Ordinary Differential Equations – 1 (ODE-1)

Exercise 6

Question 1

Find the first-order approximation in $\varepsilon \approx 0$ over any finite interval in x around 0 for the solution of the Cauchy problem $y' = \sin x \sin y + \sin 2x \sin 2y$, $y(0) = \varepsilon$.

Question 2

Find the first-order approximation in $\varepsilon \approx 0$ over any finite interval in x around 0 for the solution of the Cauchy problem $\begin{cases} y' = \sin(y - 1 + \varepsilon) + \sin\varepsilon, \\ y(0) = 1 + \varepsilon, \qquad \varepsilon \approx 0. \end{cases}$

Question 3

Let $x \in \mathbb{R}^n$, $b(t) \in \mathbb{R}^n, A(t) \in \mathbb{R}^{n\times n}$ be integrable vector and matricial functions in $[\alpha, \beta]$ satisfying the inequalities $\|A(t)\| \le k(t), \|b(t)\| \le k(t)$ where $k(t)$ is integrable, $\int_\alpha^\beta k(t)dt < \infty$. Let $t_0 \in [\alpha, \beta]$ and consider the Cauchy problem $\begin{cases} \dot{x} = A(t)x + b(t), \\ x(t_0) = \xi. \end{cases}$

Prove that there exists a unique continuous function $\phi(t)$, $\phi \in C[\alpha, \beta]$, which satisfies the identity

$\phi(t) = \xi + \int_{t_0}^{t} \left[A(s)\phi(s) + b(s)\right] ds$ for each $t \in [\alpha, \beta]$ (the Caratheodory solution).

Hint: Check that the proof of the uniqueness-and-existence theorem remains literally the same.

Question 4

Consider the DE $y' = x + \sin y$, $x \in [0,1]$. Let $y(x), z(x)$ be two solutions of the DE, $y(0) = 1, z(0) = 1.1$. Estimate the difference $|\, y(x) - z(x)\,|$ for $x \in [0,1]$.

Question 5

Perform the linearization of the system $\begin{cases} y_1' = (1 + y_1)\sin y_2, \\ y_2' = 1 - y_1 - \cos y_2 \end{cases}$ at its critical point $(y_1, y_2) = (0,0)$.

Question 6

Let $u, v, \varphi : \mathbb{R} \to \mathbb{R}$ be continuous functions in $[\alpha, \beta]$, $\varphi > 0$, such that the inequality $\varphi(t) \le A + \int_\alpha^t [u(s)\varphi(s) + v(s)]ds$ holds for some $A > 0$ and any $t \in [\alpha, \beta]$. Prove that $\varphi(t) \le A \exp\left(\int_\alpha^t [u(s) + \frac{v(s)}{A}] ds \right)$. Hint: Use the Gronwall-Bellman Lemma.