *for some* $\mu_i \geq 1$.

Parameters λ_{0i} are easily found by computer simulation, successively rising the differentiator order according to (12). A set of such parameters is listed further.

5.2 Implementation of the differentiator with r-sliding homogeneous controller

The proposed output-feedback dynamical feedback has the form

$$u = \Psi(z_0, z_1, ..., z_{r-1}), \tag{16}$$

$$\begin{aligned}
&\dot{z}_0 = v_0, v_0 = -\lambda_0 L^{1/r} |z_0 - \sigma|^{(r-1)/r} \operatorname{sign}(z_0 - \sigma) + z_1, \\
&\dot{z}_1 = v_1, v_1 = -\lambda_1 L^{1/(r-1)} |z_1 - v_0|^{(r-2)/(r-1)} \operatorname{sign}(z_1 - v_0) + z_2, \\
&\ldots \\
&\dot{z}_{r-2} = v_{r-2}, v_{r-2} = -\lambda_{r-2} L^{1/2} |z_{r-2} - v_{r-3}|^{1/2} \operatorname{sign}(z_{r-2} - v_{r-3}) + z_{r-1}, \\
&\dot{z}_{r-1} = -\lambda_{r-1} L \operatorname{sign}(z_{r-1} - v_{r-2}),
\end{aligned} \tag{17}$$

where Ψ is an r-sliding homogeneous controller, $L \geq C + \sup|\Psi|K_M$, and parameters λ_i of differentiator (17) are adjusted in advance [32]. A possible choice of the differentiator parameters with $r \leq 6$ is $\lambda_{r-1} = 1.1$, $\lambda_{r-2} = 1.5$, $\lambda_{r-3} = 3$, $\lambda_{r-4} = 5$, $\lambda_{r-5} = 8$, $\lambda_{r-6} = 12$.

Taking the homogeneity weight $r - i$ for z_i , $i = 0, 1, ..., r - 1$, obtain a homogeneous differential inclusion (7), (16), (17) of the degree -1. Due to Theorem 6 the corresponding Filippov inclusion is also globally uniformly finite-time stable. Let σ measurements be corrupted by a noise being an unknown bounded Lebesgue-measurable function of time, then solutions of (3), (16), (17) are infinitely extendable in time under the assumptions of Section 3, and the following Theorems are simple consequences of Theorem 2.

Theorem 9. *Let controller (16) be r-sliding homogeneous and finite-time stable, and the parameters of the differentiator (17) be properly chosen with respect to the upper bound of* $|\Psi|$ *. Then in the absence of measurement noises the output-feedback controller (16), (17) provides for the finite-time convergence of each trajectory to the r-sliding mode* $\sigma \equiv 0$*, otherwise convergence to a set defined by the inequalities* $|\sigma| < \mu_0\delta$, $|\dot{\sigma}| < \mu_1\delta^{(r-1)/r}$, ..., $\sigma^{(r-1)} < \mu_{r-1}\delta^{1/r}$ *is ensured, where* δ *is the unknown measurement noise magnitude and* μ_0, μ_1, ..., μ_{r-1} *are some positive constants.*

In the absence of measurement noises the convergence time is bounded by a continuous function of the initial conditions in the space σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$, z_0, z_1, ..., z_{r-1} which vanishes at the origin (Theorem 1).

Theorem 10. *Under the conditions of Theorem 9 the discrete-measurement version of the controller (16), (17) provides in the absence of measurement noises for the inequalities* $|\sigma| < \mu_0\tau^r$, $|\dot{\sigma}| < \mu_1\tau^{(r-1)}$, ..., $\sigma^{(r-1)} < \mu_{r-1}\tau$ *for some* μ_0, μ_1, ..., $\mu_{r-1} > 0$.

The asymptotic accuracy provided by Theorem 10 is the best possible with discontinuous $\sigma^{(r)}$ and discrete sampling [26]. A Theorem corresponding to the case of discrete noisy sampling is also easily formulated based on Theorem 2. These results are also valid for the 2-sliding homogeneous sub-optimal controller [4, 7].

6 Proof of Theorem 3.

Only main points of the proofs are given here. Consider first the nested-sliding-mode controller. Denote $\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r}\Psi_{i-1,r}$, $i = 1, ..., r-1$. The idea of the controller is that a 1-sliding mode is established on the smooth parts of the discontinuity set Γ of (11) (Fig. 2). That sliding mode is described by the discontinuous differential equation $\varphi_{r-1,r} = 0$ of the order $r - 1$, providing in its turn for the existence of a 1-sliding mode on the surface $\varphi_{r-2,r} = 0$ embedded in the subspace $\sigma, \dot{\sigma}, ..., \sigma^{(r-2)}$. The equation $\varphi_{r-2,r} = 0$ can be

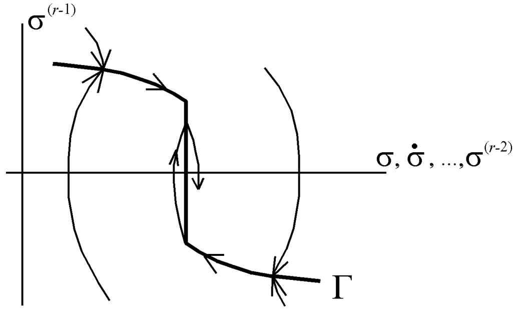

Fig. 2. The idea of r-sliding controller

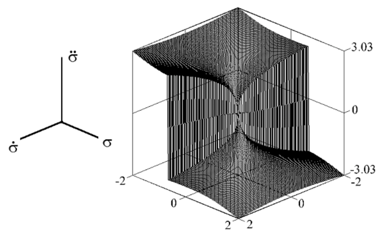

Fig. 3. The discontinuity set of the 3-sliding controller

considered as a discontinuous differential equation of the order $r-2$, providing in its turn for the appearance of the sliding mode $\varphi_{r-3,r} = 0$, etc. The last equation is $\varphi_{1,r} = \dot{\sigma} + \beta_1 |\sigma|^{(r-1)/r} \operatorname{sign} \sigma = 0$, and it provides for the finite-time establishment of the identity $\sigma \equiv 0$.

In the reality, though, no sliding mode is possible on a discontinuous surface described by the equation $\varphi_{r-1,r} = 0$. The primary sliding mode $\varphi_{r-1,r} = 0$ deteriorates when the trajectory approaches the discontinuity set of the function $\varphi_{r-1,r}$, i.e. exactly at the moment when the secondary one were to appear. The real motion takes place in some vicinity of the subset of Γ satisfying $\varphi_{r-2,r} = 0$. The larger α the smaller is the projection of this vicinity in the subspace $\sigma, \dot{\sigma}, ..., \sigma^{(r-2)}$ to the set $\varphi_{r-2,r} = 0$ (Fig. 2). Thus, the motion transfers in finite time into some vicinity of the subset satisfying $\varphi_{r-3,r} = 0$ and so on. While the trajectory approaches the r-sliding set (1), set Γ retracts to the origin in the coordinates $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$. Set Γ with $r = 3$ is shown in Fig. 3. The above qualitative reasoning makes almost obvious the contractive property of the controller. Application of Theorem 1 completes the proof.

Consider now the quasi-continuous controller. Consider first the case $r = 2$. The controller takes on the form

$$u = -\alpha(\dot{\sigma} + \beta_1 |\sigma|^{1/2} \operatorname{sign} \sigma)/(|\dot{\sigma}| + \beta_1 |\sigma|^{1/2}).$$

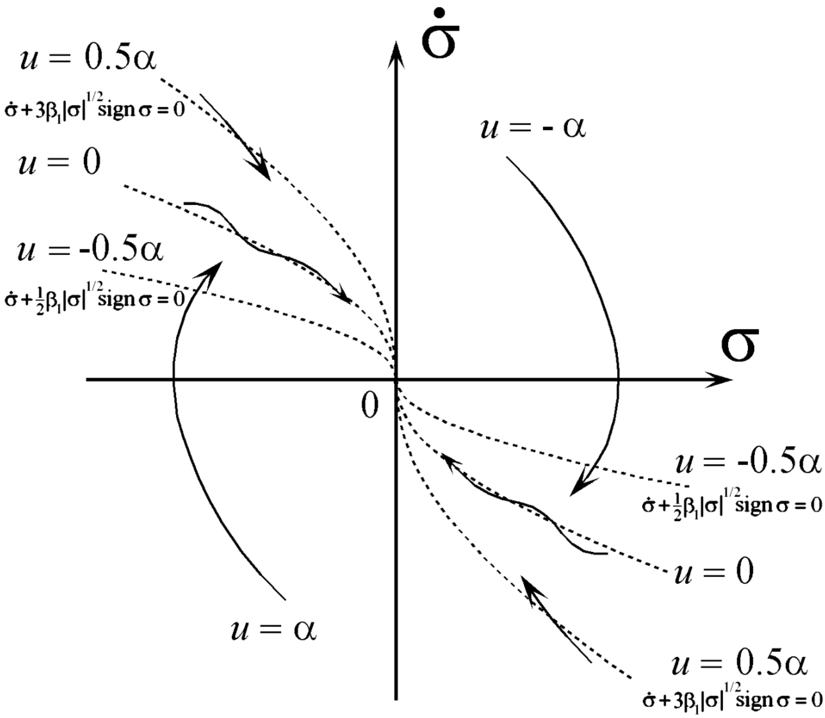

Fig. 4. 2-sliding quasi-continuous controller

The inequality $|u| \geq \frac{1}{2}\alpha$ holds outside of the region $|\Psi_{1,2}| \leq 0.5$ bounded by the curves $\dot\sigma + \frac{1}{2}\beta_1|\sigma|^{1/2}\,\mathrm{sign}\,\sigma = 0$ and $\dot\sigma + 3\beta_1|\sigma|^{1/2}\,\mathrm{sign}\,\sigma = 0$. Thus, with sufficiently large α this region is a finite-time attractor of the system (Fig. 4). Obviously, the trajectories confined between these curves inevitably converge to the origin in finite time, which completes the proof for $r = 2$. The Theorem proof generalizes this reasoning to the arbitrary relative degree.

The proof is based on a few Lemmas. Recall that the homogeneity weight of $\sigma^{(i)}$ is $r - i$, $i = 0, ..., r - 1$ and the weight of t is 1.

Lemma 1. *The weight of $N_{i,r}$ equals $r - i$, $i = 0, ..., r - 1$. Each homogeneous locally-bounded function $w(\sigma, \dot\sigma, ..., \sigma^{(i)})$ of the weight $r - i$ satisfies the inequality $|w| \leq cN_{i,r}$ for some $c > 0$.*

Indeed, $N_{i,r}$ is a positive-definite locally-bounded function (Proposition 1), which implies that $w/N_{i,r}$ is bounded on a unit sphere and, therefore, everywhere.

Lemma 2. *Let $1 \leq \ i \leq \ r - 2$, then for any positive β_i, γ_i, γ_{i+1} with sufficiently large $\beta_{i+1} > 0$ the inequality $|\sigma^{(i+1)} + \beta_{i+1} N_{i,r}^{(r-i-1)/(r-i)} \Psi_{i,r}| \leq$*

$\gamma_{i+1} N_{i,r}^{(r-i-1)/(r-i)}$ *provides for the finite-time establishment and keeping of the inequality* $|\sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}| \leq \gamma_i N_{i-1,r}^{(r-i)/(r-i+1)}$.

Proof. Consider the point set $\Omega(x) = \{(\sigma, \dot{\sigma}, ..., \sigma^{(i)}) | \, |\Psi_{i,r}| \leq \xi\}$ for some fixed $\xi > 0$, $\xi < \gamma_i/(3\beta_i)$, $\xi < 1/3$. The inequality $|\Psi_{i,r}| \leq \xi$ implies $|\sigma^{(i)}| \leq 2\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}$ and, therefore, $\Omega(\xi) \subset \Omega_1(\xi)$, where $\Omega_1(\xi)$ is defined by the inequality

$$|\sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}| \leq 3\beta_i \xi N_{i-1,r}^{(r-i)/(r-i+1)}.$$

That is equivalent, in its turn, to $\varphi_- \leq \sigma^{(i)} \leq \varphi_+$, where φ_-, φ_+ are homogeneous functions of $\sigma, \dot{\sigma}, ..., \sigma^{(i-1)}$ of the weight $r - i$. Restricting φ_- and φ_+ to the unit homogeneous sphere $\rho = 1$, where

$$\rho = (\sigma^{p/r} + \dot{\sigma}^{p/(r-1)} + ... + (\sigma^{(i-1)})^{p/(r-i+1)})^{1/p}, \quad p = 2r!\,,$$

achieves some continuous (on the sphere) functions φ_{1-} and φ_{1+}. Functions φ_{1-} and φ_{1+} can be approximated on the sphere by some smooth functions φ_{2-} and φ_{2+} from beneath and from above respectively.

Any function φ defined on the homogeneous sphere $\rho = 1$ is uniquely extended to the function Φ of the weight $w > 0$ defined in the whole space $\sigma, \dot{\sigma}, ..., \sigma^{(i)}$ by the formula

$$\Phi(\sigma, \dot{\sigma}, ..., \sigma^{(i-1)}) = \rho^w \varphi(\rho^{-r}\sigma, \rho^{-r+1}\dot{\sigma}, ..., \rho^{-r+i-1}\sigma^{(i-1)}),$$

where the function ρ is defined above. Thus, functions φ_{2-} and φ_{2+} are extended by homogeneity to the continuous homogeneous functions Φ_- and Φ_+ of $\sigma, \dot{\sigma}, ..., \sigma^{(i-1)}$ of the weight $r - i$, smooth everywhere except 0, so that $\Omega(\xi) \subset \Omega_2(\xi) = \{(\sigma, \dot{\sigma}, ..., \sigma^{(i)}) | \, \Phi_- \leq \sigma^{(i)} \leq \Phi_+\}$.

Prove now that $\Omega_2(\xi)$ is invariant and attracts the trajectories with large β_{i+1}. The "upper" boundary of $\Omega_2(\xi)$ is given by the equation $\pi_+ = \sigma^{(i)} - \Phi_+ = 0$. The inequality $|\Psi_{i,r}| \geq \xi$ is assured outside of $\Omega_2(\xi)$. Suppose that at the initial moment $\pi_+ > 0$ and, therefore, $|\Psi_{i,r}| \geq \xi$. Taking into account that $\dot{\Phi}_+$ is homogeneous of the weight $r - i - 1$ and, according to Lemma 1, $|\dot{\Phi}_+| \leq \kappa N_{i,r}^{(r-i-1)/(r-i)}$, and $|\pi_+| \leq \kappa_1 N_{i,r}$ for some κ, $\kappa_1 > 0$, achieve differentiating that with sufficiently large β_{i+1}

$$\begin{aligned} \dot{\pi}_+ &\leq (-\beta_{i+1}\xi + \gamma_{i+1}) N_{i,r}^{(r-i-1)/(r-i)} - \dot{\Phi}_+ \\ &\leq (-\beta_{i+1}\xi + \gamma_{i+1} + \kappa) N_{i,r}^{(r-i-1)/(r-i)} \\ &\leq (-\beta_{i+1}\xi + \gamma_{i+1} + \kappa)(\kappa_1^{-1}\pi_+)^{(r-i-1)/(r-i)}. \end{aligned}$$

Hence π_+ vanishes in finite time with β_{i+1} large enough. Thus, the trajectory inevitably enters the region $\Omega_2(\xi)$ in finite time. Similarly, the trajectory enters $\Omega_2(\xi)$ if the initial value of $\pi_- = \sigma^{(i)} - \Phi_- = 0$ is negative and, therefore, $|\Psi_{i,r}| \leq -\xi$. Obviously, $\Omega_2(\xi)$ is invariant.

Choosing Φ_- and Φ_+ sufficiently close to φ_{2-} and φ_{2+} on the homogeneous sphere and β_{i+1} large enough, achieve from Lemma 1 that $\Omega_2(\xi) \subset \Omega_1(\gamma_i/(3\beta_i))$ and the statement of Lemma 2. ■

Since $N_{0,r} = |\sigma|, \varphi_{0,r} = \sigma$, Lemma 2 is replaced by the next simple Lemma with $i = 0$.

Lemma 3. . *The inequality* $|\dot{\sigma} + \beta_1|\sigma|^{(r-1)/r} \operatorname{sign} \sigma| \leq \gamma_1|\sigma|^{(r-1)/r}$ *provides with* $0 \leq \gamma_1 < \beta_1$ *for the establishment in finite time and keeping of the identity* $\sigma \equiv 0$.

The proof of the Theorem is now finished by using a similar approach to Lemma 2 proof that for any $\gamma > 0$ with sufficiently large α the inequality $|\sigma^{(r-1)} + \beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}| \leq \gamma N_{r-2,r}^{1/2}$ is established in finite time and kept afterwards. ■

7 Simulation example

Car control.

Consider a simple kinematic model of car control [39]

$$\dot{x} = v\cos\varphi,\ \dot{y} = v\sin\varphi,$$

$$\dot{\varphi} = \frac{v}{l}\tan\theta,$$

$$\dot{\theta} = u,$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 5). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while x, y and φ are assumed to be measured in real time. Note that the actual control here is θ and $\dot{\theta} = u$ is used as a new control in order to avoid discontinuities of θ. Any practical implementation of the controller developed here would require some real-time coordinate transformation with φ approaching $\pm\pi/2$. Define

$$\sigma = y - g(x).$$

Let $v = \text{const} = 10\,\text{m/s}$, $l = 5\,\text{m}$, $g(x) = 10\sin 0.05x + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$. The relative degree of the system is 3 and both 3-sliding homogeneous controllers from the above lists may be applied here. Unfortunately the transient features of the nested-sliding-mode controller feature some chattering ($\alpha = 20$, Fig. 6). Thus, the relative degree is artificially increased up to 4, and the 4-sliding controller is used [30]. Also the quasi-continuous 3-sliding controller can be applied here. It was taken $\alpha = 1$, $L = 400$. The resulting output-feedback controller (16) - (17) is

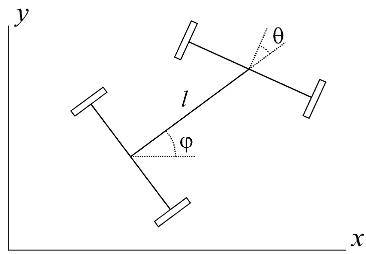

Fig. 5. Kinematic car model

$$u = -[z_2 + 2(|z_1| + |z_0|^{2/3})^{-1/2}(z_1 + |z_0|^{2/3}\,\mathrm{sign}\,\sigma)]/[|z_2| + 2(|z_1| + |z_0|^{2/3})^{1/2}]$$

$$\begin{aligned}\dot z_0 &= v_0, v_0 = -14.7361|z_0 - \sigma|^{2/3}\,\mathrm{sign}(z_0 - \sigma) + z_1,\\ \dot z_1 &= v_1, v_1 = -30|z_1 - v_0|^{1/2}\,\mathrm{sign}(z_1 - v_0) + z_2,\\ \dot z_2 &= -440\,\mathrm{sign}(z_2 - v_1).\end{aligned} \tag{19}$$

The controller parameter α is convenient to find by simulation [26, 7, 31]. The differentiator parameter $L = 400$ is taken deliberately large, in order to provide for better performance in the presence of measurement errors ($L = 25$ is also sufficient, but was found to be much worse with sampling noises), other differentiator parameters are $\lambda_2 = 1.1$, $\lambda_1 = 1.5$, $\lambda_0 = 3$. The control was applied only from $t = 1$, in order to provide some time for the differentiator convergence.

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-4}$. In the absence of noises the tracking accuracies $|\sigma| \leq 5.4 \cdot 10^{-7}$, $|\dot\sigma| \leq 2.4 \cdot 10^{-4}$, $|\ddot\sigma| \leq 0.042$ were obtained. With $\tau = 10^{-5}$ the accuracies $|\sigma| \leq 5.6 \cdot 10^{-10}$, $|\dot\sigma| \leq 1.4 \cdot 10^{-5}$, $|\ddot\sigma| \leq 0.0042$ were attained, which mainly corresponds to the asymptotic representation stated in Theorem 10. The car trajectory, 3-sliding tracking errors, steering angle θ and its derivative u are shown in Fig. 7a, b, c, d respectively. It is seen from Fig. 7c that the control u remains continuous until the entrance into the 3-sliding mode. The steering angle θ remains rather smooth and is quite feasible.

In the presence of output noise with the magnitude 0.01 m the tracking accuracies $|\sigma| \leq 0.02$, $|\dot\sigma| \leq 0.14$, $|\ddot\sigma| \leq 1.3$ were obtained. With the measurement noise of the magnitude 0.1 the accuracies changed to $|\sigma| \leq 0.20$, $|\dot\sigma| \leq 0.62$, $|\ddot\sigma| \leq 2.8$, which corresponds to the asymptotic representation stated by Theorem 9. The performance of the controller with the measurement error magnitude 0.1 m is shown in Fig. 8. It is seen from Fig. 8c that

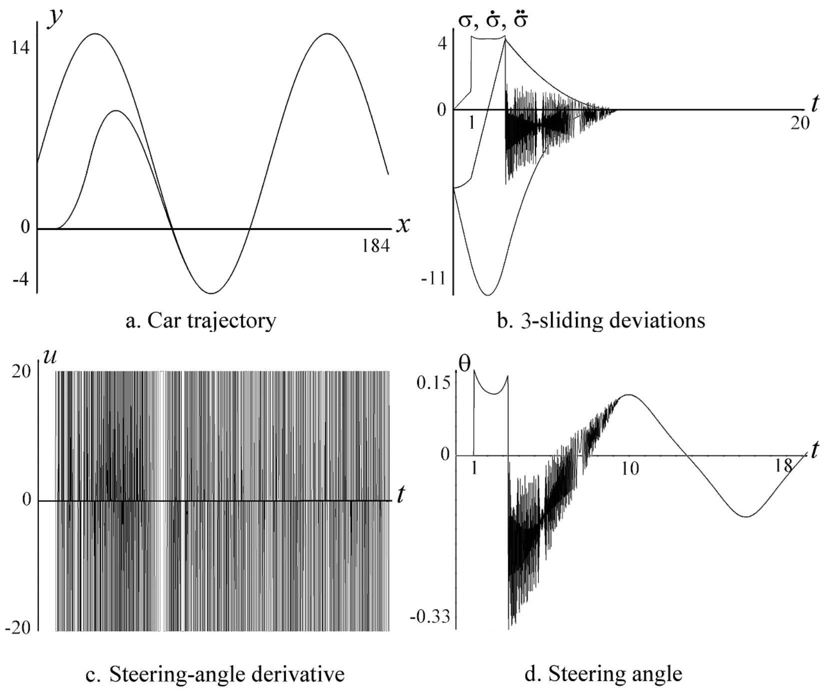

Fig. 6. Nested-SM control of the car

the control u is a continuous function of t. The steering angle vibrations have the magnitude of about 7° and the frequency 1 Hz, which is also quite feasible. The performance does not change, when the frequency of the noise varies in the range 100 - 100000. The advantages of the new controller are obvious (compare Figs. 6, 7). Simulation shows that the nested-SM controller [30, 31] is also more sensitive to the parameter choice and noises.

8 Conclusions

Stability features and asymptotic accuracy are studied of homogeneous differential inclusions with negative homogeneous degree. In particular, the uniform global finite-time stability is robust with respect to small homogeneous perturbations, if the homogeneity degree is negative. The corresponding r-sliding homogeneity notion is introduced, simplifying and standardizing design and convergence proofs of r-sliding mode controllers.

A sliding-mode SISO controller of a new type is obtained, for it provides for the finite-time stable sliding motion on the zero-dynamics manifold of high

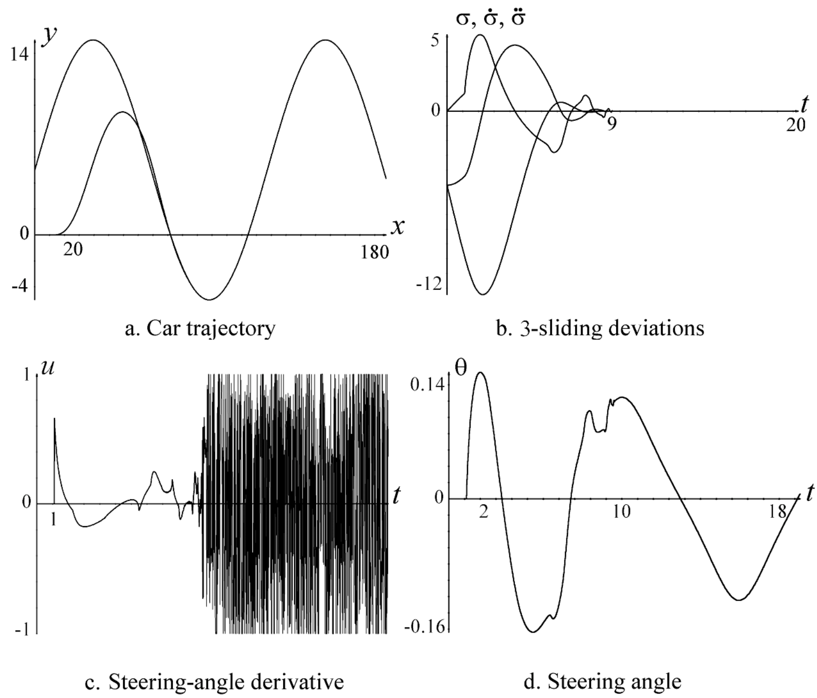

Fig. 7. 3-sliding quasi-continuous control

relative degree by means of control continuous everywhere except this manifold. The control is continuous everywhere except the high-order sliding mode itself. Since ideal sliding mode is never established in practice, the control remains a continuous function of time. As a result the chattering effect is significantly reduced.

The real-time exact differentiator of the appropriate order is combined with the proposed controller providing for the full SISO control based on the input measurements only, when the only information on the controlled uncertain process is actually its relative degree. All the features of the controller with the full state availability are preserved including the finite-time stability and the accuracy asymptotic representation. Both the proposed controller and its output-feedback version are very robust with respect to measurement noises. Only boundedness of the noise is needed, no frequency considerations are relevant.

No exact model of the process is needed. The only requirements are that the relative degree of the controlled uncertain process be known and the boundedness restrictions (6) hold. Local validity of (6) provides for the local applicability of controllers. The simulation shows that it is probably the first practically applicable output-feedback r-sliding controller with $r > 2$.

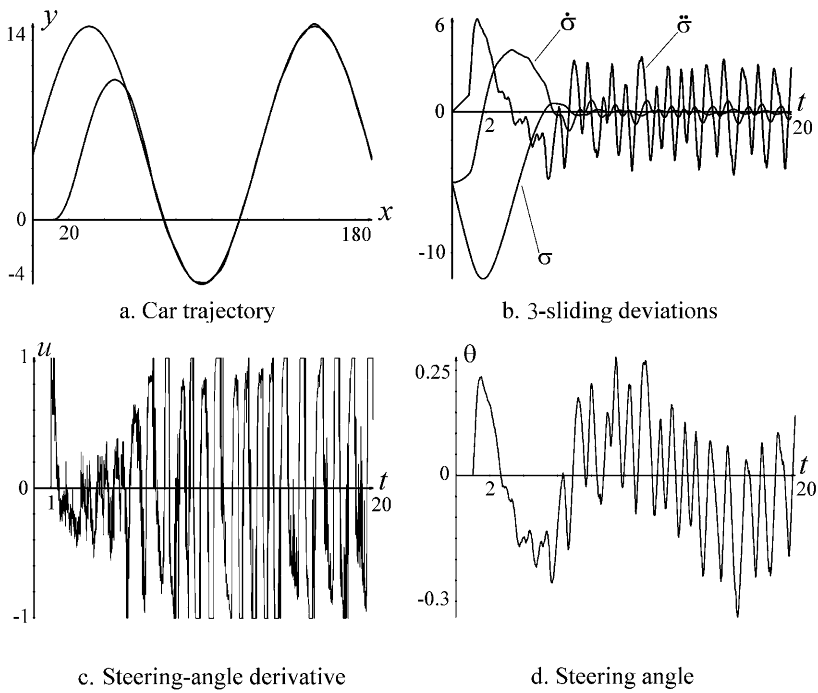

Fig. 8. Performance with the input noise magnitude 0.1m

The results of this chapter provide actually for the almost-complete theoretical and practical solution of the SISO output regulation problem, when the relative degree of the output is well defined and known. Therefore, the SISO case of the known relative degree should be considered trivial. Luckily, the condition is often violated. First of all, one may say that we never know the actual relative degree, due to the presence of some fast unaccounted-for dynamics of actuators and sensors [15, 16, 17, 8], also the mathematical model cannot be fully adequate, due to the presence of small delays, hysteresis, distributed parameters etc. The cited results by Fridman and Boiko, some new unpublished research, as well as author's own practical experience [29] show that in these cases the ideal sliding mode is replaced by an approximate one ("real sliding" mode), and the approach still preserves its most important features. In particular, if the relative degree is artificially increased in order to overcome the chattering effect, the chattering might still appear, but in the "soft" form, i.e. as infinitesimally small low-energy vibrations of physical variables. To the contrary, if the relative degree is not well defined just as the control goal is obtained, the approach cannot be directly applied. For example, the stabilization problem can be reformulated as the problem of keeping the constraint $\sigma = ||x|| = 0$, but then the relative degree of σ is almost never well defined at 0. Another difficult problem is the instability of the zero dynamics [49]. However, as usual, sliding mode control contributes here, provided an appropriate auxiliary constraint is chosen.

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