Ordinary Differential Equations – 1 (ODE-1)

Exercise 7

Question 1

Prove that all solutions of the DE $\dot{y} = tye^{-y^2} + y$ are extendable in time to the whole axis t.

Question 2

Prove that no solution of the DE $\dot{y} = |t|\, y^3$ is extendable in time to the whole axis t.

Question 3

Consider the DE $\dot{y} = A(t)y, y \in \mathbb{R}^2$, where the elements of the matrix $A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$ are continuous functions of the time $t \in \mathbb{R}$. Let the matrix $Y(t) \in \mathbb{R}^{2\times 2}$ be the fundamental matrix for the DE. It means that the columns of $Y(t) = \begin{bmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{bmatrix}$ are fundamental solutions of the DE.

a. Prove that both $Y(t)$ and its transpose $Y^T(t) = \begin{bmatrix} y_{11}(t) & y_{21}(t) \\ y_{12}(t) & y_{22}(t) \end{bmatrix}$ are invertible.

b. Prove that $\left(Y^T(t)\right)^{-1}$ is a fundamental matrix for the conjugate DE $\dot{y} = -A^T(t)y$.

Question 4

Calculate the matrix exponent $\exp(A)$ for the following matrices A

$$a.\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ b.\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ c.\ A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix},\ d.\ A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Question 4

Let $y_1(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}, y_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

a. Calculate the Wronskian $W[y_1(t), y_2(t)]$.

b. What are the intervals of the linear independence of $y_1(t), y_2(t)$?