Ordinary Differential Equations – 1 (ODE-1)

Exercise 8

Question 1

Calculate the matrix exponent $\exp(A)$ for the following matrices A

$$a.\ A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \qquad b.\ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad c.\ A = \begin{pmatrix} 2 & -7 \\ 7 & 2 \end{pmatrix}, \qquad d.\ A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 3 & 3 \end{pmatrix},$$

$$e.\ A = \begin{pmatrix} -3 & 5 \\ -5 & 7 \end{pmatrix}.$$

Question 2

Find the general solution over $\mathbb{R}$ and $\mathbb{C}$, and solve the Cauchy problem for each of the following DEs:

$$a.\ \begin{cases} \dot{y}_1 = 5y_1 - y_2 & y_1(0) = 2, \\ \dot{y}_2 = 3y_1 + y_2 & y_2(0) = -1; \end{cases} \qquad b.\ \begin{cases} \dot{y}_1 = 2y_1 - y_2 & y_1(0) = 1, \\ \dot{y}_2 = 3y_1 - 2y_2 & y_2(0) = 1; \end{cases} \qquad c.\ \begin{cases} \dot{y}_1 = y_1 - y_2 & y_1(0) = 0, \\ \dot{y}_2 = 5y_1 - 3y_2 & y_2(0) = 1. \end{cases}$$

Question 3

Let the rate $\dot{x}(t)$ of the numerically measured love $x(t)$ of the first partner to the second be strictly proportional to the second partner's love $y(t)$ and vice versa with the proportionality coefficients $\alpha, \beta \in \mathbb{R}$ respectively. Negative values of x, y correspond to hatred.

Introduce the initial conditions $x(0) = x_0, y(0) = y_0$.

a. Compose the system of DEs describing these love dynamics.
b. Describe the love $x(t), y(t)$ to be obtained in the limit at $t = \infty$ in the following cases:

1. $x_0 > 0, y_0 > 0, \alpha > 0, \beta > 0$: love grows,
2. $x_0 > 0, y_0 < 0, \alpha > 0, \beta > 0$: initial hatred of one turns into love,
3. $x_0 > 0, y_0 > 0, \alpha > 0, \beta < 0$: the initial love turns into hatred for one of the partners.