Lesson 6: Exact Equations

Determine whether or not each of equations below are exact. If it is exact find the solution by two methods.

1. \((2x + 4y) + (2x - 2y)y' = 0\)

**Solution.** The differential equation is not exact since

\[(2x + 4y)'_y = 4,\]

while

\[(2x - 2y)'_x = 2.\]

2. \((3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0\)

**Solution.** The differential equation is exact since

\[(3x^2 - 2xy + 2)'_y = -2x,\]

\[(6y^2 - x^2 + 3)'_x = -2x.\]

Let us find a solution.

**Method 1.** Integrating the first equation, we have

\[\int (3x^2 - 2xy + 2)dx = x^3 - x^2y + 2x + h(y).\]

To find \(h(y)\) we have

\[-x^2 + h'(y) = 6y^2 - x^2 + 3.\]

Therefore

\[h'(y) = 6y^2 + 3,\]

and

\[h(y) = \int (6y^2 + 3)dy = 2y^3 + 3y.\]

Therefore, the solution is

\[x^3 - x^2y + 2x + 2y^3 + 3y = c.\]

**Method 2.** By integrating on curve \((0,0) (x, y)\) we have

\[
\int_0^x (3u^2 - 2u \times 0 + 2)du + \int_0^y (6v^2 - x^2 + 3)dv = x^3 + 2x + 2y^3 - x^2y + 3y = c.
\]

3. \((2xy^2 + 2y) + (2x^2y + 2x)y' = 0\)

**Solution.**

\[(2xy^2 + 2y)'_y = 4xy + 2,\]

\[(2x^2y + 2x)'_x = 4xy + 2.\]

That is the differential equation is exact. Find a solution by two methods.

**Method 1.** Integrating the first equation we have

\[
\int (2xy^2 + 2y)dx = x^2y^2 + 2xy + h(y).
\]
Let us find $h(y)$. We have:

$$2x^2y + 2x + h'(y) = 2x^2y + 2x.$$ 

That is

$$h'(y) = 0,$$

and the solution is

$$x^2y^2 + 2xy = c.$$

**Method 2.** By integrating on the curve we have

$$\int_0^x (2u \times 0^2 + 2 \times 0)du + \int_0^y (2x^2v + 2x)dv = x^2y^2 + 2xy = c.$$ 

4. 

$$\frac{dy}{dx} = \frac{ax + by}{bx + cy}.$$

**Solution.** Rewrite

$$\frac{ax + by}{bx + cy} + y' = 0.$$ 

Then,

$$\left(\frac{ax + by}{bx + cy}\right)' \neq 0$$

in general, while

$$(1)'y = 0.$$ 

That is the differential equation is not exact. With what $c$ it is exact?

5. $(e^x \sin y - 2y\sin x)dx + (e^x \cos y + 2\cos x)dy = 0$

**Solution.**

$$(e^x \sin y - 2y\sin x)' = e^x \cos y - 2\sin x,$$

$$(e^x \cos y + 2\cos x)' = e^x \cos y - 2\sin x.$$ 

Therefore, the differential equation is exact.

**Method 1.**

$$\int (e^x \sin y - 2y\sin x)dx = e^x \sin y + 2y \cos x + h(y).$$ 

Next,

$$e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x.$$ 

Therefore,

$$h'(y) = 0,$$

and

$$e^x \sin y + 2y \cos x = c$$

is a solution.

**Method 2.**

$$\int_0^x (e^u \times \sin 0 - 2 \times 0 \sin u)du + \int_0^y (e^v \cos v + 2 \cos x)dv = e^x \sin y + 2y \cos x = c.$$ 

6. $(e^x \sin y + 3y)dx - (3x - e^x \sin y)dy = 0.$
7. \((ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x)dx + (xe^{xy} \cos 2x - 3)dy = 0.\)

Solution.
\[
(ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x)'_y = e^{xy} \cos 2x + xy e^{xy} \cos 2x - 2xe^{xy} \sin 2x.
\]
\[
(xe^{xy} \cos 2x - 3)'_x = e^{xy} \cos 2x + xy e^{xy} \cos 2x - 2xe^{xy} \sin 2x.
\]
Thus, the differential equation is exact.
Then,
\[
\int (xe^{xy} \cos 2x - 3)dy = \cos 2x e^{xy} - 3y + h(x)
\]
By differentiating in \(x\) we have
\[
-2 \sin 2e^{xy} + y \cos 2e^{xy} + h'(x) = ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x
\]
Therefore,
\[
h(x) = x^2,
\]
and the solution is
\[
\cos 2xe^{xy} - 3y + x^2 = c.
\]
8. \((y/x + 6x)dx + (\log x - 2)dy = 0, \; x > 0.\)

Solution.
\[
(y/x + 6x)'_y = 1/x.
\]
\[
(\log x - 2)'_x = 1/x.
\]
The differential equation is exact.
Therefore (\(x\) is positive),
\[
\int (y/x + 6x) = y \log x + 6x^2 + h(y).
\]
Differentiation in \(y\) yields
\[
\log x + h'(y) = \log x - 2,
\]
\[
h(y) = -2y.
\]
The solution is
\[
y \log x + 6x^2 - 2y = c.
\]
9. \((x \log y + xy)dx + (y \log x + xy)dy = 0, \; x > 0, \; y > 0.\)

Solution.
\[
(x \log y + xy)'_y = \frac{x}{y} + x
\]
\[
(y \log x + xy)'_x = \frac{y}{x} + y
\]
The differential equation is not exact.
10. \[
\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0.
\]
Solution. 
\[ \left( \frac{x}{(x^2 + y^2)^{3/2}} \right)'_y = -3yx(x^2 + y^2)^{1/2} \]
\[ \left( \frac{y}{(x^2 + y^2)^{3/2}} \right)'_x = -3yx(x^2 + y^2)^{5/2}. \]

Therefore, the differential equation is exact.

Then
\[ \int \frac{x}{(x^2 + y^2)^{3/2}} \, dx = \frac{1}{2} \int \frac{d(x^2 + y^2)}{(x^2 + y^2)^{3/2}} = -\frac{1}{(x^2 + y^2)^{1/2}} + h(y) \]
Next,
\[ \frac{y}{(x^2 + y^2)^{3/2}} + h'(y) = \frac{y}{(x^2 + y^2)^{3/2}}. \]
Therefore, the solution is
\[ \frac{1}{(x^2 + y^2)^{1/2}} = c. \]

Find an integrating factor and solve the following equations
11. \((3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0.\)

Solution. We have:
\[ (3x^2y + 2xy + y^3)'_y = 3x^2 + 2x + 3y^2, \]
\[ (x^2 + y^2)'_x = 2x. \]
Therefore,
\[ \frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = 3. \]
Thus, an integrating factor is
\[ I(x, y) = I(x) = \exp \left( \int 3dx \right) = e^{3x}. \]
Then, the differential equation
\[ e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0 \]
is exact.

The solution of the differential equation is
\[ \int_0^y e^{3x}(x^2 + y^2)dv = e^{3x} \left( x^2y + \frac{y^3}{3} \right). \]

12. \(y' = e^{2x} + y - 1.\)

Solution. Let us write the equation in the form
\[ (e^{2x} + y - 1)dx - dy = 0. \]
We have
\[ (e^{2x} + y - 1)'_y = 1 \]
\[ (-1)'_x = 0. \]
Therefore, \[
\frac{M'_y(x, y) - N'_x(x, y)}{N(x, y)} = -1.
\]
Then, an integrating factor is \(I(x, y) = I(x) = e^{-x},\)
and the differential equation
\((e^x + e^{-x}y - e^{-x})dx - e^{-x}dy = 0.\)
is exact.

13. \(dx + (x/y - \sin y)dy = 0.\)

**Solution.** We have
\((1)'_y = 0,\)
\((x/y - \sin y)'_x = \frac{1}{y},\)
Therefore
\[
\frac{N'_x(x, y) - M'_y(x, y)}{M(x, y)} = \frac{1}{y}.
\]
Then, an integrating factor is
\(I(x, y) = I(y) = \exp \left( \int \frac{dy}{y} \right) = y,\)
and the differential equation
\(ydx + (x - y \sin y)dy = 0\)
is exact.

14. \(ydx + (2xy - e^{-2y})dy = 0.\)

**Solution.**
\((y)'_y = 1,\)
\((2xy - e^{-2y})'_x = 2y\)
Therefore
\[
\frac{N'_x(x, y) - M'_y(x, y)}{M(x, y)} = \frac{2y - 1}{y}.
\]
Then, an integrating factor
\(I(x, y) = \int \frac{2y - 1}{y} dy = 2y - \log |y|,\)
and the differential equation
\((2y^2 - y \log |y|)dx + (4xy^2 - 2ye^{-2y} + 2xy \log |y| - 2y \log |y|e^{-2y})dy = 0\)
is exact.

15. \(e^x dx + (e^x \cot y + 2y \sec y)dy = 0.\)

**Solution.**
\((e^x)'_y = 0\)
\[(e^x \cot y + 2y \sec y)'_x = e^x \cot y\]

Therefore
\[
\frac{N'_x(x, y) - M'_y(x, y)}{M(x, y)} = \cot y,
\]

and the differential equation
\[
\cot ye^x dx + (e^x \cot^2 y + 2y \csc y) dy = 0
\]
is exact.

16. \[4(\frac{x^3}{y^2}) + (3/y)]dx + [3(\frac{x}{y^2}) + 4y]dy = 0.

Solution.
\[
4(\frac{x^3}{y^2}) + (3/y)'_y = -8x^3/y^3 - 3/y^2
\]
\[
3(\frac{x}{y^2}) + 4y)'_x = 3/y^2
\]

Therefore
\[
\frac{N'_x(x, y) - M'_y(x, y)}{M(x, y)} = \frac{8x^3/y^3 + 6/y^2}{4x^3/y^2 + 3/y} = 2y,
\]

and the differential equation
\[
[8(\frac{x^3}{y}) + 6]dx + [6(\frac{x}{y}) + 8y^2]dy = 0
\]
is exact.

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