Ordinary Differential Equations – 1 (ODE-1)

Exercise 5

Question 1 Consider the Cauchy problem

$$y' = 2y - 1, y(0) = 1$$

a. Find the solution $y = \varphi(x)$ and calculate $\varphi(0.1), \varphi(0.2), \varphi(0.3)$

b. Estimate the same values $\varphi(0.1), \varphi(0.2), \varphi(0.3)$ using the Euler approximation method with the integration step $h = 0.1$.

Question 2 Consider the Cauchy problem

$$y' = 1 - x + y, y(x_0) = y_{0.}$$

a. Prove that $y(x) = \varphi(x) = (y_0 - x_0)e^{x - x_0} + x$ is the exact solution.

b. Apply the Euler method with the step $h > 0$ and obtain $y_k(x) = (1+h)y_{k-1} + h - hx_{k-1}$, $k = 1,2,...$

c. Prove that $y_n(x) = (1+h)^n(y_0 - x_0) + x_n$. Prove that $y_n(x) \to \varphi(x)$ for each fixed x and $h = (x - x_0)/n,\ n \to \infty$.

Question 3 Solve the Cauchy problem

$$\begin{cases} y'' + \sin\left(x^2\right)y' - \arctan(x)y = \arctan(x) \\ y(0) = -1,\ y'(0) = 0 \end{cases}$$

Question 4

Consider the DE $y^{(n)} = f(x, y)$ for a smooth function $f(x, y)$. For which orders n can it simultaneously have both solutions $y_1 = x$ and $y_2 = x + x^4$?

Question 5

Let function $F(x, y)$ be non-increasing in y for each fixed x.

a. Show that, if $f(x), g(x)$ satisfy the DE $y' = F(x, y)$, then the numeric inequality $a < b$ implies $|f(a) - g(a)| \geq |f(b) - g(b)|$.

b. Prove that the statement a. implies right-hand uniqueness of the solutions, i.e. the Couchy problem $y' = F(x, y),\ y(a) = y_0$, has not more than one solution for $x \geq a$.